Proof Theory and Proof Search of Classical and Intuitionistic Tense Logic

Zhe Lin

(Jointed work with Minghui Ma, Yu Peng, Xue Ge)

Department of Philosophy, Xiamen University, China

New Frontiers in Proofs and Computation 2021

- Intuitionistic tense logics and its cut free sequent caclulus.
- Double negation translation between classical and intuitionistic tense logics.
- Cut free sequent caclulus of classical tense logics.
- Decidability and Proof Search of classical and intuitionistic tense logics.
- Results on fusions of classical and intuitionistic tense logics.

Definition

Let **Var** $= \{p_i \mid i \in \omega\}$ be a denumerable set of propositional variables. The set of all formulas $\mathcal{F}$ is defined inductively as follows:

$$\mathcal{F} \ni \alpha ::= p \mid \top \mid \bot \mid (\alpha_1 \wedge \alpha_2) \mid (\alpha_1 \vee \alpha_2) \mid (\alpha_1 \to \alpha_2) \mid \Diamond\alpha \mid \Box\alpha \mid \blacklozenge\alpha \mid \blacksquare\alpha$$

where $p \in$ **Var**. We use the abbreviation $\neg\alpha := \alpha \to \bot$.

Definition

The Hilbert-style axiomatic system **IK.t** for Ewald's intuitionistic tense logic consists of the following axiom schemata and rules:

(1) Axiom Schemata:
(Int) All axioms of intuitionistic propositional calculus.
$(\mathrm{K}_{\Diamond\Box})$ $\Box(\alpha \to \beta) \to (\Diamond\alpha \to \Diamond\beta)$.
$(\mathrm{K}_{\blacklozenge\blacksquare})$ $\blacksquare(\alpha \to \beta) \to (\blacklozenge\alpha \to \blacklozenge\beta)$.

(2) Inference rules:

$$\frac{\alpha \to \beta \quad \alpha}{\beta}(\mathrm{MP}) \quad \frac{\Diamond\alpha \to \beta}{\alpha \to \blacksquare\beta}(\mathrm{Adj}_1) \quad \frac{\blacklozenge\alpha \to \beta}{\alpha \to \Box\beta}(\mathrm{Adj}_2)$$

Definition

Let (,) $\circ$ and $\bullet$ be structural operators for $\wedge$, $\Diamond$ and $\blacklozenge$ respectively. The set of all formula structures $\mathcal{FS}$ are defined inductively as follows:

$$\mathcal{FS} \ni \Gamma ::= \alpha \mid \Gamma_1, \Gamma_2 \mid \circ\Gamma \mid \bullet\Gamma$$

A *sequent* is an expression of the form $\Gamma \Rightarrow \alpha$ and a *context* is a formula structure $\Gamma[-]$ with a designated position $[-]$.

The Gentzen sequent calculus GIK.t for the intuitionistic modal logic IK.t consists of the following axiom and rules:

(1) Axiom:

$$(\mathrm{Id})\ \alpha \Rightarrow \alpha$$

(2) Logical rules

$$\frac{\Gamma[\top] \Rightarrow \beta}{\Gamma[\Delta] \Rightarrow \beta}(\top) \quad \frac{\Delta \Rightarrow \bot}{\Gamma[\Delta] \Rightarrow \alpha}(\bot)$$

$$\frac{\Gamma[\alpha_1, \alpha_2] \Rightarrow \beta}{\Gamma[\alpha_1 \wedge \alpha_2] \Rightarrow \beta}(\wedge\mathrm{L}) \quad \frac{\Gamma_1 \Rightarrow \alpha_1 \quad \Gamma_2 \Rightarrow \alpha_2}{\Gamma_1, \Gamma_2 \Rightarrow \alpha_1 \wedge \alpha_2}(\wedge\mathrm{R})$$

$$\frac{\Gamma[\alpha_1] \Rightarrow \beta \quad \Gamma[\alpha_2] \Rightarrow \beta}{\Gamma[\alpha_1 \vee \alpha_2] \Rightarrow \beta}(\vee\mathrm{L}) \quad \frac{\Gamma \Rightarrow \alpha_i}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}(\vee\mathrm{R})(i = 1, 2)$$

$$\frac{\Delta \Rightarrow \alpha_1 \quad \Gamma[\alpha_2] \Rightarrow \beta}{\Gamma[\Delta, \alpha_1 \to \alpha_2] \Rightarrow \beta}(\to\mathrm{L}) \quad \frac{\alpha_1, \Gamma \Rightarrow \alpha_2}{\Gamma \Rightarrow \alpha_1 \to \alpha_2}(\to\mathrm{R})$$

(3) Structural rules:

$$\frac{\Gamma[\Delta_i] \Rightarrow \beta}{\Gamma[\Delta_1, \Delta_2] \Rightarrow \beta}(\mathrm{Wk})(i = 1, 2) \quad \frac{\Gamma[\alpha, \alpha] \Rightarrow \beta}{\Gamma[\alpha] \Rightarrow \beta}(\mathrm{Conf})$$

$$\frac{\Gamma[\circ\Delta_1, \circ\Delta_2] \Rightarrow \beta}{\Gamma[\circ(\Delta_1, \Delta_2)] \Rightarrow \beta}(\mathrm{Con}_\circ) \quad \frac{\Gamma[\bullet\Delta_1, \bullet\Delta_2] \Rightarrow \beta}{\Gamma[\bullet(\Delta_1, \Delta_2)] \Rightarrow \beta}(\mathrm{Con}_\bullet)$$

$$\frac{\Gamma[\Delta_1, \Delta_2] \Rightarrow \beta}{\Gamma[\Delta_2, \Delta_1] \Rightarrow \beta}(\mathrm{Ex})$$

$$\frac{\Gamma[\Delta_1, (\Delta_2, \Delta_3)] \Rightarrow \beta}{\Gamma[(\Delta_1, \Delta_2), \Delta_3] \Rightarrow \beta}(\mathrm{As}_1) \quad \frac{\Gamma[(\Delta_1, \Delta_2), \Delta_3] \Rightarrow \beta}{\Gamma[\Delta_1, (\Delta_2, \Delta_3)] \Rightarrow \beta}(\mathrm{As}_2)$$

(4) Cut rule:

$$\frac{\Delta \Rightarrow \alpha \quad \Gamma[\alpha] \Rightarrow \beta}{\Gamma[\Delta] \Rightarrow \beta}(\mathrm{Cut})$$

