HYPERBOLIC SYSTEMS OF BALANCE LAWS

WITH STIFF SOURCE

CAUCHY PROBLEM FOR SYSTEMS OF CONSERVATION LAWS

$$\begin{cases} \partial_t U + \partial_x F(U) = 0 & -\infty < x < \infty,\ t > 0 \\ U(x,0) = U_0(x) & -\infty < x < \infty \end{cases}$$

$DF(U)$: eigenvalues $\lambda_1(U) < \cdots < \lambda_n(U)$

eigenvectors $R_1(U), \ldots, R_n(U)$

$TV\, U_0(\cdot) = \delta \ll 1 \implies \exists$ BV solution $U(x,t)$

$$TV\, U(\cdot,t) \leq c\delta, \qquad 0 \leq t < \infty$$

GLIMM, LIU, BRESSAN + coauthors

CAUCHY PROBLEM FOR SYSTEMS OF BALANCE LAWS

$$\begin{cases} \partial_t U + \partial_x F(U) + G(U) = 0 & -\infty < x < \infty, \; t > 0 \\ U(x,0) = U_0(x) & -\infty < x < \infty \end{cases}$$

$\mathrm{TV}\, U_0(\cdot) = \delta \ll 1 \Rightarrow \exists$ BV solution $U(x,t)$ on $[0,T)$

$$\mathrm{TV}\, U(\cdot,t) \le c\delta e^{\rho t}, \quad 0 \le t < T$$

solution may be extended for as long as

$$\mathrm{TV}\, U(\cdot,t) \ll 1$$

STRONG DISSIPATION

$$\begin{cases} \partial_t U + \partial_x F(U) + G(U) = 0 & -\infty < x < \infty,\ t > 0 \\ U(x,0) = U_0(x) & -\infty < x < \infty \end{cases}$$

$$R(U) = [\, R_1(U) \cdots R_n(U)]$$

$$A = R(0)^{-1} DG(0)\, R(0)$$

- A strictly diagonal dominant
- $TV\, U_0(\cdot) = \delta \ll 1$

$\Rightarrow \exists$ BV solution $U(x,t)$ on $[0,\infty)$

$$TV\, U(\cdot,t) \le c\delta e^{-\nu t}, \quad 0 \le t < \infty$$

PARTIAL DISSIPATION

$$\begin{cases} \partial_t U + \partial_x F(U) + G(U) = 0 & -\infty < x < \infty, \ t > 0 \\ U(x,0) = U_0(x) & -\infty < x < \infty \end{cases}$$

- There exists convex entropy $\eta(U)$
- Dissipative source: $D\eta(U)\, G(U) \geq |G(U)|^2$
- $A_{ii} > 0, \quad i = 1, \dots, n$
- $TV\, U_0(\cdot) = \delta \ll 1$
- $\int_{-\infty}^{\infty} (1+x^2)\, |U_0(x)|^2 dx = \varepsilon^2 \ll 1$

$\Rightarrow \exists$ BV solution $U(x,t)$ on $[0,\infty)$

$$TV\, U(\cdot,t) \leq c_1 \delta e^{-\nu t} + c_2 \varepsilon$$

$$TV\, U(\cdot,t) \longrightarrow 0, \quad \text{as } t \to \infty$$

STIFF SOURCE – ZERO RELAXATION LIMIT

$$\begin{cases} \partial_t U + \partial_x F(U) + \frac{1}{\mu} G(U) = 0, & -\infty < x < \infty,\ t > 0 \\ U(x,0) = U_0(x) & -\infty < x < \infty \end{cases}$$

TASKS:

- For fixed U_0, existence of solution $U_\mu(x,t)$ for all $\mu > 0$.
- $TV\, U_\mu(\cdot,t)$ bounded small for $t \geq 0$ uniformly in $\mu > 0$
- Limits of $\{U_\mu\}$ as $\mu \to 0$ – uniqueness??

