# Resemblance and Collapsing

**Anton Freund, TU Darmstadt**

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# Resemblance and large ordinals

**Idea.** Statements of the form "the ordinal $\alpha$ resembles some $\beta > \alpha$" are strong and force $\alpha$ to be large (note that $\alpha \cong \beta$ is impossible).

Some **large cardinal properties** can be seen as manifestations of this idea. In this talk, we explore two manifestations in the context of computable ordinals / ordinal analysis / second order arithmetic:

- **patterns of resemblance** (due to Timothy Carlson),
- **ordinal collapsing functions**, as used to describe, e. g., the Bachmann-Howard ordinal (Feferman, Buchholz, Rathjen, . . . ).

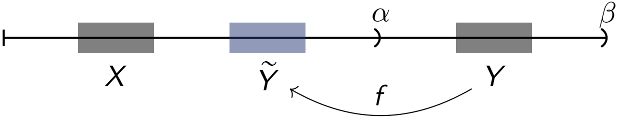
# Carlson's patterns of resemblance

We consider the language $\{\leqslant, \leqslant_1\}$ and focus on structures with universe $M \subseteq \mathrm{Ord}$ and $\leqslant$ interpreted as the usual inequality. Define

$$\alpha \leqslant_1 \beta \quad :\Leftrightarrow \quad (\alpha, \leqslant, \leqslant_1) \preceq_{\Sigma_1} (\beta, \leqslant, \leqslant_1)$$

by recursion on $\beta \geqslant \alpha$, where $\preceq_{\Sigma_1}$ refers to $\Sigma_1$-elementarity.

Equivalently, $\alpha \leqslant_1 \beta$ holds if all finite $X \subseteq \alpha$ and $Y \subseteq \beta\backslash\alpha$ admit an $\{\leqslant, \leqslant_1\}$-isomorphism $f : X \cup Y \to X \cup \widetilde{Y}$ with $\widetilde{Y} \subseteq \alpha$.



# Characterizing $\varepsilon_0$

Recall the ordinal $\varepsilon_0 = \min\{\beta \mid \omega^\beta = \beta\}$, recursively generated by

$$\alpha \quad ::= \quad 0 \quad \mid \quad \omega^{\alpha_0} + \ldots + \omega^{\alpha_n} \quad (\alpha_0 \geqslant \ldots \geqslant \alpha_n).$$

**Theorem (Gentzen).**
Peano arithmetic has proof-theoretic ordinal $\varepsilon_0$.

For a detailed appraisal of proof-theoretic ordinals, see, e. g., Michael Rathjen's "The Realm of Ordinal Analysis" (LC '97).

**Theorem (Carlson).** $\varepsilon_0 = \min\{\alpha \mid \alpha \leqslant_1 \beta \text{ for all } \beta > \alpha\}$

# Larger ordinals

Far beyond $\varepsilon_0$, we find the proof-theoretic ordinal of $\Pi^1_1$-$\mathsf{CA}_0$ (the strongest of the "big five" systems from reverse mathematics).
It can be characterized via either of the following modifications, as shown by Carlson and Gunnar Wilken:

(1) replace the language $\{\leqslant, \leqslant_1\}$ by $\{0, +, \leqslant, \leqslant_1\}$, or

(2) consider $\{\leqslant, \leqslant_1, \leqslant_2\}$ and simultaneously for $i = 1, 2$ declare

$$\alpha \leqslant_i \beta \quad :\Leftrightarrow \quad (\alpha, \leqslant, \leqslant_1, \leqslant_2) \preceq_{\Sigma_i} (\beta, \leqslant, \leqslant_1, \leqslant_2).$$

Carlson conjectures that we obtain the proof-theoretic ordinal of full second-order arithmetic by extending (2) to all $i \in \mathbb{N}$.

# Characterizing $\varepsilon_0$, again

Consider $\alpha \mapsto D(\alpha) := 1 + \alpha^2$ and $\mathsf{supp}_\alpha : D(\alpha) \to \mathcal{P}_{\mathsf{fin}}(\alpha)$ with

$$\mathsf{supp}_\alpha(0) = \emptyset \quad \text{and} \quad \mathsf{supp}_\alpha(1 + \alpha \cdot \beta + \beta') = \{\beta, \beta'\}.$$

Call $\vartheta : D(\alpha) \to \alpha$ a **Bachmann-Howard collapse** if

(1) $\gamma < \delta < D(\alpha)$ implies $\vartheta(\gamma) < \vartheta(\delta)$, under the side condition that we have $\mathsf{supp}_\alpha(\gamma) \subseteq \vartheta(\delta)$, and

(2) we have $\mathsf{supp}_\alpha(\gamma) \subseteq \vartheta(\gamma)$.

**Theorem.** For $D(\alpha) = 1 + \alpha^2$ we have
$\varepsilon_0 = \min\{\alpha \,|\, \text{there is a Bachmann-Howard collapse } \vartheta : D(\alpha) \to \alpha\}$.

# The strength of collapsing functions

The definition of a Bachmann-Howard collapse $\vartheta : D(\alpha) \to \alpha$ makes sense whenever $D$ is a dilator in the sense of Girard (i. e., an endofunctor of ordinals that preserves pullbacks and direct limits).