(5) Modal rules:

$$\frac{\Gamma[\circ\alpha] \Rightarrow \beta}{\Gamma[\Diamond\alpha] \Rightarrow \beta}(\Diamond\mathrm{L}) \quad \frac{\Gamma \Rightarrow \alpha}{\circ\Gamma \Rightarrow \Diamond\alpha}(\Diamond\mathrm{R}) \quad \frac{\Gamma[\bullet\alpha] \Rightarrow \beta}{\Gamma[\blacklozenge\alpha] \Rightarrow \beta}(\blacklozenge\mathrm{L}) \quad \frac{\Gamma \Rightarrow \alpha}{\bullet\Gamma \Rightarrow \blacklozenge\alpha}(\blacklozenge\mathrm{R})$$

$$\frac{\Gamma[\alpha] \Rightarrow \beta}{\Gamma[\circ\blacksquare\alpha] \Rightarrow \beta}(\blacksquare\mathrm{L}), \quad \frac{\circ\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \blacksquare\alpha}(\blacksquare\mathrm{R}). \quad \frac{\Gamma[\alpha] \Rightarrow \beta}{\Gamma[\bullet\Box\alpha] \Rightarrow \beta}(\Box\mathrm{L}), \quad \frac{\bullet\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \Box\alpha}(\Box\mathrm{R}).$$

$$\frac{\Gamma[\circ(\Delta_1, \bullet\Delta_2)] \Rightarrow \beta}{\Gamma[\circ\Delta_1, \Delta_2] \Rightarrow \beta}(\mathrm{K}_{\circ\bullet}). \quad \frac{\Gamma[\bullet(\Delta_1, \circ\Delta_2)] \Rightarrow \beta}{\Gamma[\bullet\Delta_1, \Delta_2] \Rightarrow \beta}(\mathrm{K}_{\bullet\circ}).$$

Lemma

If $\vdash_{\mathrm{GIK.t}} \Gamma[\Delta, \Delta] \Rightarrow \beta$ *is derivable without any application of (Cut), then* $\vdash_{\mathrm{GIK.t}} \Gamma[\Delta] \Rightarrow \beta$ *is derivable without any application of (Cut).*

Theorem

If $\vdash_{\mathrm{GIK.t}} \Gamma \Rightarrow \beta$*, then* $\vdash_{\mathrm{GIK.t}} \Gamma \Rightarrow \beta$ *without any application of (Cut).*

$$\dfrac{\Delta \Rightarrow \alpha \qquad \dfrac{\dfrac{\Sigma'[\circ(\Sigma_1, \alpha^{n_1}), \circ(\Sigma_2, \alpha^{n_2})] \Rightarrow \beta}{\Sigma'[\circ(\Sigma_1, \alpha^{n_1}, \Sigma_2, \alpha^{n_2})] \Rightarrow \beta}\,(\mathrm{Con}_\circ)}{\Sigma'[\circ(\Sigma_1, \Sigma_2, \alpha)] \Rightarrow \beta}\,(\mathrm{Conf}\times(\text{n-1}))}{\Sigma'[\circ(\Sigma_1, \Sigma_2, \Delta)] \Rightarrow \beta}\,(\mathrm{Cut})$$

$$\dfrac{\dfrac{\dfrac{\Delta \Rightarrow \alpha \qquad \dfrac{\dfrac{\Delta \Rightarrow \alpha \qquad \dfrac{\Sigma'[\circ(\Sigma_1, \alpha^{n_1}), \circ(\Sigma_2, \alpha^{n_2})] \Rightarrow \theta}{\Sigma'[\circ(\Sigma_1, \alpha), \circ(\Sigma_2, \alpha^{n_2})] \Rightarrow \theta}\,(\mathrm{Conf}\times(n_1 - 1))}{\Sigma'[\circ(\Sigma_1, \Delta), \circ(\Sigma_2, \alpha^{n_2})] \Rightarrow \theta}\,(\mathrm{Cut})}{\Sigma'[\circ(\Sigma_1, \Delta), \circ(\Sigma_2, \alpha)] \Rightarrow \theta}\,(\mathrm{Conf}\times(n_2 - 1))}{\Sigma'[\circ(\Sigma_1, \Delta), \circ(\Sigma_2, \Delta)] \Rightarrow \theta}\,(\mathrm{Cut})}{\Sigma'[\circ(\Sigma_1, \Delta, \Sigma_2, \Delta)] \Rightarrow \theta}\,(\mathrm{Con}\circ)}{\Sigma'[\circ(\Sigma_1, \Sigma_2, \Delta)] \Rightarrow \theta}\,\text{Lemma 4}$$

Definition

A formula is called sp-formulas if it is strictly positive modal formulas constructed from propositional variables and $\top$ using $\wedge$, $\Diamond$, $\blacklozenge$. A sp-implication is a formula of the of the form $\alpha \to \beta$ where α and β are sp-formulas. GIK.t enriched with a set of axioms which can be axiomatised by sp-implications, denoted by GIK.t$\oplus$SP is called a sp-extension of GIK.t. Sp-axioms are sequents in the form of $\alpha \Rightarrow \beta$ where $\alpha \to \beta$ is a sp-implication. (Kikot, Stanislav, Kurucz, Agi, Tanaka, Yoshihito, Wolter, Frank and Zakharyaschev, Michael (2019))

Every sp-axiom can be replaced by a general structural rule.

Definition

The set of generalized contexts is defined as follows:

$$\mathcal{GCT} \ni \bar{\Xi}[] ::= [] \mid \circ(\bar{\Xi}[]) \mid \bullet(\bar{\Xi}[]) \mid (\bar{\Xi}[], \bar{\Xi}[])$$

For a context $\bar{\Xi}[]$ of arity n, we denote the simultaneous substitution of n formula structures $\Gamma_1, \ldots, \Gamma_n$ in left-to-right order by $\bar{\Xi}[\Gamma_1; \ldots; \Gamma_n]$. The result of this simultaneous substitution is a formula structure.

Definition

Given two contexts $\bar{\Xi}[]$ of arity n and $\bar{\Xi}'[]$ of arity m. Let $\pi_1, \ldots, \pi_m \in \{1, \ldots, n\}$. We define the general structural rule schema as follows:

$$\frac{\Gamma[\bar{\Xi}[\Gamma_1; \ldots; \Gamma_n]] \Rightarrow \psi}{\Gamma[\bar{\Xi}'[\Gamma_{\pi_1}; \ldots; \Gamma_{\pi_n}]] \Rightarrow \psi} \quad (GSR)$$

One denote GIK.t enriched with (GSR) by GSGIK.t.

Theorem

If $\vdash_{\mathrm{GSGIK.t}} \Gamma \Rightarrow \beta$*, then* $\vdash_{\mathrm{GSGIK.t}} \Gamma \Rightarrow \beta$ *without any application of (Cut).*

Corollary

If $\vdash_{\mathrm{GSGIK.t}} \Gamma \Rightarrow \beta$*, then there is a derivation of* $\Gamma \Rightarrow \beta$ *in* $\mathrm{GSGIK.t}$ *such that all formulas appearing in the derivation are subformulas of formulas in* $\Gamma \Rightarrow \beta$*.*

Definition

The set of general formula contexts is defined as follows:

$$\mathcal{GFC} \ni \bar{\varphi}\langle\rangle ::= \langle\rangle \mid \Diamond(\bar{\varphi}\langle\rangle) \mid \blacklozenge(\bar{\varphi}\langle\rangle) \mid (\bar{\varphi}\langle\rangle \wedge \bar{\varphi}\langle\rangle)$$

Example

Given two formula context $\blacklozenge(\Diamond\langle\rangle \wedge \langle\rangle)$ and $\langle\rangle \wedge \blacklozenge\langle\rangle$, performing simultaneous substitution, one obtains $\blacklozenge(\Diamond\langle\varphi\rangle \wedge \langle\psi\rangle) = \blacklozenge(\Diamond\varphi \wedge \psi)$ and $\langle\varphi\rangle \wedge \blacklozenge\langle\psi\rangle = \varphi \wedge \blacklozenge\psi$, respectively. Clearly $\blacklozenge(\Diamond\varphi \wedge \psi) \Rightarrow \varphi \wedge \blacklozenge\psi$ is a sp-axiom.