CHEN – LEVERMORE – LIU

MODEL SYSTEM

$$\begin{cases} \partial_t U - \partial_x V = 0 \\ \partial_t V - \partial_x S(U) + \frac{1}{\mu} V = 0 \end{cases}$$

$S(U) = D\varphi(U)$, $\varphi(U)$ convex

entropy: $\varphi(U) + \frac{1}{2}|V|^2$

$n = 1$:

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v + \partial_x p(u) + \frac{1}{\mu} v = 0 \end{cases}$$

SPECIAL SYSTEM

$$
\begin{cases}
\partial_t u - \partial_x v = 0 \\[2ex]
\partial_t v + \partial_x \left(\frac{1}{u}\right) + \frac{1}{\mu} v = 0
\end{cases}
$$

$$u(x,0) = u_0(x) \quad , \quad v(x,0) = v_0(x)$$

- $TV\, u_0(\cdot) + TV\, v_0(\cdot) = \delta$

$\Rightarrow \exists$ BV solution $(u_\mu(x,t);\, v_\mu(x,t))$ on $[0,\infty)$

$$TV\, u_\mu(\cdot,t) + TV\, v_\mu(\cdot,t) \leq c\delta \quad , \quad 0 \leq t < \infty$$

$$\int_{-\infty}^{\infty} | u_\mu(x,\sigma) - u_\mu(x,\tau) |\, dx \leq c_1 \delta (\sigma - \tau) \quad , \quad 0 \leq \tau < \sigma < \infty$$

$$\int_{-\infty}^{\infty} | v_\mu(x,\sigma) - v_\mu(x,\tau) |\, dx \leq [c_1 \delta + c_2 \tfrac{1}{\mu} e^{-\frac{\lambda t}{\mu}}] (\sigma - \tau)$$

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MAIN RESULT

$$\begin{cases} \partial_t U - \partial_x V = 0 \\ \partial_t V - \partial_x S(U) + \frac{1}{\mu} V = 0 \end{cases} \qquad -\infty < x < \infty, \ t > 0$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \qquad -\infty < x < \infty$$

$$\int_{-\infty}^{\infty} \{ |U_0(x)|^2 + |V_0(x)|^2 + |U_0'(x)|^2 + |V_0'(x)|^2 \} dx = \varepsilon^2 << 1$$

$\Rightarrow$ There are $\varepsilon_0, c_0, c_1, \lambda$ such that, for any $\mu > 0$ and $\varepsilon < \varepsilon_0$ $\exists$ BV solution (U_μ, V_μ) with

$$TV\, U_\mu(\cdot, t) + TV\, V_\mu(\cdot, t) \leq c_0 \varepsilon$$

$$\int_{-\infty}^{\infty} |U_\mu(x,\sigma) - U_\mu(x,\tau)| \, dx \leq c_0 \varepsilon (\sigma - \tau)$$

$$\int_{-\infty}^{\infty} |V_\mu(x,\sigma) - V_\mu(x,\tau)| \, dx \leq c_1 \left[\varepsilon + \frac{1}{\mu} e^{-\lambda t/\mu} \|V_0\|_{L_1} \right] (\sigma - \tau)$$

REDISTRIBUTION OF DAMPING FAILS

$$\begin{cases} \partial_t U - \partial_x V = 0 \\ \partial_t V - \partial_x S(U) + \frac{1}{\mu} V = 0 \end{cases} \qquad -\infty < x < \infty, \ t > 0$$

$$U(x,0) = U_0(x), \ V(x,0) = V_0(x), \qquad -\infty < x < \infty$$

$$\Phi(x,t) = \frac{1}{\mu} \int_{-\infty}^{x} U(y,t)\, dy$$

$$\Phi_0(x) = \frac{1}{\mu} \int_{-\infty}^{x} U_0(y)\, dy$$

$$Y(x,t) = V(x,t) + \frac{1}{2} \Phi(x,t)$$

$$\begin{cases} \partial_t U - \partial_x Y + \frac{1}{2\mu} U = 0 \\ \partial_t Y - \partial_x S(U) + \frac{1}{2\mu} Y = \frac{1}{4\mu} \Phi \end{cases}$$