**Theorem (F).** The following are equivalent over $\mathsf{RCA}_0$:

(1) for every dilator $D$ there is an ordinal $\Omega$ that admits a Bachmann-Howard collapse $\vartheta : D(\Omega) \to \Omega$,

(2) $\Pi^1_1$-comprehension (the strongest of the "big five" principles from reverse mathematics).

# Relativizing patterns to dilators

Consider a dilator $D$ that satisfies a certain normality condition. Each ordinal $\gamma < D(\alpha)$ has a unique representation

$$\gamma = (\sigma; \gamma_0, \ldots, \gamma_{n-1})_D \qquad (\star)$$

for a "constructor" $\sigma \in D(n)$ and "arguments" $\gamma_0, \ldots, \gamma_{n-1} < \alpha$. Let $\mathcal{L}_D$ be the extension of $\{\leqslant, \leqslant_1\}$ by an $(n+1)$-ary relation $(\star)$ for each constructor $\sigma$. Define $\leqslant_1^D$ as $\leqslant_1$, with $\mathcal{L}_D$ replacing $\{\leqslant, \leqslant_1\}$.

**Theorem (F).** The following are equivalent over $\mathsf{ATR}_0^{\mathsf{set}}$:

(1) for every normal dilator $D$ there is an ordinal $\Omega \leqslant_1^D D(\Omega + 1)$,

(2) $\Pi^1_1$-comprehension.

# Proof sketch 1/3: towards resemblance

Use $\Pi^1_1$-comprehension to get an admissible $\Omega$ closed under $D$.

For $\gamma = (\sigma; \gamma_0, \ldots, \gamma_{n-1}, \Omega)_D$ between $D(\Omega)$ and $D(\Omega + 1)$, put

$$\gamma[\eta] := (\sigma; \gamma_0, \ldots, \gamma_{n-1}, \eta)_D \quad \text{for} \quad \eta \geqslant \gamma^* := \sup\{\gamma_i + 1 \,|\, i < n\}.$$

By induction on $\gamma$ as above, show that

$$C_D(\gamma) := \{\eta < \Omega \,|\, \eta \geqslant \gamma^* \text{ and } \eta \leqslant_1^D \gamma[\eta]\}$$

is $\Omega$-club (admissibles support diagonal intersections etc.).

Conclude $\Omega \leqslant_1^D \gamma[\Omega] = \gamma$ for all $\gamma < D(\Omega + 1)$.

# Proof sketch 2/3: defining a collapse

By the theorem before, $\Pi^1_1$-comprehension reduces to the statement that any dilator $D$ has a collapse $\vartheta : D(\Omega) \to \Omega$.

Define a normal $\Sigma D$ and embedding $\xi : D(\Omega) \to \Sigma D(\Omega+1)$ by

$$\Sigma D(\gamma) := \Sigma_{\beta<\gamma} 1 + D(\beta), \quad \xi(\alpha) := \Sigma D(\Omega) + 1 + \alpha.$$

Given $\Omega \leqslant_1^D \Sigma D(\Omega+1)$, we have $\Omega \leqslant_1^D \xi(\alpha)[\Omega] < \Sigma D(\Omega+1)$. By $\Sigma_1$-elementarity, we get an $\eta < \Omega$ as in

$$\vartheta(\alpha) := \min\{\eta < \Omega \,|\, \eta \geqslant \xi(\alpha)^* \text{ and } \eta \leqslant_1^D \xi(\alpha)[\eta]\}$$

(think $X = \mathrm{supp}_\Omega(\alpha)$, $Y = \{\Omega, \xi(\alpha[\Omega])\}$ and $\tilde{Y} = \{\eta, \xi(\alpha)[\eta]\}$).

# Proof sketch 3/3: the Bachmann-Howard property

Consider $\alpha < \beta < D(\Omega)$ with

$$\mathsf{supp}_\Omega(\alpha) \subseteq \vartheta(\beta) = \min\{\eta < \Omega \,|\, \eta \geqslant \xi(\beta)^* \text{ and } \eta \leqslant_1^D \xi(\beta)[\eta]\}.$$

For $\eta := \vartheta(\beta)$ we get $\eta \leqslant_1^D \xi(\alpha)[\eta] < \xi(\beta)[\eta]$. We now invoke $\Sigma_1$-elementarity, to find an $\eta_0 < \eta$ with

$$\eta^0 \geqslant \xi(\alpha)^* \quad \text{and} \quad \eta_0 \leqslant_1^D \xi(\alpha)[\eta_0].$$

This yields $\vartheta(\alpha) \leqslant \eta_0 < \vartheta(\beta)$, as required by the definition of Bachmann-Howard collapse.

# Conclusion and Thanks

Short of paradox, we obtain strong principles if we declare that $\alpha$ resembles $\beta > \alpha$, in the sense that

(1) we have $\alpha \leqslant_1 \beta$ (Carlson's patterns of resemblance),

(2) there is an ordinal collapsing function $\vartheta : \beta \to \alpha$.

Statements (1) and (2) are intimately linked (Carlson & Wilken). The link becomes particularly elegant on a general level, where (1) and (2) are relativized to dilators. For details and references, see


- A. Freund, *Patterns of resemblance and Bachmann-Howard fixed points*, `arXiv:2012.10292`.