Conservation theorems on semi-classical arithmetic

Makoto Fujiwara

(joint work with Taishi Kurahashi)

JSPS Research Fellow (PD), Meiji University

New Frontiers in Proofs and Computation,
Online (virtual alternative of Hangzhou),
15 September 2021

This work is supported by JSPS KAKENHI Grant Numbers JP19K14586, 19J01239 and JP20K14354.

Abstract

Fact.

PA is Π_2-conservative over HA.

There are several ways to prove this fact:

1. The negative translation + the Dialectica interpretation (essentially, Gödel 1958);
2. The negative translation + the Friedman A-translation (H. Friedman 1978);
3. A generalized Gödel-Gentzen negative translation with substitution (Ishihara 2000 and 2012).

In this talk, we generalize this result in the context of semi-classical arithmetic (arithmetic in-between PA and HA), by extending the second and third methods.

Framework

- We work with a standard formulation of HA, which has function symbols for all primitive recursive functions.
- We work in the language containing all the logical constants $\forall, \exists, \rightarrow, \wedge, \vee, \perp$. Note $\neg\varphi :\equiv \varphi \rightarrow \perp$.
- PA $:=$ HA $+$ LEM.

Definition

The classes Σ_k and Π_k of formulas of HA are defines as follows:

- Σ_0, as well as Π_0, is the class of quantifier-free formulas;
- Π_{k+1} is the class of formulas of form $Q_1\overline{x_1}\cdots Q_{k+1}\overline{x_{k+1}}\,\varphi_{\mathrm{qf}}$;
- Σ_{k+1} is the class of formulas of form $Q'_1\overline{x_1}\cdots Q'_{k+1}\overline{x_{k+1}}\,\varphi_{\mathrm{qf}}$;

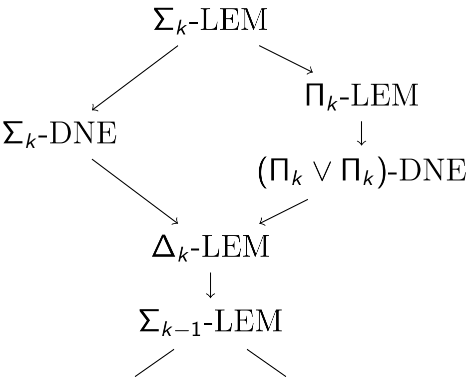
where Q_i represents $\forall$ for odd i and $\exists$ for even i and Q'_i represents $\exists$ for odd i and $\forall$ for even i. A formula φ is of **prenex normal form** if $\varphi \in \Sigma_k \cup \Pi_k$ for some k.

Definition

- $\Gamma\text{-}\mathrm{LEM}$: $\forall x\,(\varphi(x) \vee \neg\varphi(x))$ where $\varphi(x) \in \Gamma(x)$.
- $\Gamma\text{-}\mathrm{DNE}$: $\forall x\,(\neg\neg\varphi(x) \to \varphi(x))$ where $\varphi(x) \in \Gamma(x)$.

Fact. $\Sigma_k\text{-}\mathrm{LEM} \longrightarrow \Sigma_k\text{-}\mathrm{DNE} \longrightarrow \Sigma_{k-1}\text{-}\mathrm{LEM}$.

Arithmetical Hierarchy of Logical Principles (Akama-Berardi-Hayashi-Kohlenbach 2004)



For a logical principle P, HA $+$ P forms "semi-classical" arithmetic.

Friedman's Method

Let HA^* denote HA in the extended language where a predicate symbol $*$ of arity 0, which behaves as a "place holder", is added.

$$\mathrm{PA} \vdash \forall x \exists y \varphi_{\mathrm{qf}}(x, y)$$

$\Longrightarrow$ Negative translation

$$\mathrm{HA} \vdash \forall x \neg\neg \exists y \varphi_{\mathrm{qf}}(x, y)$$

$\Longrightarrow$

$$\mathrm{HA} \vdash \neg\neg \exists y \varphi_{\mathrm{qf}}(x, y)$$

$\Longrightarrow$ A-translation $[\varphi_{\mathrm{p}} \vee * / \varphi_{\mathrm{p}}]$

$$\mathrm{HA}^* \vdash \neg_* \neg_* \exists y \varphi^*_{\mathrm{qf}}(x, y)$$

$\Longrightarrow$ Substitution $[\exists y \varphi_{\mathrm{qf}}(x,y) / *]$

$$\mathrm{HA} \vdash (\exists y\, (\varphi_{\mathrm{qf}}(x, y) \vee \exists y \varphi_{\mathrm{qf}}(x, y)) \to \exists y \varphi_{\mathrm{qf}}(x, y)) \to \exists y \varphi_{\mathrm{qf}}(x, y).$$