$$TV\, \Phi(\cdot, t) = \frac{1}{\mu} \int_{-\infty}^{\infty} |U(y,t)|\, dy$$

PARTITION OF INITIAL DATA

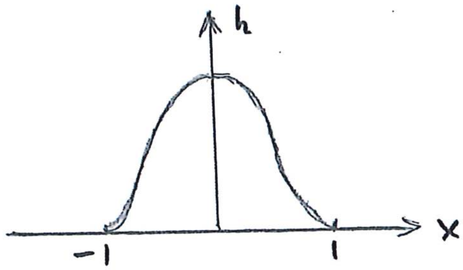
$$U_0(x) = P_0(x) + W_0(x), \quad V_0(x) = Q_0(x) + \Sigma_0(x), \quad \Phi_0(x) = \Psi_0(x) + \Omega_0(x)$$

$$P_0(x) = \frac{1}{\mu}\int_{-\infty}^{\infty} h\left(\frac{x-\xi}{\mu}\right) U_0(\xi)\, d\xi$$

$$Q_0(x) = \frac{1}{\mu}\int_{-\infty}^{\infty} h\left(\frac{x-\xi}{\mu}\right) V_0(\xi)\, d\xi$$

$$\Psi_0(x) = \frac{1}{\mu}\int_{-\infty}^{\infty} h\left(\frac{x-\xi}{\mu}\right) \Phi_0(\xi)\, d\xi = \frac{1}{\mu}\int_{-\infty}^{x} P_0(y)\, dy$$

$$h(-x) = h(x)$$



$$\int_{-1}^{1} h(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} (1+x^2)\left\{ |P_0(x)|^2 + |P_0'(x)|^2 + \mu^2 |P_0''(x)|^2 + \mu^4 |P_0'''(x)|^2 \right\} dx \le c\,\varepsilon^2$$

$$\int_{-\infty}^{\infty} (1+x^2)\left\{ |Q_0(x)|^2 + |Q_0'(x)|^2 + \mu^2 |Q_0''(x)|^2 + \mu^4 |Q_0'''(x)|^2 \right\} dx \le c\,\varepsilon^2$$

$$\int_{-\infty}^{\infty} (1+x^2)\left\{ |W_0(x)|^2 + |\Sigma_0(x)|^2 + |\Omega_0(x)|^2 \right\} dx \le c\,\mu^2 \varepsilon^2$$

PARTITION OF STATE VECTOR

$$U(x,t) = P(x,t) + W(x,t), \quad V(x,t) = R(x,t) + Z(x,t), \quad \Phi(x,t) = \Psi(x,t) + \Omega(x,t)$$

$$\begin{cases} \partial_t P = \mu \partial_x^2 S(P) + e^{-\frac{t}{\mu}} \partial_x Q_0 \\ P(x,0) = P_0(x) \end{cases}$$

$$R(x,t) = \mu \partial_x S(P(x,t)) + e^{-\frac{t}{\mu}} Q_0(x)$$

$$\Psi(x,t) = \frac{1}{\mu} \int_{-\infty}^{x} P(y,t)\, dy$$

$$\begin{cases} \partial_t W - \partial_x Z = 0 \\ \partial_t Z - \partial_x [S(P+W) - S(P)] + \frac{1}{\mu} Z = -\mu \partial_t \partial_x S(P) \end{cases}$$

$$W(x,0) = W_0(x), \quad Z(x,0) = \Sigma_0(x) - \mu \partial_x S(P_0(x))$$

ENERGY ESTIMATES FOR THE POROUS MEDIA EQUATION

$$\begin{cases} \partial_t P = \mu \partial_x^2 S(P) + e^{-\frac{t}{\mu}} \partial_x Q_0 \\ P(x,0) = P_0(x) \end{cases}$$

