Functional intrinsic volumes and mixed Monge–Ampère measures

Monika Ludwig
joint work with Andrea Colesanti and Fabian Mussnig

Technische Universität Wien

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Intrinsic Volumes

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  $V_0, \ldots, V_n : \mathcal{K}^n \to \mathbb{R}$ intrinsic volumes

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$$V_j(K) = \int_{Graff(n,j)} V_0(K \cap E)\,\mathrm{d}\mu_j(E) = \int_{Gr(n,j)} V_j(K|E)\,\mathrm{d}\nu_j(E)$$

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  for $K \in \mathcal{K}^n$
- $K \in \mathcal{K}^n$ with smooth boundary

$$V_j(K) = \frac{\binom{n}{j}}{n\kappa_{n-j}} \int_{\mathbb{S}^{n-1}} s_j(K, y)\,\mathrm{d}y = \frac{\binom{n}{j}}{n\kappa_{n-j}} \int_{\mathrm{bd}\,K} H_{n-j-1}(K, x)\,\mathrm{d}x$$

Valuations on Convex Bodies

- $\mathrm{Z}\colon \mathcal{K}^n \to \mathbb{R}$ is a (real-valued) **valuation** $\iff$

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

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- Hilbert's Third Problem: Dehn 1902, ...
- Classification of valuations:

Blaschke 1937, **Hadwiger** 1949, Schneider 1971, Groemer 1972, McMullen 1977, Betke & Kneser 1985, Klain 1995, Ludwig 1999, Reitzner 1999, Alesker 1999, Hug 2005, Bernig 2006, Fu 2006, Haberl 2006, Schuster 2006, Tsang 2010, Wannerer 2010, Abardia 2011, Parapatits 2011, Faifman 2013, Solanes 2014, Wang 2014, Böröczky 2015, Li 2015, Ma 2016, Colesanti 2017, Mussnig 2017, Jochemko 2018, Sanyal 2018, Zeng 2019, Xia 2019, ...

Valuations on Convex Bodies

Theorem (Hadwiger 1952)

$\mathrm{Z}\colon \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation and rotation invariant valuation*

$$\iff$$

$\exists c_0, \ldots, c_n \in \mathbb{R}:$

$$\mathrm{Z}(K) = c_0 V_0(K) + \cdots + c_n V_n(K)$$

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Corollary

$\mathrm{Z}\colon \mathcal{K}^n \to \mathbb{R}$ *is a non-trivial, continuous, j-homogeneous, translation and rotation invariant valuation*

$$\Longleftrightarrow$$

$j \in \{0, \dots, n\}$ *and* $\exists c \in \mathbb{R}$:

$$\mathrm{Z}(K) = c\, V_j(K)$$

for every $K \in \mathcal{K}^n$.

Valuations on Function Spaces

- $\mathcal{F}(X) := \{f : X \to \mathbb{R}\}$ space of real-valued functions on X
- $f \vee g := \max\{f, g\}$, $f \wedge g := \min\{f, g\}$

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Examples

- Valuations on convex bodies (via indicator or support functions)
- Valuations on star sets (via indicator or radial functions)

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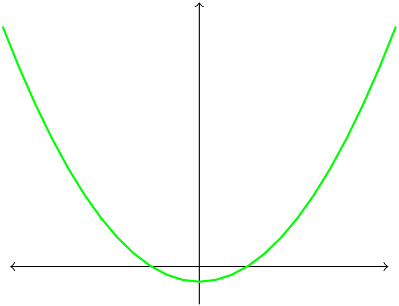
Question (L. 2010):

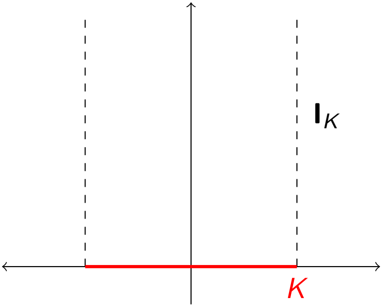
- Classification of interesting valuations on classical function spaces