$\Longrightarrow$

$$\mathrm{HA} \vdash \forall x \exists y \varphi_{\mathrm{qf}}(x, y) \quad \square$$

Substitution Lemma

Lemma

Let X be a set of HA*-sentences and φ be a* HA^**-formula. If* $\mathsf{HA}^* + X \vdash \varphi$*, then* $\mathsf{HA} + X \vdash \varphi[\psi/*]$ *for any* HA*-formula ψ such that the free variables of ψ are not bounded in φ, where $\varphi[\psi/*]$ is the* HA*-formula obtained from φ by replacing all the occurrences of $*$ in φ with ψ.*

Gentzen's negative translation

Definition (cf. Troelstra 1973)

For each formula φ, its negative translation φ^N is defined inductively be the following clauses:

- For φ_{p} prime such that $\varphi_{\mathrm{p}} \not\equiv \bot$, $(\varphi_{\mathrm{p}})^N :\equiv \neg\neg\varphi_{\mathrm{p}}$;
- $\bot^N :\equiv \bot$;
- $(\varphi_1 \circ \varphi_2)^N :\equiv \varphi_1^N \circ \varphi_2^N$ for $\circ \in \{\wedge, \rightarrow\}$;
- $(\varphi_1 \vee \varphi_2)^N :\equiv \neg\neg\left(\varphi_1^N \vee \varphi_2^N\right)$;
- $(\forall x \varphi)^N :\equiv \forall x \varphi^N$;
- $(\exists x \varphi)^N :\equiv \neg\neg\exists x \varphi^N$.

Remark. By induction on the structure of formulas, one can show $\mathrm{FV}(\varphi) = \mathrm{FV}\left(\varphi^N\right)$ for all formulas φ.

Proposition

For any HA-formula φ, if PA $\vdash \varphi$, then HA $\vdash \varphi^N$.

Proof. By induction on the length of the derivations. □

Proposition

For any HA-formula φ, if $\mathsf{PA} \vdash \varphi$, then $\mathsf{HA} \vdash \varphi^N$.

Proof. By induction on the length of the derivations. □

Lemma

1. *For any* HA-*formula* $\varphi \in \Sigma_k$, $\mathsf{HA} + \Sigma_k\text{-DNE} \vdash \varphi^N \leftrightarrow \varphi$.
2. *For any* HA-*formula* $\varphi \in \Pi_k$, $\mathsf{HA} + \Sigma_{k-1}\text{-DNE} \vdash \varphi^N \leftrightarrow \varphi$.

Proof. By simultaneous induction on k. □

Proposition

For any HA-formula φ, if PA $\vdash \varphi$, then HA $\vdash \varphi^N$.

Proof. By induction on the length of the derivations. □

Lemma

1. *For any* HA-*formula* $\varphi \in \Sigma_k$, HA $+ \Sigma_k\text{-DNE} \vdash \varphi^N \leftrightarrow \varphi$.
2. *For any* HA-*formula* $\varphi \in \Pi_k$, HA $+ \Sigma_{k-1}\text{-DNE} \vdash \varphi^N \leftrightarrow \varphi$.

Proof. By simultaneous induction on k. □

By the second clause of the above lemma, if PA $\vdash \forall x \exists y \varphi(x, y)$ where $\varphi(x, y) \in \Pi_k$, then HA $+ \Sigma_{k-1}\text{-LEM} \vdash \forall x \neg\neg \exists y \varphi(x, y)$.