$$|\partial_x P(x,t)| \le \frac{c\varepsilon}{\sqrt{\mu}} , \quad |\partial_t P(x,t)| \le \frac{c\varepsilon}{\sqrt{\mu}}$$

$$t\,|\partial_t P(x,t)| \le c\varepsilon$$

$$TV\, P(\cdot,t) \le c\varepsilon$$

$$TV\, \partial_x S(P(\cdot,t)) \le \frac{c\varepsilon}{\mu}$$

$$TV\, \partial_x^2 S(P(\cdot,t)) \le \frac{c\varepsilon}{\mu^2}$$

$$\int_{-\infty}^{\infty} |\partial_t P(x,t)|\, dx \le c\varepsilon$$

$$\int_0^{\infty}\int_{-\infty}^{\infty} (1+x^2)\, |\partial_t \partial_x S(P(x,t))|^2\, dx\, dt \le \frac{c\varepsilon^2}{\mu}$$

ENERGY ESTIMATES FOR THE REDUCED BALANCE LAW

$$\begin{cases} \partial_t W - \partial_x Z = 0 \\ \partial_t Z - \partial_x [S(P+W) - S(P)] + \frac{1}{\mu} Z = -\mu \partial_t \partial_x S(P) \end{cases}$$

$$W(x,0) = W_0(x), \quad Z(x,0) = \Sigma_0(x) - \mu \partial_x S(P_0(x))$$

$$\eta(W,Z) = \varphi(P+W) - \varphi(P) - S(P) \cdot W + \tfrac{1}{2}|Z|^2$$

$$H(W,Z,\Omega) = \eta(W,Z) + \nu|\Omega|^2 + 2\nu\Omega \cdot Z$$

$$\partial_t [(1+x^2)\eta(W,Z)] + \cdots\cdots$$

$$\partial_t [(1+x^2) H(W,Z,\Omega)] + \cdots$$

$$\partial_t [t(1+x^2)\eta(W,Z)] + \cdots\cdot$$

$$\int_{-\infty}^{\infty} (1+x^2)\{|W(x,t)|^2 + |Z(x,t)|^2\} dx \le c\mu^2\varepsilon^2$$

$$\int_{-\infty}^{\infty} |W(x,t)| dx \le c\mu\varepsilon$$

BOUNDS ON THE VARIATION

$$\begin{cases} \partial_t W - \partial_x Z = 0 \\ \partial_t Z - \partial_x [S(P+W) - S(P)] + \frac{1}{\mu} Z = -\mu \partial_t \partial_x S(P) \end{cases}$$

$$\Xi = Z + \mu \partial_x S(P) - \frac{1}{2} \int_0^t e^{-\frac{t-\tau}{2\mu}} \partial_x S(P)\, d\tau + \frac{1}{2}\Omega - \frac{1}{2} e^{-\frac{t}{2\mu}} \Omega_0$$

$$\begin{cases} \partial_t W - \partial_x \Xi + \frac{1}{2\mu} W = \frac{1}{\mu} \Theta \\ \partial_t \Xi - \partial_x [S(P+W) - S(P)] + \frac{1}{2\mu} \Xi = \frac{1}{4\mu} \Omega \end{cases}$$

$$\Theta = -\mu^2 \partial_x^2 S(P) + \frac{1}{2}\mu \int_0^t e^{-\frac{t-\tau}{2\mu}} \partial_x^2 S(P)\, d\tau + \frac{1}{2} e^{-\frac{t}{2\mu}} W_0$$

$$TV\, \Omega(\cdot, t) = \frac{1}{\mu} \int_{-\infty}^{\infty} |W(x,t)|\, dx \le c\varepsilon$$

$$TV\, \Theta(\cdot, t) \le c\varepsilon$$