Let (GSA) be $f(\Xi[\Gamma_1; \ldots; \Gamma_n]) \Rightarrow f(\Xi'[\Gamma_{\pi_1}; \ldots; \Gamma_{\pi_n}])$. Clearly (GSA) is a rsp-axiom. Let GSGIK'.t be GIK.t enriched with axiom (GSA).

Theorem

$\vdash_{\mathrm{GSGIK'.t}} \Gamma \Rightarrow \varphi$ *iff* $\vdash_{\mathrm{GSGIK.t}} \Gamma \Rightarrow \varphi$

The axioms and their corresponding structural rules are presented in the following table.

Axioms	Structural Rules	Axioms	Structural Rules
$\top \Rightarrow \Diamond\top$	$\frac{\Gamma[\circ\top]\Rightarrow\alpha}{\Gamma[\top]\Rightarrow\alpha}$	$\top \Rightarrow \blacklozenge\top$	$\frac{\Gamma[\bullet\top]\Rightarrow\alpha}{\Gamma[\top]\Rightarrow\alpha}$
$\Box\alpha \Rightarrow \alpha$	$\frac{\Gamma[\bullet(\Delta)]\Rightarrow\psi}{\Gamma[\Delta]\Rightarrow\psi}$	$\blacksquare\alpha \Rightarrow \alpha$	$\frac{\Gamma[\circ(\Delta)]\Rightarrow\psi}{\Gamma[\Delta]\Rightarrow\psi}$
$\alpha \Rightarrow \Box\Diamond\alpha$	$\frac{\Gamma[\circ(\Delta)]\Rightarrow\psi}{\Gamma[\bullet(\Delta)]\Rightarrow\psi}$	$\alpha \Rightarrow \blacksquare\blacklozenge\alpha$	$\frac{\Gamma[\bullet(\Delta)]\Rightarrow\psi}{\Gamma[\circ(\Delta)]\Rightarrow\psi}$
$\Box\Box\alpha \Rightarrow \Box\alpha$	$\frac{\Gamma[\bullet\bullet(\Delta)]\Rightarrow\psi}{\Gamma[\bullet(\Delta)]\Rightarrow\psi}$	$\blacksquare\blacksquare\alpha \Rightarrow \blacksquare\alpha$	$\frac{\Gamma[\circ\circ(\Delta)]\Rightarrow\psi}{\Gamma[\circ(\Delta)]\Rightarrow\psi}$
$\Diamond\alpha \Rightarrow \Box\Diamond\alpha$	$\frac{\Gamma[\circ(\Delta)]\Rightarrow\psi}{\Gamma[\bullet\circ(\Delta)]\Rightarrow\psi}$	$\blacklozenge\alpha \Rightarrow \blacksquare\blacklozenge\alpha$	$\frac{\Gamma[\bullet(\Delta)]\Rightarrow\psi}{\Gamma[\circ\bullet(\Delta)]\Rightarrow\psi}$
$\Box\alpha \to \Box\beta \Rightarrow \Box(\alpha \to \beta)$	$\frac{\Gamma[\circ(\Delta_1,\Delta_2)]\Rightarrow\psi}{\Gamma[\circ\Delta_1,\circ\Delta_2]\Rightarrow\psi}$	$\blacksquare\alpha \to \blacksquare\beta \Rightarrow \blacksquare(\alpha \to \beta)$	$\frac{\Gamma[\bullet(\Delta_1,\Delta_2)]\Rightarrow\psi}{\Gamma[\bullet\Delta_1,\bullet\Delta_2]\Rightarrow\psi}$
$\Diamond\alpha \Rightarrow \Box\alpha$	$\frac{\Gamma[\Delta]\Rightarrow\psi}{\Gamma[\bullet\circ\Delta]\Rightarrow\psi}$	$\blacklozenge\alpha \Rightarrow \blacksquare\alpha$	$\frac{\Gamma[\bullet\Delta]\Rightarrow\psi}{\Gamma[\circ\bullet\Delta]\Rightarrow\psi}$
$\Box\alpha \Rightarrow \Box\Box\alpha$	$\frac{\Gamma[\bullet(\Delta)]\Rightarrow\psi}{\Gamma[\bullet\bullet(\Delta)]\Rightarrow\psi}$	$\blacksquare\alpha \Rightarrow \blacksquare\blacksquare\alpha$	$\frac{\Gamma[\circ(\Delta)]\Rightarrow\psi}{\Gamma[\circ\circ(\Delta)]\Rightarrow\psi}$

Table: Structural Axioms and Rules

Sequent calculus for classical tense logic denoted by GK.t is obtained by GIK.t enriching with the double negation elimination axiom (Dne) $\neg\neg\alpha \Rightarrow \alpha$. Let IG be a sequent calculi GSGIK.t of a rsp-extension of IK.t and G its classical correspondent. The Kolmogorov translation from G formulas to IG formulas are defined as follows.

Definition

We define a translation *ko*() from *G* formulas to *IG* formulas as follows:

- $ko(p) = \neg\neg p$;
- $ko(\bot) = \bot$;
- $ko(\top) = \top$;
- $ko(\Diamond\beta) = \neg\neg\Diamond ko(\beta)$;
- $ko(\blacklozenge\beta) = \neg\neg\blacklozenge ko(\beta)$
- $ko(\Box\beta) = \neg\neg\Box ko(\beta)$;
- $ko(\blacksquare\beta) = \neg\neg\blacksquare ko(\beta)$;
- $ko(\alpha \vee \beta) = \neg\neg(ko(\alpha) \vee ko(\beta))$;
- $ko(\beta \wedge \chi) = \neg\neg(ko(\beta) \wedge ko(\chi))$;
- $ko(\neg\beta) = \neg ko(\beta)$;
- $ko(\alpha \to \beta) = \neg\neg(ko(\alpha) \to ko(\beta))$, if $\beta \neq \bot$.

Lemma

$\vdash_{\mathrm{G}} \Gamma \Rightarrow \beta$ *iff* $\vdash_{\mathrm{IG}} ko(\Gamma) \Rightarrow ko(\beta)$.