The Friedman A-translation

Definition

For a HA-formula φ, we define φ^* as a formula obtained from φ by replacing all the prime formulas φ_{p} in φ with $\varphi_{\mathrm{p}} \vee *$. In particular, $\perp^* :\equiv (\perp \vee *)$, which is equivalent to $*$ over HA*. In what follows, $\neg_* \varphi$ denotes $\varphi \to *$.

Remark. By induction on the structure of formulas, one can show $\mathrm{FV}(\varphi) = \mathrm{FV}(\varphi^*)$ for all HA-formulas φ.

A Key Lemma for our Relativization

In what follows, let $\mathsf{HA}^* + \Sigma_k\text{-}\mathrm{LEM}$ denote HA^* augmented with $\Sigma_k\text{-}\mathrm{LEM}$ for "HA-formulas".

Lemma

1. For any HA-formula $\varphi \in \Sigma_k$, $\mathsf{HA}^* + \Sigma_{k-1}\text{-}\mathrm{LEM} \vdash \varphi^* \leftrightarrow \varphi \vee *$.
2. For any HA-formula $\varphi \in \Pi_k$, $\mathsf{HA}^* + \Sigma_k\text{-}\mathrm{LEM} \vdash \varphi^* \leftrightarrow \varphi \vee *$.

Proof. By simultaneous induction on k.

A Relativized Soundness of the A-translation

Theorem

For any HA-formula φ, if $\mathrm{HA} + \Sigma_k\text{-}\mathrm{LEM} \vdash \varphi$, then $\mathrm{HA}^* + \Sigma_k\text{-}\mathrm{LEM} \vdash \varphi^*$.

Proof. It suffices to show $\mathrm{HA}^* + \Sigma_k\text{-}\mathrm{LEM} \vdash \varphi^*$ for $\varphi \in \Sigma_k$.
Fix $\varphi :\equiv \exists x \varphi_1 \vee \neg \exists x \varphi_1$ with $\varphi_1 \in \Pi_{k-1}$.
Since φ^* is (equivalent to) $\exists x \, ({\varphi_1}^*) \vee \neg_* \exists x \, ({\varphi_1}^*)$, by the previous lemma, $\mathrm{HA}^* + \Sigma_{k-1}\text{-}\mathrm{LEM}$ proves

$$\exists x \, (\varphi_1 \vee *) \vee \neg_* \exists x \, (\varphi_1 \vee *) \, ,$$

which is equivalent to

$$\exists x \, (\varphi_1 \vee *) \vee \neg_* \exists x \varphi_1 ,$$

which is derived from $\exists x \varphi_1 \vee \neg \exists x \varphi_1$ over HA^*.
Thus $\mathrm{HA}^* + \Sigma_k\text{-}\mathrm{LEM}$ proves φ^*. □

Relativizing Friedman's Proof

Theorem

PA is Π_{k+2}-conservative over HA $+ \Sigma_k\text{-LEM}$.

Proof.

$$\text{PA} \vdash \forall x \exists y \varphi(x,y) \text{ where } \varphi(x,y) \in \Pi_k$$

$\Longrightarrow$ Negative translation

$$\text{HA} + \Sigma_{k-1}\text{-LEM} \vdash \forall x \neg\neg\exists y \varphi(x,y)$$

$\Longrightarrow$

$$\text{HA} + \Sigma_{k-1}\text{-LEM} \vdash \neg\neg\exists y \varphi(x,y)$$

$\Longrightarrow$ A-translation

$$\text{HA}^* + \Sigma_{k-1}\text{-LEM} \vdash \neg_*\neg_*\exists y \varphi^*(x,y)$$

$\Longrightarrow$ Key Lemma

$$\text{HA}^* + \Sigma_{k}\text{-LEM} \vdash \neg_*\neg_*\exists y\, (\varphi(x,y) \vee *)$$

$\Longrightarrow$ Substitution $[\exists y \varphi(x,y)/*]$

$$\text{HA} + \Sigma_{k}\text{-LEM} \vdash$$

$$(\exists y\, (\varphi(x,y) \vee \exists y \varphi(x,y)) \to \exists y \varphi(x,y)) \to \exists y \varphi(x,y).$$

$\Longrightarrow$

$$\text{HA} + \Sigma_{k}\text{-LEM} \vdash \forall x \exists y \varphi(x,y) \quad \square$$