Valuations on the Classical Function Spaces

- Valuations on Sobolev and BV functions:
  L.: AIM 2011, AJM 2012; Wang: IUMJ 2014; Ma: SCM 2016
- Valuations on L_p and Orlicz functions:
  Tsang: IMRN 2010, TAMS 2012; L.: AAM 2013;
  Ober: JMAA 2014; Kone: AAM 2014; Li & Ma: JFA 2017
- **Valuations on convex functions:**
  Cavallina & Colesanti: AGMS 2015; Colesanti, L. & Mussnig: IMRN 2017, CVPDE 2017, IUMJ 2020, JFA 2020; Alesker: AG 2019; Knoerr JFA 2021; Mussnig: AiM 2019, CJM 2021, JGA 2021
- Valuations on quasi-concave functions:
  Colesanti & Lombardi: 2017; Colesanti, Lombardi & Parapatits: 2017
- Valuations on continuous and Lipschitz functions:
  Villanueva: AiM 2016; Tradacete & Villanueva: JMAA 2017, AiM 2018, IMRN 2020; Colesanti, Pagnini, Tradacete & Villanueva: AiM 2020, JFA 2021

Valuations on Convex Functions





- Convex functions
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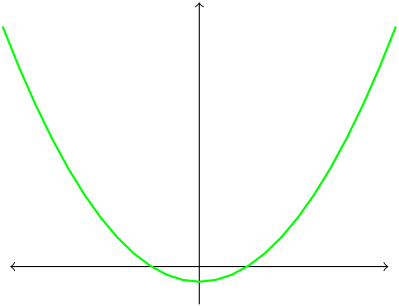
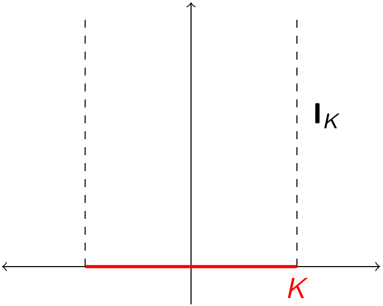
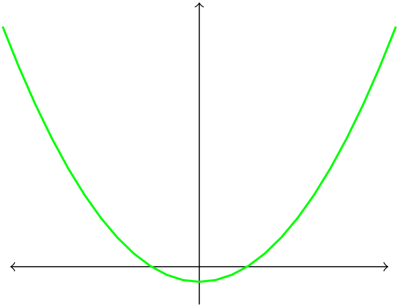
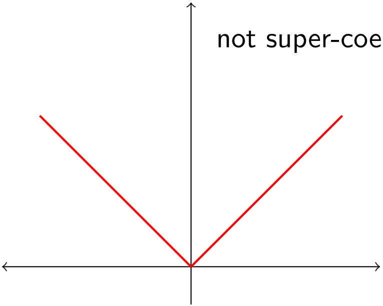




- Convex functions
  $\mathrm{Conv}(\mathbb{R}^n) := \{u \colon \mathbb{R}^n \to (-\infty, \infty] : u \text{ convex, l.s.c., proper}\}$
- u_k is epi-convergent to u in $\mathrm{Conv}(\mathbb{R}^n) \Leftrightarrow$
  - $u(x) \leqslant \liminf_{k\to\infty} u_k(x_k)$ for every (x_k) with $x_k \to x$
  - $\forall x,\ \exists\, (x_k)$ with $x_k \to x$ such that $u(x) = \lim_{k\to\infty} u_k(x_k)$

Valuations on Super-coercive Convex Functions





- $u \in \mathrm{Conv}(\mathbb{R}^n)$ super-coercive
  $\Leftrightarrow \lim_{|x| \to +\infty} \frac{u(x)}{|x|} = +\infty$

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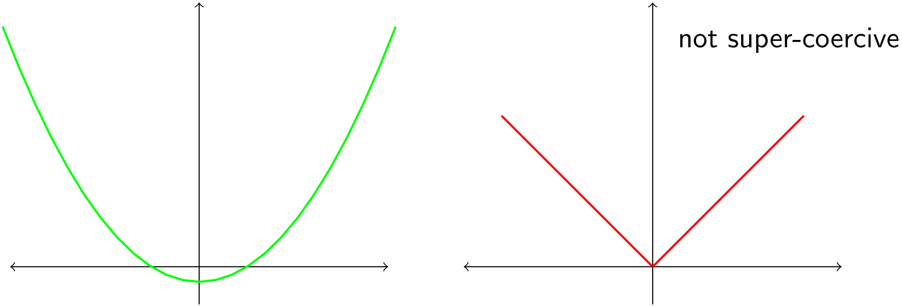

- $u \in \mathrm{Conv}(\mathbb{R}^n)$ super-coercive
  $\Leftrightarrow \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty$
- $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) := \big\{u \in \mathrm{Conv}(\mathbb{R}^n) : u \text{ super-coercive}\big\}$

