Connecting Constructive Notions of Ordinals in Homotopy Type Theory

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What are ordinals?

One answer: **Numbers for ranking/ordering**

$$0, \quad 1, \quad 2, \quad \ldots, \quad \omega, \quad \omega+1, \quad \ldots, \quad \omega\cdot 2, \quad \omega\cdot 2+1, \quad \ldots, \quad \omega\cdot 3, \quad \ldots$$

$$\omega^2, \quad \ldots, \quad \omega^2\cdot 3+\omega\cdot 7+13, \quad \ldots, \quad \omega^\omega, \quad \ldots, \quad \varepsilon_0 = \omega^{\omega^{\omega^{\cdots}}}, \quad \ldots, \quad \varepsilon_{17}, \quad \ldots$$

Another answer: **Sets with an order** $<$ which is

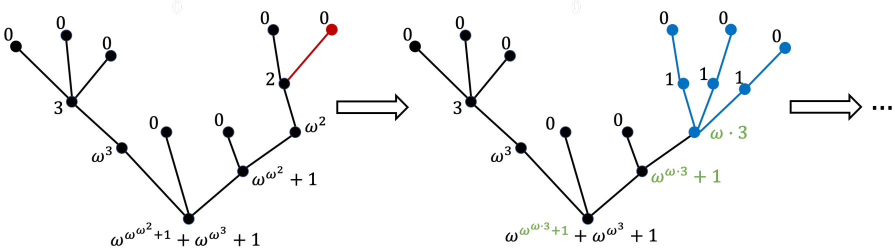
- **transitive**: $(a < b) \to (b < c) \to (a < c)$
- **wellfounded**: every sequence $a_0 > a_1 > a_2 > a_3 > \ldots$ terminates
- and **trichotomous**: $(a < b) \vee (a = b) \vee (b < a)$
- . . . or **extensional** (instead of trichotomous):

$$(\forall a. a < b \leftrightarrow a < c) \to b = c$$

What are ordinals good for?

Some examples:

- Justifying recursive definitions, e.g., the Ackermann function
- Consistency proof e.g. of Peano's axioms [Gentzen 1936]
- Termination of processes, e.g., [Goodstein 1944], [Turing 1949], Hydra game [Kirby&Paris 1982]: All hydras eventually die.



Ordinals in constructive type theory

Problem/feature of a constructive setting: different definitions differ!

Consider the following constructive notions of "ordinals":

- Cantor normal forms
- Brouwer trees
- Wellfounded & extensional & transitive orders

Why can we call them "ordinals"? Pros and cons? What's the connection?

We study them in **homotopy type theory** (HoTT):

(i) axiomatic framework for ordinals and ordinal arithmetic

(ii) connections between the three notions and their arithmetic operations

What is HoTT? Why HoTT?

HoTT = MLTT + HITs + UA

Martin-Löf type theory (MLTT)

- Dependent functions $(x : A) \to B(x)$
- Dependent pairs $\Sigma(x : A).B(x)$
- Inductive types, e.g. $\mathbb{N}$, List, ...
- Universes $\mathcal{U}_i : \mathcal{U}_{i+1}$
- Identity type $a =_A b$
  - Propositions isProp$(A) :\equiv (x : A) \to (y : A) \to x =_A y$
  - Sets isSet$(A) :\equiv (x : A) \to (y : A) \to$ isProp$(x =_A y)$
  - ...

Proof assistants based on variants of MLTT: Agda, Coq, Nuprl, ...

What is HoTT? Why HoTT?

HoTT = MLTT + HITs + UA

Higher inductive types (HITs)

- Generalization of inductive types
- Constructors for elements (or points) and identity proofs (or paths)
- Example: propositional truncation $\|A\|$
  - Point constructor $|-| : A \to \|A\|$
  - Path constructor trunc : isProp($\|A\|$)
  - Recursion principle $(A \to P) \to \|A\| \to P$ for any proposition P
  - Mere existence $\exists(x : A).B(x) := \|\Sigma(x : A).B(x)\|$
- Circle, interval, quotient, Cauchy reals, patch theory (version control), ...

We define Brouwer trees as a **quotient inductive-inductive type**.

What is HoTT? Why HoTT?

HoTT = MLTT + HITs + UA

Univalence Axiom (UA)

- "Isomorphic structures are identical"; thus, $X \cong Y \to P(X) \to P(Y)$
- Independent from MLTT, but provable in cubical type theory (CTT)
- Not needed in this talk

Cubical Agda is an extension of Agda support features of CTT, including HITs and UA.

Most results in this talk have been formalized in Cubical Agda.

What do we expect of "ordinals"?

When does $(\mathcal{O}, <)$ deserve to be called "ordinals"?

(a) **Wellfoundedness**: Every decreasing sequence terminates / Can do transfinite induction.

(b) **Arithmetic**: Can do addition, multiplication, exponentiation, ...
(But what does that mean? Why are they correct?)

(c) **Trichotomy**: $(a < b) \vee (a = b) \vee (b < a)$ Not necessary!!

