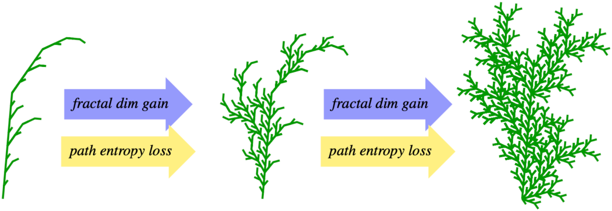
Compactness and measure in second order arithmetic



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WKL :	Every infinite tree has a path
P^+ :	Every positive tree has a positive perfect subtree
P :	Every positive tree has a perfect subtree
P^- :	Every positive tree has an infinite countable family of paths
WWKL :	Every positive tree has a path

Table: Compactness principles derived by weakening weak König's lemma

Our paper: http://arxiv.org/abs/2104.12066

References

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- Barmpalias and Wang: http://arxiv.org/abs/2104.12066

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Trees of random reals are ubiquitous in measure-theoretic constructions in computability theory and, as recently suggested by Chong, Li, Wang, and Yang., essential in study of compactness in reverse mathematics.

Our goal is:

- to establish essential computational properties of the pathwise-random trees (formally defined below)
- to apply our analysis to the classification of compactness principles in second order arithmetic.

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Provable in RCA_0: $\mathrm{WKL} \to \mathrm{P}^+ \to \mathrm{P} \to \mathrm{P}^- \to \mathrm{WWKL}$

Theorem (Models)

Each of the following extensions of RCA has an ω-model:

(a) $\mathrm{WWKL} + \neg\mathrm{P}^-$: every positive tree has a path but some positive tree only has finitely many paths.

(b) $\mathrm{P} + \neg\mathrm{P}^+$: every positive tree has a perfect subtree but some positive tree has no positive perfect subtree.

(c) $\mathrm{P}^+ + \neg\mathrm{WKL}$: every positive tree has a positive perfect subtree but some infinite tree has no path.

Definition

The deficiency of σ is $|\sigma| - K(\sigma)$;
the deficiency of a real x is the supremum of the deficiencies of $x \upharpoonright_n, n \in \mathbb{N}$.
The deficiency of a set of reals is the supremum of the deficiencies of its members.

Definition (Pathwise randomness)

A tree T is:

- pathwise-random if the deficiency of each $\sigma \in T$ is bounded above by a constant.
- weakly pathwise-random if all of its paths are random.
- proper if it has infinitely many paths.

Theorem

If z is random and computes or enumerates a pathwise-random tree of unbounded width, then $z \geq_T \emptyset'$.

Hirschfeldt, Jockusch, and Schupp (2021) obtained a similar statement for perfect trees and 2-randoms.

Corollary

The set of paths through the van Lambalgen array:

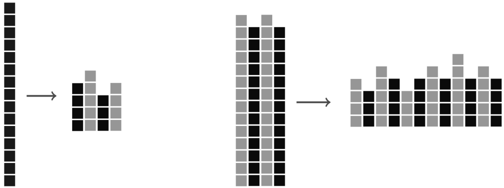
$$A_x := \{ x_{1^n * 0} : n \in \mathbb{N} \}$$

of any $x \not\geq_T \emptyset'$ has infinite randomness deficiency.

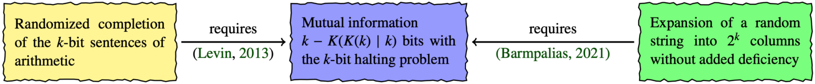
Corollary

If F is a computable space of trees, for every computable measure ν on F the class of proper pruned pathwise-random members of F is ν-null.

Finitary consequences



- Effectively splitting a random source into k many random sources without a significant increase in the randomness deficiency is about as hard as computing the k-bit halting problem.
- Levin (2013): randomly guessing a completion of the k-bit segment of PA is about as improbable as randomly guessing the k-bit halting problem.



Digression on the proof of WWKL $\not\to$ P^- (algorithms)

Proof base: Randomness cannot be used in order to produce infinitely many random reals with a fixed upper bound on their deficiency.

We explore the finite version of this fact and its limits.

Theorem (Random production of incompressible strings I)

There exists no randomized algorithm and $\epsilon > 0, c \in \mathbb{N}$ such that for each input k, with probability $> \epsilon$ the output is a set of k strings ρ with $K(\rho) > |\rho| - c$ of the same length ℓ_k which depends only on k.

Theorem (Random production of incompressible strings II)

There exists a randomized algorithm which, almost surely and on almost all inputs k, outputs a set of k strings ρ of equal length ℓ_k with $K(\rho) > |\rho|$, where $\ell_k \in (2^k, 2^{k+1})$ is a random variable.

$P \not\to P^+$

Theorem

The following hold for each z:

(a) there exists a perfect pathwise-random tree T such that no T-c.e. positive tree is pathwise-random.

(b) if no z-c.e. positive tree is pathwise-random, then there exists a perfect pathwise-z-random tree T such that no $(z \oplus T)$-c.e. positive tree is pathwise-random.

(c) the tree T of clause (b) can be found inside any given positive tree.

Forcing with sets of sets of positive measure.

Involving hitting sets and notions from Poisson point processes.

$P^+ \not\to$ WKL

Patey has shown that every positive tree contains a perfect subtree which does not compute any PA degree.

Theorem (Patey)

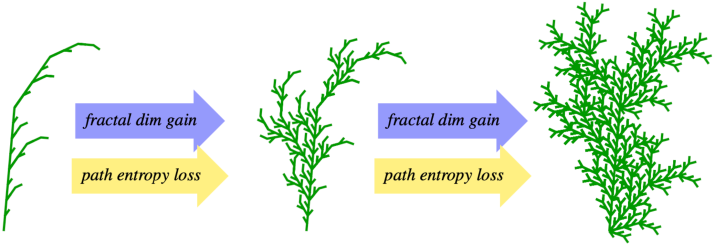
Every positive tree contains a perfect subtree which does not compute a complete extension of PA.

Theorem

There exists a positive perfect pathwise-random tree which does not compute any complete extension of Peano Arithmetic. In fact, given any non-computable z, every positive tree has a positive perfect subtree $T \not\geq_T z$ which does not compute any complete extension of Peano Arithmetic.

Independently obtained by Greenberg, Miller, Nies (2021).

Open problems



- build a model for $\mathrm{P}^- + \neg\mathrm{P}$
- build measures ν such that ν-randomness gives the above separations.

Thanks for listening!