Lemma

If $\vdash_{\mathrm{IG}} \Gamma_1[\beta], \Gamma_2 \Rightarrow \bot$, *then* $\vdash_{\mathrm{IG}} \Gamma_1[\neg\neg\beta], \Gamma_2 \Rightarrow \bot$

Lemma

If $\vdash_{\mathrm{IG}} \Gamma[\beta] \Rightarrow \neg\chi$, *then* $\vdash_{\mathrm{IG}} \Gamma[\neg\neg\beta] \Rightarrow \neg\chi$

Lemma

$\vdash_{\mathrm{IG}} ko(\neg\neg\beta) \Rightarrow ko(\beta)$

Definition

We define a translation $gg()$ from G formulae to IG formulae as follows:

- $gg(p) = \neg\neg p$;
- $gg(\bot) = \bot$;
- $gg(\top) = \top$;
- $gg(\Diamond\beta) = \neg\neg\Diamond gg(\beta)$;
- $gg(\blacklozenge\beta) = \neg\neg\blacklozenge gg(\beta)$
- $gg(\Box\beta) = \Box gg(\beta)$;
- $gg(\blacksquare\beta) = \blacksquare gg(\beta)$;
- $gg(\alpha \vee \beta) = \neg\neg(gg(\alpha) \vee gg(\beta))$;
- $gg(\beta \wedge \chi) = gg(\beta) \wedge gg(\chi)$;
- $gg(\alpha \to \beta) = gg(\alpha) \to gg(\beta)$.

Theorem

$\vdash_{\mathrm{G}} \Gamma \Rightarrow \beta$ *iff* $\vdash_{\mathrm{IG}} gg(\Gamma) \Rightarrow gg(\beta)$

Definition

We define a translation $ku()$ from G formulae to IG formulae as follows:

- $ku(p) = p$;
- $ku(\bot) = \bot$;
- $ku(\top) = \top$;
- $ku(\lozenge\beta) = \lozenge ku(\beta)$;
- $ku(\blacklozenge\beta) = \blacklozenge ku(\beta)$
- $ku(\Box\beta) = \Box\neg\neg ku(\beta)$;
- $ku(\blacksquare\beta) = \blacksquare\neg\neg ku(\beta)$;
- $ku(\alpha \vee \beta) = ku(\alpha) \vee ku(\beta)$;
- $ku(\alpha \wedge \beta) = ku(\alpha) \wedge ku(\beta)$;
- $ku(\alpha \to \beta) = ku(\alpha) \to ku(\beta)$.

Theorem

$\vdash_{\mathrm{G}} \Gamma \Rightarrow \beta$ *iff* $\vdash_{\mathrm{IG}} ku^{(\neg\neg)}(\Gamma) \Rightarrow \neg\neg ku(\beta)$

The Glivenko translation can be simply defined as $g(\alpha) = \neg\neg\alpha$ for any formula α. It can be extended to formula structure. The Glivenko theorem holds for a intuitionistic sequent system IS and its classical corespondent S means that for any sequent $\Gamma \Rightarrow \alpha$, $\vdash_{\mathrm{S}} \Gamma \Rightarrow \alpha$ iff $\vdash_{\mathrm{S}} \Gamma, \neg\alpha \Rightarrow \bot$.

It is known that $\nvdash_{\mathrm{MIPC}} \neg\neg\Box(p \vee \neg p)$. So $\nvdash_{\mathrm{IK.t}} \neg\neg\Box(p \vee \neg p)$. Thus $\nvdash_{\mathrm{GIK.t}} \top \Rightarrow \neg\neg\Box(p \vee \neg p)$. So $\nvdash_{\mathrm{GIK.t}} \neg\neg\top \Rightarrow \neg\neg\Box(p \vee \neg p)$. However $\vdash_{\mathrm{GK.t}} \top \Rightarrow \Box(p \vee \neg p)$. Hence the Glivenko translation is not holds for $\mathrm{GIK.t}$ and $\mathrm{GK.t}$. Let us consider an extension of GIK.t enriching with the following two axioms

(g1) $\neg\Box\alpha \Rightarrow \neg\neg\Diamond\neg\alpha$

(g2) $\neg\blacksquare\alpha \Rightarrow \neg\neg\blacklozenge\neg\alpha$.

The resulting system is denoted by $\mathrm{GGIK.t}$. Define the classical correspondence of $\mathrm{GGK.t}$ as $\mathrm{GIK.t}$. Let $\mathrm{G} \in \mathrm{EXT}(\mathrm{GGK.t})$ and $\mathrm{IG} \in \mathrm{EXT}(\mathrm{GGIK.t})$.

Theorem

$\vdash_{\mathrm{G}} \Gamma \Rightarrow \beta$ *iff* $\vdash_{\mathrm{IG}} g(\Gamma) \Rightarrow g(\beta)$.

Corollary

$\vdash_{\mathrm{G}} \Gamma \Rightarrow \beta$ *iff* $\vdash_{\mathrm{IG}} \Gamma, \neg\beta \Rightarrow \bot$.

Lemma

For each $\mathrm{G} \in \mathrm{EXT}(\mathrm{GK.t})$*, the Glivenko theorem holds for G relative to its intuitionistic correspondent IG, if and only if IG* $\mathrm{IG} \in \mathrm{EXT}(\mathrm{GGIK.t})$

Proof.

The if part follows from Lemma above. Conversely since $\vdash_{\mathrm{GK.t}} \neg\Box\alpha \Rightarrow \Diamond\neg\alpha$, $\vdash_{\mathrm{G}} \neg\Box\alpha \Rightarrow \Diamond\neg\alpha$ for any formula α. Then by G theorem one gets $\vdash_{\mathrm{IG}} \neg\neg\neg\Box\alpha \Rightarrow \neg\neg\Diamond\neg\alpha$. Since $\vdash_{\mathrm{G}} \neg\Box\alpha \Rightarrow \neg\neg\neg\Box\alpha$. Thus by (Cut) $\vdash_{\mathrm{IG}} \neg\Box\alpha \Rightarrow \neg\neg\Diamond\neg\alpha$. Similarly, one gets $\vdash_{\mathrm{IG}} \neg\blacksquare\alpha \Rightarrow \neg\neg\blacklozenge\neg\alpha$. Therefore IG is an extension of $\mathrm{GGIK.t}$.

□

Definition

The Gentzen sequent calculus GK.t for tense logic K.t is obtained from replacing $(\mathrm{K}_{\circ\bullet})$ and $(\mathrm{K}_{\bullet\circ})$ by the following

$$\frac{\circ\Delta_1, \Delta_2 \Rightarrow \bot}{\Gamma[\Delta_1, \bullet\Delta_2] \Rightarrow \beta}(\mathrm{Dual}_{\circ\bullet}) \quad \frac{\bullet\Delta_1, \Delta_2 \Rightarrow \bot}{\Gamma[\Delta_1, \circ\Delta_2] \Rightarrow \beta}(\mathrm{Dual}_{\bullet\circ})$$

and adding the following

$$\frac{\Gamma[\alpha\{\neg\neg\gamma\}] \Rightarrow \beta}{\Gamma[\alpha] \Rightarrow \beta}(\neg\neg L) \quad \frac{\Gamma[\alpha] \Rightarrow \beta\{\neg\neg\gamma\}}{\Gamma[\alpha] \Rightarrow \beta\{\gamma\}}(\neg\neg R)$$

Theorem

If $\vdash_{\mathrm{GSGK.t}} \Gamma \Rightarrow \beta$*, then* $\vdash_{\mathrm{GSGK.t}} \Gamma \Rightarrow \beta$ *without any application of (Cut).*

Theorem

If $\vdash_{\mathrm{GSGK.t}} \Gamma \Rightarrow \beta$, then $\vdash_{\mathrm{GSGK.t}} \Gamma \Rightarrow \beta$ without any application of (Cut).