Valuations on $\mathrm{Conv_{sc}}(\mathbb{R}^n)$

- $\mathrm{Z} : \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is rotation invariant
  $\Leftrightarrow \mathrm{Z}(u \circ \vartheta^{-1}) = \mathrm{Z}(u)$ for all $\vartheta \in \mathrm{SO}(n)$ and $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$

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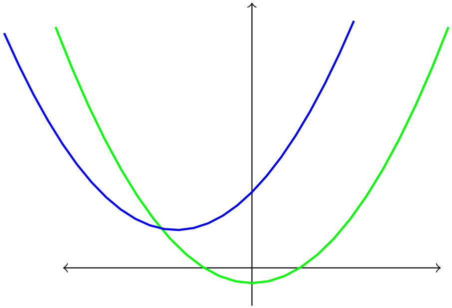
- $\mathrm{Z} : \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is epi-translation invariant
  $\Leftrightarrow \mathrm{Z}(u \circ \tau^{-1} + c) = \mathrm{Z}(u)$ for all translations $\tau : \mathbb{R}^n \to \mathbb{R}^n$ and $c \in \mathbb{R}$



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Theorem (Colesanti, L. & Mussnig, JFA 2020)

$\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation invariant valuation*

$$\Longrightarrow$$

$$\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n$$

where $\mathrm{Z}_j\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation invariant valuation that is is epi-homogeneous of degree* j.

Functional Intrinsic Volumes

Theorem (Colesanti, L. & Mussnig 2020+)

For $\zeta \in D_j^n$, there exists a unique, continuous, epi-translation and rotation invariant valuation $\mathrm{V}_{j,\zeta} \colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ such that

$$\mathrm{V}_{j,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-j} \, \mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$.

- For $j \in \{0, \dots, n-1\}$,
  $$D_j^n := \Big\{\zeta \in C_b((0,\infty)) \colon \lim_{s \to 0^+} s^{n-j}\zeta(s) = 0, \lim_{s \to 0^+} \int_s^\infty t^{n-j-1}\zeta(t)\, \mathrm{d}t \text{ finite}\Big\}$$
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- $[\mathrm{D}^2 u(x)]_k$ kth elementary symmetric function of the eigenvalues of the Hessian matrix $\mathrm{D}^2 u(x)$
- Hessian measures: Trudinger & Wang (Annals 1999), Colesanti & Hug (TAMS 2000)
- Hessian valuations: Colesanti, L. & Mussnig (IUMJ 2020)
- Singular Hessian valuations, Moreau-Yosida approximation

The Hadwiger Theorem on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$

Theorem (Colesanti, L. & Mussnig 2020+)

$\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation and rotation invariant valuation*

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Corollary

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Duality and Finite-valued Convex Functions

- Legendre transform (convex conjugate):

$$u^*(y) := \sup_{x\in\mathbb{R}^n} (\langle x, y\rangle - u(x))$$

- $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) := \{v \in \mathrm{Conv}(\mathbb{R}^n) : v(x) < +\infty$ for all $x \in \mathbb{R}^n\}$
- $^* : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ continuous bijection

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- $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ continuous valuation
  $\Leftrightarrow \mathrm{Z}^* : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$, defined by

$$\mathrm{Z}^*(u) := \mathrm{Z}(u^*),$$

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  $\Leftrightarrow \mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ dually epi-translation invariant:

$$\mathrm{Z}(v + \ell + c) = \mathrm{Z}(v)$$

  for all linear functions $\ell : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$

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Theorem (Colesanti, L. & Mussnig 2020+)

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for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

The Steiner Formula on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$

Theorem (Colesanti, L. & Mussnig 2021+)

Let $\zeta \in D^n_n$. For $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $r > 0$,

$$\mathrm{V}^*_{n,\zeta}(v + r\,h_{B^n}) = \sum_{j=0}^{n} r^{n-j}\kappa_{n-j}\,\mathrm{V}^*_{j,\zeta_j}(v),$$

where $\zeta_j \in D^n_j$ is given for $s > 0$ by

$$\zeta_j(s) := \frac{1}{\kappa_{n-j}}\left(\frac{\zeta(s)}{s^{n-j}} - (n-j)\int_s^\infty \frac{\zeta(t)}{t^{n-j+1}}\,\mathrm{d}t\right).$$

- $h_{B^n}(x) = |x|$ for $x \in \mathbb{R}^n$; support function of B^n

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$$\zeta_j(s) := \frac{1}{\kappa_{n-j}} \left(\frac{\zeta(s)}{s^{n-j}} - (n-j) \int_s^\infty \frac{\zeta(t)}{t^{n-j+1}}\, \mathrm{d}t \right).$$