Ishihara's method

- Ishihara 2000 defined classes $\mathcal{Q}, \mathcal{R}, \mathcal{J}, \mathcal{K}$ of formulas of IQC, and by using the generalized negative translation and the substitution lemma, showed that if $\varphi \in \mathcal{K}$ and Γ is a set of formulas closed under \$, then $\Gamma \vdash_c \varphi \Rightarrow \Gamma \vdash_i \varphi$.
- Since $\Pi_2 \subseteq \mathcal{K}$, applying the result in the language of arithmetic, one can obtain the fact that PA is Π_2-conservative over HA (Ishihara 2012).
- In what follows, by extending Ishihara's arguments in the context of semi-classical arithmetic, we show a relativized conservation theorem.

The Generalized Negative Translation

Definition (Ishihara 2000)

Let $\neg_\$\varphi$ denote $\varphi \to \$$. For each formula φ, its \$-negative translation $\varphi^\$$ is defined inductively be the following clauses:

- For φ_p prime such that $\varphi_\mathrm{p} \not\equiv \bot$, $(\varphi_\mathrm{p})^\$:\equiv \neg_\$\neg_\$\varphi_\mathrm{p}$;
- $\bot^\$:\equiv \$$;
- $(\varphi_1 \circ \varphi_2)^\$:\equiv \varphi_1^\$ \circ \varphi_2^\$$ for $\circ \in \{\wedge, \to\}$;
- $(\varphi_1 \vee \varphi_2)^\$:\equiv \neg_\$\neg_\$ \left(\varphi_1^\$ \vee \varphi_2^\$\right)$;
- $(\forall x \varphi)^\$:\equiv \forall x \varphi^\$$;
- $(\exists x \varphi)^\$:\equiv \neg_\$\neg_\$\exists x \varphi^\$$.

Remark. $\vdash_i \left(\varphi^G\right)^* \leftrightarrow \varphi^\$[*/\$]$ since $\vdash_i (\bot \vee *) \leftrightarrow *$. In particular, $\varphi^\$[\bot/\$]$ is Gentzen's negative translation φ^N of φ.

Proposition

1. For any HA-formula φ, $\mathrm{HA}^{\$} \vdash \neg_{\$}\neg_{\$}\varphi^{\$} \leftrightarrow \varphi^{\$}$;
2. For any HA-formula φ, if $\mathrm{PA} \vdash \varphi$, then $\mathrm{HA}^{\$} \vdash \varphi^{\$}$.

Proof. Same as for Gentzen's negative translation.

What is done in Ishihara 2000

Definition 6. We define simultaneously classes $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{J}$ of formulas as follows. Let P range over prime formulas distinct from $\bot$, Q and Q' over $\mathcal{Q}$, R and R' over $\mathcal{R}$, and J and J' over $\mathcal{J}$, respectively. Then $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{J}$ are simultaneously inductively generated by the clauses

1. $\bot, P, Q \wedge Q', Q \vee Q', \forall x Q, \exists x Q, J \to Q \in \mathcal{Q}$;
2. $\bot, R \wedge R', R \vee R', \forall x R, J \to R \in \mathcal{R}$;
3. $\bot, P, J \wedge J', J \vee J', \exists x J, R \to J \in \mathcal{J}$.

Proposition 7.

1. *If A is in $\mathcal{Q}$, then $\vdash_{\mathrm{i}} A \to A^{\$}$.*
2. *If A is in $\mathcal{R}$, then $\vdash_{\mathrm{i}} \neg_{\$}\neg A \to A^{\$}$.*
3. *If A is in $\mathcal{J}$, then $\vdash_{\mathrm{i}} A^{\$} \to \neg_{\$}\neg_{\$} A$.*

Source: H. Ishihara, A note on the Gödel-Gentzen translation, Mathematical Logic Quarterly, 46(1), p. 136, 2000.