- $(\mathrm{Dual}_{\circ\bullet})$ and $(\mathrm{Dual}_{\bullet\circ})$ are both derivable in GSGIK.t.
- If $\vdash_{\mathrm{GSGK.t}} ko(\Gamma) \Rightarrow ko(\beta)$, then there is a derivation of $\Gamma \Rightarrow \beta$ without any application of $(\neg\neg\ \mathrm{L})$ and $(\neg\neg\ \mathrm{R})$.
- If $\vdash_{\mathrm{GSGK.t}} ko(\Gamma) \Rightarrow ko(\beta)$, then there is a cut free derivation of If $ko(\Gamma) \Rightarrow ko(\beta)$ in GSGK.t.
- $\vdash_{\mathrm{GSGK.t}} ko(\Gamma) \Rightarrow ko(\beta)$ implies $\vdash_{\mathrm{GSGK.t}} \Gamma \Rightarrow \beta$ by rule $(\neg\neg\ \mathrm{L})$ and $(\neg\neg\ \mathrm{R})$.

Definition

We define the following calculation:

$$(\alpha)^{\neg} = \neg\alpha \quad \textit{if} \quad \alpha \neq \neg\alpha^{'}; \quad (\alpha)^{\neg} = \alpha^{'} \quad \textit{if} \quad \alpha = \neg\alpha^{'}$$

In addition, we construct another sequent calculus GSGK$'$.t without (Cut) based on GSGK.t by replacing $(\rightarrow \mathrm{L})$ and $(\rightarrow \mathrm{R})$ with the following rules

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma, (\alpha)^{\neg} \Rightarrow \bot}(\neg\mathrm{L}^{'}) \quad \frac{\alpha, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow (\alpha)^{\neg}}(\neg\mathrm{R}^{'})$$

and replacing $(\neg\neg\mathrm{L})$ and $(\neg\neg\mathrm{R})$ with the following two rules respectively

$$\frac{\Gamma[\alpha\{\theta\}] \Rightarrow \beta}{\Gamma[\alpha\{\neg\neg\theta\}] \Rightarrow \beta}(\neg\neg\mathrm{L}') \quad \frac{\Gamma \Rightarrow \alpha\{\theta\}}{\Gamma \Rightarrow \alpha\{\neg\neg\theta\}}(\neg\neg\mathrm{R}')$$

Definition

We define $(\alpha)^{\neg\neg}$ be a formula obtained from α by replacing all subformula with the form of $\neg\neg\theta$ by θ in α.

Lemma

If $\vdash_{\mathsf{GSGK.t}} \Gamma \Rightarrow \alpha$*, then* $\vdash_{\mathsf{GSGK'.t}} (\Gamma)^{\neg\neg} \Rightarrow (\alpha)^{\neg\neg}$.

Theorem

$\vdash_{\mathsf{GSGK.t}} \Gamma \Rightarrow \alpha$ *iff* $\vdash_{\mathsf{GSGK'.t}} \Gamma \Rightarrow \alpha$.

Theorem

(subformula property) Let T *be the set of all subformulas of formulas in* $\Gamma \Rightarrow \beta$*. Define* $\neg T = \{\neg\alpha\}$ *and* $T^* = T \cup \neg T$*. If* $\vdash_{\mathsf{GSGK'.t}} \Gamma \Rightarrow \beta$*, then there is a derivation of* $\Gamma \Rightarrow \beta$ *such that all formulas appearing in the derivations belong to* T^**.*

Condition	Axioms	Axioms(Diamonds)	Structural Rules
Reflexive	$\Box\alpha \Rightarrow \alpha; \blacksquare\alpha \Rightarrow \alpha$	$\alpha \Rightarrow \Diamond\alpha; \alpha \Rightarrow \blacklozenge\alpha$	$\frac{\Gamma[\star\Delta]\Rightarrow\beta}{\Gamma[\Delta]\Rightarrow\beta} \star \in \{\circ, \bullet\}$
Pathetic	$\alpha \Rightarrow \Box\alpha; \alpha \Rightarrow \blacksquare\alpha$	$\Diamond\alpha \Rightarrow \alpha; \blacklozenge\alpha \Rightarrow \alpha$	$\frac{\Gamma[\Delta]\Rightarrow\beta}{\Gamma[\star\Delta]\Rightarrow\beta} \star \in \{\circ, \bullet\}$
Functional injective	$\Diamond\alpha \Rightarrow \Box\alpha; \blacklozenge\alpha \Rightarrow \blacksquare\alpha$	$\blacklozenge\Diamond\alpha \Rightarrow \alpha; \Diamond\blacklozenge\alpha \Rightarrow \alpha$	$\frac{\Gamma[\Delta]\Rightarrow\beta}{\Gamma[\circ\bullet\Delta]\Rightarrow\beta}; \frac{\Gamma[\Delta]\Rightarrow\beta}{\Gamma[\bullet\circ\Delta]\Rightarrow\beta}$
Surjective	$\Box\alpha \Rightarrow \Diamond\alpha; \blacksquare\alpha \Rightarrow \blacklozenge\alpha$	$\alpha \Rightarrow \blacklozenge\Diamond\alpha; \alpha \Rightarrow \Diamond\blacklozenge\alpha$	$\frac{\Gamma[\circ\bullet\Delta]\Rightarrow\beta}{\Gamma[\Delta]\Rightarrow\beta}; \frac{\Gamma[\bullet\circ\Delta]\Rightarrow\beta}{\Gamma[\Delta]\Rightarrow\beta}$
Symmetric	$\alpha \Rightarrow \Box\Diamond\alpha; \Diamond\Box\alpha \Rightarrow \alpha$	$\Diamond\alpha \Rightarrow \blacklozenge\alpha; \blacklozenge\alpha \Rightarrow \Diamond\alpha$	$\frac{\Gamma[\circ\Delta]\Rightarrow\beta}{\Gamma[\bullet\Delta]\Rightarrow\beta} \frac{\Gamma[\circ\Delta]\Rightarrow\beta}{\Gamma[\bullet\Delta]\Rightarrow\beta}$
Transitive	$\Box\alpha \Rightarrow \Box\Box\alpha; \blacksquare\alpha \Rightarrow \blacksquare\blacksquare\alpha$	$\Diamond\alpha \Rightarrow \Diamond\Diamond\alpha; \blacklozenge\alpha \Rightarrow \blacklozenge\blacklozenge\alpha$	$\frac{\Gamma[\circ\Delta]\Rightarrow\beta}{\Gamma[\circ\circ\Delta]\Rightarrow\beta}; \frac{\Gamma[\bullet\Delta]\Rightarrow\beta}{\Gamma[\bullet\bullet\Delta]\Rightarrow\beta}$
Dense	$\Box\Box\alpha \Rightarrow \Box\alpha; \blacksquare\blacksquare\alpha \Rightarrow \blacksquare\alpha$	$\Diamond\alpha \Rightarrow \Diamond\Diamond\alpha; \blacklozenge\alpha \Rightarrow \blacklozenge\blacklozenge\alpha$	$\frac{\Gamma[\circ\circ\Delta]\Rightarrow\beta}{\Gamma[\circ\Delta]\Rightarrow\beta}; \frac{\Gamma[\bullet\bullet\Delta]\Rightarrow\beta}{\Gamma[\bullet\Delta]\Rightarrow\beta}$
Euclidean	$\Diamond\Box\alpha \Rightarrow \Box\alpha; \blacklozenge\blacksquare\alpha \Rightarrow \blacksquare\alpha$	$\blacklozenge\Diamond\alpha \Rightarrow \blacklozenge\alpha; \Diamond\blacklozenge\alpha \Rightarrow \Diamond\alpha$	$\frac{\Gamma[\bullet\Delta]\Rightarrow\beta}{\Gamma[\circ\bullet\Delta]\Rightarrow\beta}; \frac{\Gamma[\circ\Delta]\Rightarrow\beta}{\Gamma[\bullet\circ\Delta]\Rightarrow\beta}$
Confluent	$\Diamond\Box\alpha \Rightarrow \Box\Diamond\alpha$	$\blacklozenge\Diamond\alpha \Rightarrow \Diamond\blacklozenge\alpha$	$\frac{\Gamma[\circ\bullet\Delta]\Rightarrow\beta}{\Gamma[\bullet\circ\Delta]\Rightarrow\beta}$
Divergent	$\blacklozenge\blacksquare\alpha \Rightarrow \blacksquare\blacklozenge\alpha$	$\Diamond\blacklozenge\alpha \Rightarrow \blacklozenge\Diamond\alpha$	$\frac{\Gamma[\bullet\circ\Delta]\Rightarrow\beta}{\Gamma[\circ\bullet\Delta]\Rightarrow\beta}$