Theorem (Steiner)

For $K \in \mathcal{K}^n$ *and* $r > 0$,

$$V_n(K + r\, B^n) = \sum_{j=0}^{n} r^{n-j} \kappa_{n-j} V_j(K).$$

The Steiner Formula on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$

Theorem (Colesanti, L. & Mussnig 2021+)

Let $\zeta \in D^n_n$. *For* $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ *and* $r > 0$,

$$\mathrm{V}^*_{n,\zeta}(v + r\, h_{B^n}) = \sum_{j=0}^{n} r^{n-j} \kappa_{n-j}\, \mathrm{V}^*_{j,\zeta_j}(v),$$

where $\zeta_j \in D^n_j$ *is given for* $s > 0$ *by*

$$\zeta_j(s) := \frac{1}{\kappa_{n-j}} \left(\frac{\zeta(s)}{s^{n-j}} - (n-j) \int_s^\infty \frac{\zeta(t)}{t^{n-j+1}}\, \mathrm{d}t \right).$$

- Proof using Reilly's formulas
- Second proof using the functional Hadwiger theorem

The Steiner Formula on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$

Theorem (Colesanti, L. & Mussnig 2021+)

Let $\zeta \in D_n^n$*. For* $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ *and* $r > 0$,

$$\mathrm{V}_{n,\zeta}^*(v + r\, h_{B^n}) = \sum_{j=0}^{n} r^{n-j} \kappa_{n-j}\, \mathrm{V}_{j,\zeta_j}^*(v),$$

where $\zeta_j \in D_j^n$ *is given for* $s > 0$ *by*

$$\zeta_j(s) := \frac{1}{\kappa_{n-j}} \left(\frac{\zeta(s)}{s^{n-j}} - (n-j) \int_s^\infty \frac{\zeta(t)}{t^{n-j+1}} \,\mathrm{d}t \right).$$

Corollary

For $\zeta \in D_j^n$ *and* $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$,

$$\mathrm{V}_{j,\zeta}^*(v) = \frac{j!}{n!} \frac{\mathrm{d}^{n-j}}{\mathrm{d}r^{n-j}}\Big|_{r=0} \mathrm{V}_{n,\alpha}^*(v + r\, h_{B^n}),$$

where $\alpha \in C_c([0,\infty))$ *is given for* $s > 0$ *by*

$$\alpha(s) := \binom{n}{j} \left(s^{n-j} \zeta(s) + (n-j) \int_s^\infty t^{n-j-1} \zeta(t) \,\mathrm{d}t \right).$$

New Representation Formulas

Theorem (Colesanti, L. & Mussnig, JFA 2020)

For $\zeta \in D_n^n$ and $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$,

$$\mathrm{V}_{n,\zeta}^*(v) = \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{dMA}(v;x).$$

Moreover,

$$\mathrm{V}_{n,\zeta}^*(v) = \int_{\mathbb{R}^n} \zeta(|x|)\det(\mathrm{D}^2 v(x))\,\mathrm{d}x$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$.

- $\mathrm{MA}(v;\cdot)$ Monge–Ampère measure of $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$
  - $\partial v(x)$ subdifferential of v at x

    $$\partial v(x) := \{y \in \mathbb{R}^n \colon v(z) \geqslant v(x) + \langle y, z-x\rangle \text{ for } z \in \mathbb{R}^n\}$$

  - For $B \subset \mathbb{R}^n$ Borel,

    $$\partial v(B) := \bigcup_{x\in B} \partial v(x) \text{ and } \mathrm{MA}(v;B) := |\partial v(B)|$$

Mixed Monge–Ampère Measures

- Polarization

$$\mathrm{MA}(v_1,\dots,v_n;\cdot) := \frac{1}{n!}\sum_{k=1}^{n}\sum_{1\leqslant i_1<\dots<i_k\leqslant n}(-1)^{n-k}\mathrm{MA}(v_{i_1}+\dots+v_{i_k};\cdot)$$

for $v_1,\dots,v_n \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$

Mixed Monge–Ampère Measures

- Polarization

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  for $v_1,\dots,v_n \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$
- jth Hessian measures for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ (with density $[\mathrm{D}^2 v]_j$ for $v \in C^2(\mathbb{R}^n)$)

$$\binom{n}{j}\mathrm{MA}(v[j], q[n-j];\cdot)$$

  where $q(x) = \frac{1}{2}x^2$ for $x \in \mathbb{R}^n$

Mixed Monge–Ampère Measures

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- jth Hessian measures for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ (with density $[\mathrm{D}^2 v]_j$ for $v \in C^2(\mathbb{R}^n)$)

$$\binom{n}{j}\mathrm{MA}(v[j], q[n-j];\cdot)$$

where $q(x) = \frac{1}{2}x^2$ for $x \in \mathbb{R}^n$

- New family of mixed Monge–Ampère measures

$$\mathrm{MA}_j(v;\cdot) := \mathrm{MA}(v[j], h_{B^n}[n-j];\cdot)$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$