To extend these arguments in the context of semi-classical arithmetic, we first define relativized classes $\mathcal{R}_k$ and $\mathcal{J}_k$ as follows:

Definition

Define $\mathcal{R}_0 := \mathcal{J}_0 := \Sigma_0 \, (= \Pi_0)$. For each k, we define simultaneously classes $\mathcal{R}_{k+1}$ and $\mathcal{J}_{k+1}$ as follows. Let F range over formulas in Σ_k, F' over Π_k, R and R' over those in $\mathcal{R}_{k+1}$, and J and J' over those in $\mathcal{J}_{k+1}$. Then $\mathcal{R}_{k+1}$ and $\mathcal{J}_{k+1}$ are inductively generated by the clauses

1. $F, R \wedge R', R \vee R', \forall x R, J \to R \in \mathcal{R}_{k+1}$;
2. $F', J \wedge J', J \vee J', \exists x J, R \to J \in \mathcal{J}_{k+1}$.

Lemma

$\mathsf{HA}^{\$} \vdash \varphi_{\mathrm{qf}}^{\$} \leftrightarrow \varphi_{\mathrm{qf}} \vee \$$.

Proof. By induction on the structure of quantifier-free formulas of HA. □

Lemma

$\mathsf{HA}^{\$} \vdash \varphi_{\mathrm{qf}}^{\$} \leftrightarrow \varphi_{\mathrm{qf}} \lor \$.$

Proof. By induction on the structure of quantifier-free formulas of HA. □

Lemma

1. *For* $\varphi \in \Sigma_k$, $\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \varphi^{\$} \leftrightarrow \varphi \lor \$;$
2. *For* $\varphi \in \Pi_k$, $\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \varphi^{\$} \leftrightarrow \varphi \lor \$.$

Proof. By simultaneous induction on k. □

Lemma

$\mathsf{HA}^\$ \vdash \varphi_{\mathrm{qf}}^\$ \leftrightarrow \varphi_{\mathrm{qf}} \vee \$$.

Proof. By induction on the structure of quantifier-free formulas of HA. □

Lemma

1. *For* $\varphi \in \Sigma_k$, $\mathsf{HA}^\$ + \Sigma_k\text{-}\mathrm{LEM} \vdash \varphi^\$ \leftrightarrow \varphi \vee \$$;
2. *For* $\varphi \in \Pi_k$, $\mathsf{HA}^\$ + \Sigma_k\text{-}\mathrm{LEM} \vdash \varphi^\$ \leftrightarrow \varphi \vee \$$.

Proof. By simultaneous induction on k. □

Lemma

1. *If* $\varphi \in \mathcal{R}_{k+1}$, *then* $\mathsf{HA}^\$ + \Sigma_k\text{-}\mathrm{LEM}$ *proves* $\neg_\$ \neg \varphi \to \varphi^\$$;
2. *If* $\varphi \in \mathcal{J}_{k+1}$, *then* $\mathsf{HA}^\$ + \Sigma_k\text{-}\mathrm{LEM}$ *proves* $\varphi^\$ \to \neg_\$ \neg_\$ \varphi$.

Definition

Define $\mathcal{Q}_0 := \Sigma_0 \, (= \Pi_0)$. For each k, we define a class $\mathcal{Q}_{k+1}$ as follows. Let P range over prime formulas, Q and Q' over those in $\mathcal{Q}_{k+1}$, and J over those in $\mathcal{J}_{k+1}$. Then $\mathcal{Q}_{k+1}$ is inductively generated by the clause

$$P, Q \wedge Q', Q \vee Q', \forall x Q, \exists x Q, J \to Q \in \mathcal{Q}_{k+1}.$$

Lemma

Let k be a natural number and φ be a HA*-formula. If $\varphi \in \mathcal{Q}_{k+1}$, then* $\mathrm{HA}^{\$} + \Sigma_k\text{-}\mathrm{LEM} \vdash \varphi \to \varphi^{\$}$.

Definition

For each k, we define a class $\mathcal{V}_k$ as follows. Let J range over formulas in $\mathcal{J}_k$, V and V' over those in $\mathcal{V}_k$. Then $\mathcal{V}_k$ is inductively generated by the clause

$$J, V \wedge V', \forall x V \in \mathcal{V}_k.$$

Theorem

For any HA-formulas $\varphi \in \mathcal{V}_{k+1}$ and $\psi \in \mathcal{Q}_{k+1}$, if $\mathsf{PA} \vdash \psi \to \varphi$, then $\mathsf{HA} + \Sigma_k\text{-}\mathrm{LEM} \vdash \psi \to \varphi$.