Definition

The Hilbert-style axiomatic system **wIK.t** for intuitionistic tense logic consists of the following axiom schemata and rules:

(1) Axiom Schemata:

(Int) All axioms of intuitionistic propositional calculus.

$(\mathrm{Dual}_{\Diamond\Box})$ $\Box(\neg\alpha) \to (\neg\Diamond\alpha)$.

$(\mathrm{Dual}_{\blacklozenge\blacksquare})$ $\blacksquare(\neg\alpha) \to (\neg\blacklozenge\alpha)$.

(2) Inference rules:

$$\frac{\alpha \to \beta \quad \alpha}{\beta}(\mathrm{MP}) \quad \frac{\Diamond\alpha \to \beta}{\alpha \to \blacksquare\beta}(\mathrm{Adj}_1) \quad \frac{\blacklozenge\alpha \to \beta}{\alpha \to \Box\beta}(\mathrm{Adj}_2)$$

The Gentzen sequent calculus GwIK.t for tense logic wIK.t is obtained from replacing ($\mathrm{K}_{\circ\bullet}$) and ($\mathrm{K}_{\bullet\circ}$) by

$$\frac{\circ\Delta_1, \Delta_2 \Rightarrow \bot}{\Gamma[\Delta_1, \bullet\Delta_2] \Rightarrow \beta}(\mathrm{Dual}_{\circ\bullet}) \quad \frac{\bullet\Delta_1, \Delta_2 \Rightarrow \bot}{\Gamma[\Delta_1, \circ\Delta_2] \Rightarrow \beta}(\mathrm{Dual}_{\bullet\circ})$$

Definition

Let T be a set of formulas containing $\bot, \top$. Define $b(T)$ be the $\wedge, \vee, \neg$ closure of T and $c(T)$ be the $\wedge, \vee$ closure of T respectively. We define a translation from $b(T)$ to $b(T)$ as follows. Let $n2c(T) = \{\neg\neg\alpha | \alpha \in c(k_T(T))\}$ and $\overline{T} = c(k_T(b(T)) \cup T)$.

$$k_T(\alpha) = \neg\neg\alpha \text{ where } \alpha \in T,$$

$$k_T(\neg\alpha) = \neg k_T(\alpha)$$

$$k_T(\alpha_1 \wedge \alpha_2) = \neg\neg(k_T(\alpha_1) \wedge k_T(\alpha_2))$$

$$k_T(\alpha_1 \vee \alpha_2) = \neg\neg(k_T(\alpha_1) \vee k_T(\alpha_2))$$

Lemma

For any $\alpha \in k_T(b(T))$, there is a $\beta \in n2c(T)$ such that $\vdash_{\mathrm{GwIK.t}} \alpha \Leftrightarrow \beta$.

Corollary

Let T be a finite set of formulas. $\overline{T}$ is finite up to GwIK.t-equivalent.

Lemma

If $\vdash_{\mathsf{GwIK.t}} \Gamma[\Delta] \Rightarrow \beta$, then there is a $\gamma \in \overline{T}$ such that $\vdash_{\mathsf{GwIK.t}} \Delta \Rightarrow \gamma$ and $\vdash_{\mathsf{GwIK.t}} \Gamma[\gamma] \Rightarrow \beta$ where T be the set of all subformulas of formulas in $\Gamma[\Delta] \Rightarrow \beta$.

Let $\vdash_{\mathsf{GwIK.t}} \Gamma \Rightarrow \beta$, T be the set of all subformulas of formulas in $\Gamma \Rightarrow \beta$.

- there is a derivation of $\Gamma \Rightarrow \beta$ in GwIK.t such that all formulas appearing in the derivations belongs to $\overline{T}$
- there is a derivation of $\Gamma \Rightarrow \beta$ in GwIK.t such that all rules are restricted to at most containing three formulas.
- One can construct a CFG which derives same sequents with GwIK.t
- By CYK-algorithm, GwIK.t is decidable.

Theorem

GwIK.t is decidable.

Theorem

If GSGwIK.t satisfies the interpolant property, then GSGwIK.t is decidable.