New Representation Formulas

Theorem (Colesanti, L. & Mussnig 2021+)

For $\zeta \in D_j^n$ *and* $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$,

$$\mathrm{V}_{j,\zeta}^*(v) = \int_{\mathbb{R}^n} \alpha(|x|)\,\mathrm{dMA}_j(v;x),$$

where $\alpha \in C_c([0,\infty))$ *is given by*

$$\alpha(s) := \binom{n}{j}\Big(s^{n-j}\zeta(s) + (n-j)\int_s^\infty t^{n-j-1}\zeta(t)\,\mathrm{d}t\Big).$$

New Representation Formulas

Theorem (Colesanti, L. & Mussnig 2021+)

For $\zeta \in D_j^n$ *and* $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$,

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$$\alpha(s) := \binom{n}{j}\Big(s^{n-j}\zeta(s) + (n-j)\int_s^\infty t^{n-j-1}\zeta(t)\,\mathrm{d}t\Big).$$

Moreover, for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$,

$$\mathrm{V}_{j,\zeta}^*(v) = \int_{\mathbb{R}^n} \alpha(|x|)\det\Big(\mathrm{D}^2 v(x)[j], \frac{1}{|x|}\big(I_n - \frac{x}{|x|}\otimes\frac{x}{|x|}\big)[n-j]\Big)\,\mathrm{d}x.$$

- $\mathrm{D}^2 h_{B^n}(x) = \dfrac{1}{|x|}\big(I_n - \dfrac{x}{|x|}\otimes\dfrac{x}{|x|}\big)$

New Representation Formulas

Theorem (Colesanti, L. & Mussnig 2021+)

For $\zeta \in D_j^n$ and $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$,

$$\mathrm{V}_{j,\zeta}(u) = \int_{\mathbb{R}^n} \alpha(|y|)\, \mathrm{dMA}_j^*(u; y),$$

where $\alpha \in C_c([0,\infty))$ is given by

$$\alpha(s) := \binom{n}{j}\Big(s^{n-j}\zeta(s) + (n-j)\int_s^\infty t^{n-j-1}\zeta(t)\,\mathrm{d}t\Big).$$

- $\mathrm{MA}_j^*(u;\cdot) := \mathrm{MA}_j(u^*;\cdot)$ conjugate Monge–Ampère measure

New Representation Formulas

Theorem (Colesanti, L. & Mussnig 2021+)

For $\zeta \in D_j^n$ *and* $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$,

$$\mathrm{V}_{j,\zeta}(u) = \int_{\mathbb{R}^n} \alpha(|y|)\, \mathrm{dMA}_j^*(u; y),$$

where $\alpha \in C_c([0,\infty))$ *is given by*

$$\alpha(s) := \binom{n}{j} \left(s^{n-j}\zeta(s) + (n-j) \int_s^\infty t^{n-j-1} \zeta(t)\, \mathrm{d}t \right).$$

Moreover, for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$,

$$\mathrm{V}_{j,\zeta}(u) = \frac{1}{\binom{n}{j}} \int_{\mathbb{R}^n} \alpha(|\nabla u(x)|)\, \tau_{n-j}(u,x)\, \mathrm{d}x.$$

- $\mathrm{MA}_j^*(u;\cdot) := \mathrm{MA}_j(u^*;\cdot)$ conjugate Monge–Ampère measure
- $\tau_k(u,x)$ kth elementary symmetric function of the principal curvatures of $\{u = t\}$ with $t = u(x)$ at x

The Hadwiger Theorem on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$

Theorem (Colesanti, L. & Mussnig 2021+)

$\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation and rotation invariant valuation*

$$\Longleftrightarrow$$

$\exists\, \alpha_0, \dots, \alpha_n \in C_c([0,\infty))$:

$$\mathrm{Z}(u) = \sum_{j=0}^{n} \int_{\mathbb{R}^n} \alpha_j(|y|)\, \mathrm{dMA}_j^*(u; y)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

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Thank you!