Definition

For each k, we define a class $\mathcal{V}_k$ as follows. Let J range over formulas in $\mathcal{J}_k$, V and V' over those in $\mathcal{V}_k$. Then $\mathcal{V}_k$ is inductively generated by the clause

$$J, V \wedge V', \forall x V \in \mathcal{V}_k.$$

Theorem

For any HA-formulas $\varphi \in \mathcal{V}_{k+1}$ and $\psi \in \mathcal{Q}_{k+1}$, if $\mathsf{PA} \vdash \psi \to \varphi$, then $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \varphi$.

Corollary

For any formulas $\varphi \in \Pi_{k+2}$ and ψ of prenex normal form, if $\mathsf{PA} \vdash \psi \to \varphi$, then $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \varphi$.

Proof of Theorem (1/2)

It suffices to show the assertion for $\varphi \in \mathcal{V}_{k+1}$ and $\psi \in \mathcal{Q}_{k+1}$ such that the free variables of φ is not bounded in ψ. We show this by induction on the structure of formulas in $\mathcal{V}_{k+1}$.

Case of $\varphi \in \mathcal{J}_{k+1}$:
Fix $\psi \in \mathcal{Q}_{k+1}$ s.t. the free variables of φ is not bounded in ψ.
Suppose $\mathsf{PA} \vdash \psi \to \varphi$.
Applying the \$-negative translation, we have $\mathsf{HA}^{\$} \vdash \psi^{\$} \to \varphi^{\$}$.
Since $\varphi \in \mathcal{J}_{k+1}$ and $\psi \in \mathcal{Q}_{k+1}$, we have

$$\mathsf{HA}^{\$} + \Sigma_k\text{-LEM} \vdash \psi \to \neg_{\$}\neg_{\$}\varphi.$$

By the substitution lemma, we have that $\mathsf{HA} + \Sigma_k\text{-LEM}$ proves $\psi \to ((\varphi \to \varphi) \to \varphi)$, equivalently, $\psi \to \varphi$.

Proof of Theorem (2/2)

Case of $\varphi :\equiv \varphi_1 \wedge \varphi_2 \in \mathcal{V}_{k+1}$:
Then $\varphi_1, \varphi_2 \in \mathcal{V}_{k+1}$.
Fix $\psi \in \mathcal{Q}_{k+1}$ s.t. the free variables of φ is not bounded in ψ.
Suppose $\mathsf{PA} \vdash \psi \to \varphi_1 \wedge \varphi_2$.
Then $\mathsf{PA} \vdash \psi \to \varphi_1$ and $\mathsf{PA} \vdash \psi \to \varphi_2$.
By I.H., we have $\mathsf{HA} + \Sigma_k\text{-LEM}$ proves $\psi \to \varphi_1$ and $\psi \to \varphi_2$, and hence, $\psi \to \varphi_1 \wedge \varphi_2$.

Case of $\varphi :\equiv \forall x \varphi_1 \in \mathcal{V}_{k+1}$:
Then $\varphi_1 \in \mathcal{V}_{k+1}$.
Fix $\psi \in \mathcal{Q}_{k+1}$ s.t. the free variables of φ is not bounded in ψ.
In addition, assume $x \notin \mathrm{FV}(\psi)$ without loss of generality.
Suppose $\mathsf{PA} \vdash \psi \to \forall x \varphi_1$. Then $\mathsf{PA} \vdash \psi \to \varphi_1$.
By I.H., we have that $\mathsf{HA} + \Sigma_k\text{-LEM}$ proves $\psi \to \varphi_1$, and hence, $\forall x(\psi \to \varphi_1)$.
Since $x \notin \mathrm{FV}(\psi)$, we have $\mathsf{HA} + \Sigma_k\text{-LEM} \vdash \psi \to \forall x \varphi_1$. □

Questions

1. Not only Gentzen's negative translation, but there are also other well-known negative translations which are equivalent to the former over intuitionistic logic but not so over minimal logic. Can we obtain interesting conservation results by using their generalized variants combined with the substitution lemma?
2. How is the entire structure of the relation between the second and third method? How can we study it?