Axioms	Structural Rules	Axioms	Structural Rules
$\top \Rightarrow \Diamond\top$	$\frac{\Gamma[\circ\top]\Rightarrow\alpha}{\Gamma[\top]\Rightarrow\alpha}$	$\top \Rightarrow \blacklozenge\top$	$\frac{\Gamma[\bullet\top]\Rightarrow\alpha}{\Gamma[\top]\Rightarrow\alpha}$
$\Box\alpha \Rightarrow \alpha$	$\frac{\Gamma[\bullet(\Delta)]\Rightarrow\psi}{\Gamma[\Delta]\Rightarrow\psi}$	$\blacksquare\alpha \Rightarrow \alpha$	$\frac{\Gamma[\circ(\Delta)]\Rightarrow\psi}{\Gamma[\Delta]\Rightarrow\psi}$
$\alpha \Rightarrow \Box\Diamond\alpha$	$\frac{\Gamma[\circ(\Delta)]\Rightarrow\psi}{\Gamma[\bullet(\Delta)]\Rightarrow\psi}$	$\alpha \Rightarrow \blacksquare\blacklozenge\alpha$	$\frac{\Gamma[\bullet(\Delta)]\Rightarrow\psi}{\Gamma[\circ(\Delta)]\Rightarrow\psi}$
$\Box\Box\alpha \Rightarrow \Box\alpha$	$\frac{\Gamma[\bullet\bullet(\Delta)]\Rightarrow\psi}{\Gamma[\bullet(\Delta)]\Rightarrow\psi}$	$\blacksquare\blacksquare\alpha \Rightarrow \blacksquare\alpha$	$\frac{\Gamma[\circ\circ(\Delta)]\Rightarrow\psi}{\Gamma[\circ(\Delta)]\Rightarrow\psi}$
$\Diamond\alpha \Rightarrow \Box\Diamond\alpha$	$\frac{\Gamma[\circ(\Delta)]\Rightarrow\psi}{\Gamma[\bullet\circ(\Delta)]\Rightarrow\psi}$	$\blacklozenge\alpha \Rightarrow \blacksquare\blacklozenge\alpha$	$\frac{\Gamma[\bullet(\Delta)]\Rightarrow\psi}{\Gamma[\circ\bullet(\Delta)]\Rightarrow\psi}$
$\Box\alpha \to \Box\beta \Rightarrow \Box(\alpha \to \beta)$	$\frac{\Gamma[\circ(\Delta_1,\Delta_2)]\Rightarrow\psi}{\Gamma[\circ\Delta_1,\circ\Delta_2]\Rightarrow\psi}$	$\blacksquare\alpha \to \blacksquare\beta \Rightarrow \blacksquare(\alpha \to \beta)$	$\frac{\Gamma[\bullet(\Delta_1,\Delta_2)]\Rightarrow\psi}{\Gamma[\bullet\Delta_1,\bullet\Delta_2]\Rightarrow\psi}$
$\Diamond\alpha \Rightarrow \Box\alpha$	$\frac{\Gamma[\Delta]\Rightarrow\psi}{\Gamma[\bullet\circ\Delta]\Rightarrow\psi}$	$\blacklozenge\alpha \Rightarrow \blacksquare\alpha$	$\frac{\Gamma[\bullet\Delta]\Rightarrow\psi}{\Gamma[\circ\bullet\Delta]\Rightarrow\psi}$
$\Box\alpha \Rightarrow \Box\Box\alpha$	$\frac{\Gamma[\bullet(\Delta)]\Rightarrow\psi}{\Gamma[\bullet\bullet(\Delta)]\Rightarrow\psi}$	$\blacksquare\alpha \Rightarrow \blacksquare\blacksquare\alpha$	$\frac{\Gamma[\circ(\Delta)]\Rightarrow\psi}{\Gamma[\circ\circ(\Delta)]\Rightarrow\psi}$

Table: Structural Axioms and Rules

Lemma

If $\vdash_{GK'.t} \Gamma[\Delta] \Rightarrow \beta$, then there is a $\gamma \in T$ such that $\vdash_{GK'.t} \Delta \Rightarrow \gamma$ and $\vdash_{GK'.t} \Gamma[\gamma] \Rightarrow \beta$ where T be the set of all subformulas of formulas in $\Gamma[\Delta] \Rightarrow \beta$.

Let $\vdash_{GK'.t} \Gamma \Rightarrow \beta$, T be the set of all subformulas of formulas in $\Gamma \Rightarrow \beta$.

- there is a derivation of $\Gamma \Rightarrow \beta$ in GK'.t such that all formulas appearing in the derivations belongs to T
- there is a derivation of $\Gamma \Rightarrow \beta$ in GK'.t such that all rules are restricted to at most containing three formulas.
- One can construct a CFG which derives same sequents with GK'.t
- By CYK-algorithm, GK'.t is decidable.

Theorem

GK'.t is decidable.

Theorem

If GSGK′.t satisfies the interpolant property, then GSGK′.t is decidable.

Condition	Axioms	Axioms(Diamonds)	Structural Rules
Reflexive	$\Box\alpha \Rightarrow \alpha; \blacksquare\alpha \Rightarrow \alpha$	$\alpha \Rightarrow \Diamond\alpha; \alpha \Rightarrow \blacklozenge\alpha$	$\frac{\Gamma[\star\Delta]\Rightarrow\beta}{\Gamma[\Delta]\Rightarrow\beta} \star \in \{\circ, \bullet\}$
Pathetic	$\alpha \Rightarrow \Box\alpha; \alpha \Rightarrow \blacksquare\alpha$	$\Diamond\alpha \Rightarrow \alpha; \blacklozenge\alpha \Rightarrow \alpha$	$\frac{\Gamma[\Delta]\Rightarrow\beta}{\Gamma[\star\Delta]\Rightarrow\beta} \star \in \{\circ, \bullet\}$
Functional injective	$\Diamond\alpha \Rightarrow \Box\alpha; \blacklozenge\alpha \Rightarrow \blacksquare\alpha$	$\blacklozenge\Diamond\alpha \Rightarrow \alpha; \Diamond\blacklozenge\alpha \Rightarrow \alpha$	$\frac{\Gamma[\Delta]\Rightarrow\beta}{\Gamma[\circ\bullet\Delta]\Rightarrow\beta}; \frac{\Gamma[\Delta]\Rightarrow\beta}{\Gamma[\bullet\circ\Delta]\Rightarrow\beta}$
Surjective	$\Box\alpha \Rightarrow \Diamond\alpha; \blacksquare\alpha \Rightarrow \blacklozenge\alpha$	$\alpha \Rightarrow \blacklozenge\Diamond\alpha; \alpha \Rightarrow \Diamond\blacklozenge\alpha$	$\frac{\Gamma[\circ\bullet\Delta]\Rightarrow\beta}{\Gamma[\Delta]\Rightarrow\beta}; \frac{\Gamma[\bullet\circ\Delta]\Rightarrow\beta}{\Gamma[\Delta]\Rightarrow\beta}$
Symmetric	$\alpha \Rightarrow \Box\Diamond\alpha; \Diamond\Box\alpha \Rightarrow \alpha$	$\Diamond\alpha \Rightarrow \blacklozenge\alpha; \blacklozenge\alpha \Rightarrow \Diamond\alpha$	$\frac{\Gamma[\circ\Delta]\Rightarrow\beta}{\Gamma[\bullet\Delta]\Rightarrow\beta}\ \frac{\Gamma[\circ\Delta]\Rightarrow\beta}{\Gamma[\bullet\Delta]\Rightarrow\beta}$
Transitive	$\Box\alpha \Rightarrow \Box\Box\alpha; \blacksquare\alpha \Rightarrow \blacksquare\blacksquare\alpha$	$\Diamond\alpha \Rightarrow \Diamond\Diamond\alpha; \blacklozenge\alpha \Rightarrow \blacklozenge\blacklozenge\alpha$	$\frac{\Gamma[\circ\Delta]\Rightarrow\beta}{\Gamma[\circ\circ\Delta]\Rightarrow\beta}; \frac{\Gamma[\bullet\Delta]\Rightarrow\beta}{\Gamma[\bullet\bullet\Delta]\Rightarrow\beta}$
Dense	$\Box\Box\alpha \Rightarrow \Box\alpha; \blacksquare\blacksquare\alpha \Rightarrow \blacksquare\alpha$	$\Diamond\alpha \Rightarrow \Diamond\Diamond\alpha; \blacklozenge\alpha \Rightarrow \blacklozenge\blacklozenge\alpha$	$\frac{\Gamma[\circ\circ\Delta]\Rightarrow\beta}{\Gamma[\circ\Delta]\Rightarrow\beta}; \frac{\Gamma[\bullet\bullet\Delta]\Rightarrow\beta}{\Gamma[\bullet\Delta]\Rightarrow\beta}$
Euclidean	$\Diamond\Box\alpha \Rightarrow \Box\alpha; \blacklozenge\blacksquare\alpha \Rightarrow \blacksquare\alpha$	$\blacklozenge\Diamond\alpha \Rightarrow \blacklozenge\alpha; \Diamond\blacklozenge\alpha \Rightarrow \Diamond\alpha$	$\frac{\Gamma[\bullet\Delta]\Rightarrow\beta}{\Gamma[\circ\bullet\Delta]\Rightarrow\beta}; \frac{\Gamma[\circ\Delta]\Rightarrow\beta}{\Gamma[\bullet\circ\Delta]\Rightarrow\beta}$
Confluent	$\Diamond\Box\alpha \Rightarrow \Box\Diamond\alpha$	$\blacklozenge\Diamond\alpha \Rightarrow \Diamond\blacklozenge\alpha$	$\frac{\Gamma[\circ\bullet\Delta]\Rightarrow\beta}{\Gamma[\bullet\circ\Delta]\Rightarrow\beta}$
Divergent	$\blacklozenge\blacksquare\alpha \Rightarrow \blacksquare\blacklozenge\alpha$	$\Diamond\blacklozenge\alpha \Rightarrow \blacklozenge\Diamond\alpha$	$\frac{\Gamma[\bullet\circ\Delta]\Rightarrow\beta}{\Gamma[\circ\bullet\Delta]\Rightarrow\beta}$