(d) **Extensionality**: $(\forall a. a < b \leftrightarrow a < c) \to b = c$

(e) **Suprema/limits**: Can calculate the limit of any sequence. Not necessary!!

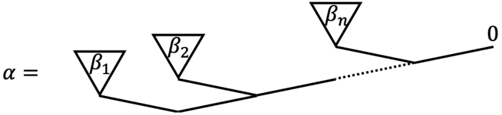
(f) **Classifiability**: Any $x : \mathcal{O}$ is a zero, a successor, or a limit. Not necessary!!

Cantor normal forms

Motivation: $\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_n}$ with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$

Let $\mathcal{T}$ be the type of *unlabeled binary trees*:

$$\begin{array}{lcl} 0 & : & \mathcal{T} \\ \omega^{-} + - & : & \mathcal{T} \to \mathcal{T} \to \mathcal{T} \end{array}$$



A tree is a *Cantor normal form* if $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ (*lexicographical order*).
Cnf is just a subset of binary trees (i.e. Σ-type).

Equivalent implementations [NF**X**G20]: (i) hereditary descending lists, and (ii) finite hereditary multisets

Cantor normal forms

Theorem. Cnf cannot calculate limits of sequences, but everything else works.

- Cnf cannot have the limit of ω, ω^ω, ω^{ω^ω}, $\omega^{\omega^{\omega^\omega}}$, $\ldots$ which is ε_0.
- If Cnf has limits of arbitrary *bounded* sequences, then WLPO holds.
- ...
- Every Cnf is a zero, a successor or a limit (of its fundamental sequence).
- Cnf has addition, multiplication and exponentiation (with base ω).

  E.g., we show
  $$a + 0 = a$$
  $$a + (b + 1) = a + b + 1$$
  $$b \text{ is-lim-of } f \to c \text{ is-lim-of } (\lambda i.a + fi) \to a + b = c$$
  where the last one is proved by defining subtraction.

Brouwer trees (a.k.a. Brouwer ordinal trees)

How about this inductive type $\mathcal{O}$ of Brouwer trees?

$$\mathsf{zero} : \mathcal{O} \qquad \mathsf{succ} : \mathcal{O} \to \mathcal{O} \qquad \mathsf{sup} : (\mathbb{N} \to \mathcal{O}) \to \mathcal{O}$$

Problem: $\mathsf{sup}(0, 1, 2, 3, \ldots) \neq \mathsf{sup}(1, 2, 3, \ldots)$

How to fix this without losing wellfoundedness, validity of arithmetic operations, and so on?

Brouwer trees quotient inductive-inductively

```
data Brw : Set where
  zero   : Brw
  succ   : Brw → Brw
  limit : (f : ℕ → Brw) {f↑ : increasing f} → Brw
  bisim : f ≈ g → limit f ≡ limit g

data _≤_ : Brw → Brw → Prop where
  ≤-zero       : zero ≤ x
  ≤-trans      : x ≤ y → y ≤ z → x ≤ z
  ≤-succ-mono : x ≤ y → succ x ≤ succ y
  ≤-cocone     : x ≤ f k → x ≤ limit f
  ≤-limiting  : (∀ k → f k ≤ x) → limit f ≤ x
```

Theorem. The order on Brw is not trichotomous, but everything else works.

- Wellfoundedness: encode-decode method to find n such that $x < f(n)$ for $x <$ limit f

Extensional wellfounded orders

The type Ord consists of pairs $(X, \prec)$ where X is a type and $\prec$ is a transitive, extensional, wellfounded relation on X.

$(X, \prec_X) \leq (Y, \prec_Y)$ is given by a *monotone* function $f : X \to Y$

such that if $y \prec_Y f\,x$, then there is $x_0 \prec_X x$ such that $f\,x_0 = y$.

Theorem.

- The order on Ord is extensional and wellfounded.
- Ord has addition (disjoint union) and multiplication (cartesian product). Exponentiation may be constructively problematic.
- Limits of increasing sequences of Ord can be calculated.
- Nothing is decidable.
  E.g. deciding whether an Ord is a successor implies LEM.

Connections between the notions

decidable — partially decidable — undecidable

$$\text{Cnf} \xhookrightarrow{\text{CtoB}} \text{Brw} \xhookrightarrow{\text{BtoO}} \text{Ord}$$

CtoB: $(\omega^a + b) \mapsto \omega^{\mathsf{CtoB}(a)} + \mathsf{CtoB}(b)$

- injective
- preserves and reflects $<$, $\leq$
- commutes with $+$, $*$, ω^x
- bounded (by ε_0)

BtoO: $A \mapsto \Sigma(Y : \mathsf{Brw}).Y < A$

- injective
- preserves $<$, $\leq$
- over-approximates $+$, $*$:
  $\mathsf{BtoO}(x + y) \geq \mathsf{BtoO}(x) + \mathsf{BtoO}(y)$
- commutes with limits
  (but not successors)
- LEM $\Rightarrow$ BtoO is a simulation
- BtoO is a simulation $\Rightarrow$ WLPO
- bounded (by Brw)