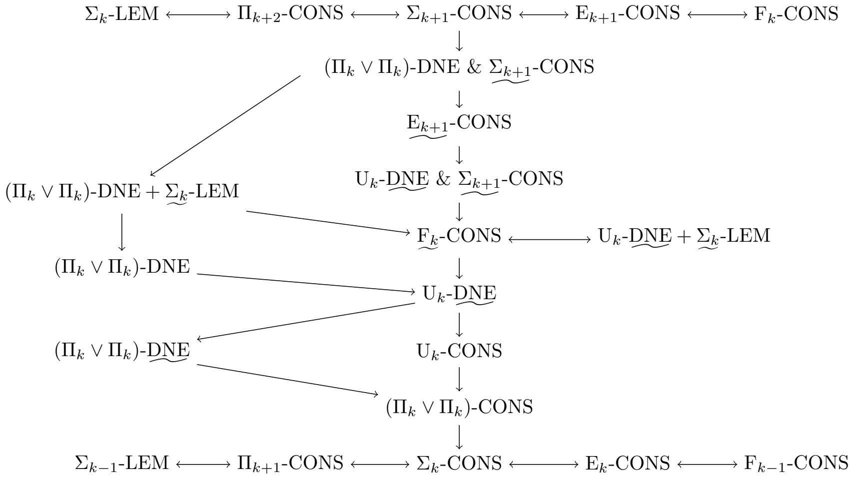
Our relativized conservation theorem is optimal:

Proposition

Let T be an extension of HA. If PA is Π_{k+2}-conservative over T, then $T \vdash \Sigma_k\text{-LEM}$.

Proof. By induction on k. The base case is trivial. For the ind. step, assume the assertion for k to show that for $k+1$.
Suppose that PA is Π_{k+3}-conservative over T.
By I.H., we have $T \vdash \Sigma_k\text{-LEM}$.
Fix an instance of $\Sigma_{k+1}\text{-LEM}$ $\varphi :\equiv \forall x(\varphi_1(x) \vee \neg\varphi_1(x))$,
where $\varphi_1(x) \in \Sigma_{k+1}(x)$.
Let $\psi_1(x) \in \Pi_{k+1}(x)$ satisfy $\mathrm{HA} + \Sigma_k\text{-DNE} \vdash \neg\varphi_1(x) \leftrightarrow \psi_1(x)$.
There exists $\xi(x) \in \Sigma_{k+2}(x)$ s.t. $\mathrm{HA} \vdash \xi(x) \leftrightarrow \varphi_1(x) \vee \psi_1(x)$.
Thus $\mathrm{HA} + \Sigma_k\text{-DNE} \vdash \varphi \leftrightarrow \forall x \xi(x)\, (\in \Pi_{k+3})$.
Since $\mathrm{PA} \vdash \varphi$, we have $\mathrm{PA} \vdash \forall x \xi(x)$.
By our assumption, we have $T \vdash \forall x \xi(x)$.
Since $\Sigma_k\text{-LEM}$ derives $\Sigma_k\text{-DNE}$, we have $T \vdash \varphi$. □

Comprehensive Scrutiny



where Γ-CONS is the assertion "PA is Γ-conservative over T" and $\underset{\sim}{\Gamma}$ denotes the class of sentences in Γ.

References

1. H. Friedman, Classically and intuitionistically provably recursive functions. In G. Müller and D. S. Scott, editors, Higher Set Theory, pp. 21–27, 1978.
2. H. Ishihara, A note on the Gödel-Gentzen translation, MLQ 46(1):135–137, 2000.
3. H. Ishihara, Some conservative extension results on classical and intuitionistic sequent calculi, In U. Berger, H. Diener, P. Schuster, and M. Seisenberger, editors, Logic, Construction, Computation, pp. 289–304, 2012.
4. M. Fujiwara and T. Kurahashi, Prenex normal form theorems in semi-classical arithmetic, The Journal of Symbolic Logic, to appear.
5. M. Fujiwara and T. Kurahashi, Conservation theorems on semi-classical arithmetic, arXiv:2107.11356.