Let$G_1, \ldots, G_n$ be any GSGIK.t, GSGwIK.t, GSGK.t discussed above. The finite fusions of $G_1, \ldots, G_n$, denoted by $G = G_1 \otimes \ldots \otimes G_n$ has the following properties.

Theorem

G *admits cut elimination.*

Theorem

G *is decidable.*

- Burgess, J.P.: Basic Tense Logic, pp. 89 – 133. Springer Netherlands, Dordrecht (1984)
- Ewald, W.: Intuitionistic tense and modal logic. Jounal of Symbolic Logic 51(1), 166 – 179 (1986)
- Figallo, A., Pelaitay, G.: An algebraic axiomatization of the ewald' s intuitionistic tense logic. Soft Comput. 18(10), 1873 – 1883 (2014)
- Fitting, M.: Nested sequents for intuitionistic logics. Notre Dame Journal of Formal Logic 1(1), 41 – 61 (2014)
- Alex K. Simpson: The proof theory and semantics of intuitionistic modal logic. Ph.D. thesis, University of Edinburgh, UK, 1994.
- Frank Wolter, Michael Zakharyaschev (1999): Intuitionistic modal logic. Logic and Foundations of Mathematics, pp. 227 – 238.
- Jouni Javinen, Wojciech Dzik, Michiro Kondo (2010): Intuiitionistic propositional lgoic with Galois connections. Logic Journal of IGPL 6(18), pp. 837 – 858.
- Joseph Boudou, Mart in Dieguez, David Fernandez-Duque, Fabian Romero (2019): Axiomatic systems and topological semantics for intuitionistic temporal logic. In European Conference on Logics in Artificial Intelligence, pp. 763 – 777.
- Wojciech Buszkowski (2011): Interpolation and FEP for logics of residuated algebras. Logic Journal of IGPL 3(19), pp. 437 – 454.

- Nicola Leone, Gerald Pfeifer, Wolfgang Faber, Thomas Eiter, Georg Gottlob, Simona Perri, Francesco Scarcello (2006): The DLV system for knowledge representation and reasoning. ACM Transactions on Computational Logic (TOCL) 3(7), pp. 499 – 562.
- Zhe Lin, Minghui Ma (2019): Gentzen sequent calculi for some intuitionistic modal logics. Logic Journal of IGPL 4(27), pp. 596 – 623.
- T.Kowalski H.Ono N.Galatos, P.Jipsen (2007): Residuated Lattices: An Algebraic Glimpse at Substructural Logics.
- David Pearce (1996): A new logical characterisation of stable models and answer sets. Springer, Berlin, Heidelberg, pp. 57 – 70.
- Gisele Fischer Servi (1984): Axiomatizations for some intuitionistic modal logics. Rendiconti Del Seminario Matematico 42(3), pp. 178 – 184.
- Patrick Maier. Intuitionistic LTL and a new characterization of safety and liveness. In 18th EACSL Annual Conference on Computer Science Logic (CSL), pages 295 – 309. Springer Berlin Heidelberg, 2004.
- Bull R. A. MIPC as the formalisation of an intuitionist concept of modality. Journal of Symbolic Logic, 31, 609-616,1966.
- Bierman G. and de Paiva V. On an intuitionistic modal logic. Studia Logica 65. 383-416, 2000.
- Ono H. On some intuitionistic modal logics. Publications of Research Institute for Mathematical Science, 13, 687-722. 1977.
- Roman Kuznets, Lutz Stra?burger. Maehara-style modal nested calculi. Archive for Mathematical Logic, Springer Verlag, 2019, 58 (3-4),

- Rosalie Iemhoff, Terminating sequent calculi for two intuitionistic modal logics, Journal of Logic and Computation, Volume 28, Issue 7, October 2018, Pages 1701 – 1712.
- Kikot, Stanislav, Kurucz, Agi, Tanaka, Yoshihito, Wolter, Frank and Zakharyaschev, Michael (2019) KRIPKE COMPLETENESS OF STRICTLY POSITIVE MODAL LOGICS OVER MEET-SEMILATTICES WITH OPERATORS. JOURNAL OF SYMBOLIC LOGIC, 84 (2). 533 - 588.
- Moortgat. M. Multimodal linguistic inference. Journal of Logic, Language and Information, 5, 349-385. 1996.
- Hiroakira Ono, Glivenko theorems revisited, Annals of Pure and Applied Logic 161 (2009) 246 – 250
- Morten Heine S?rensen and Pawel Urzyczyn. Lectures on the Curry-Howard Isomorphism, volume 149 of Stud. Logic Found. Math. Elsevier Science Inc., 2006.
- Milan Bozic and Kosta Dosen. Models for normal intuitionistic modal logics. Stud. Logica, 43(3):217 – 245, 1984.
- Guram Bezhanishvili,Glivenko Type Theorems for Intuitionistic Modal Logics,Studia Logica volume 67, pages,89 – 109,(2001).
- Tadeusz Litak, Miriam Polzer, Ulrich Rabenste, Negative Translations and Normal Modality, FSCD 2017.

Thank you