Compressible Euler-Maxwell limit for global smooth solutions to the Vlasov-Maxwell-Boltzmann system

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Goal of this talk: Under the (compressible) Euler scaling on the (Vlasov-Maxwell-Boltzmann)$_{\varepsilon>0}$ system (ϵ : Knudsen number, non-dimensional),

- to construct an *almost global* smooth solution such that

$$\sup_{0\le t\le T_\varepsilon} \|F^\varepsilon - M_{[\bar\rho,\bar u,\bar T](t,x)}\|_{(L^2_x\cap L^\infty_x)L^2_v(\mu^{-1/2})} + \sup_{0\le t\le T_\varepsilon} \|(E^\varepsilon, B^\varepsilon) - (\bar E, \bar B)\|_{L^2_x\cap L^\infty_x} \lesssim \varepsilon^{1-a}$$

with

$$M_{[\bar\rho,\bar u,\bar T]}(t,x,v) := \frac{\bar\rho(t,x)}{[2\pi\overline{T}(t,x)]^{3/2}} \exp\Big\{-\frac{|v-\overline{u}(t,x)|^2}{2\overline{T}(t,x)}\Big\},$$

$$T_\varepsilon \sim \frac{1}{\eta_0\varepsilon^a + \varepsilon^{\frac12 - a}}, \quad 0\le a<\frac12,$$

where $(\bar\rho, \bar u, \bar T, \bar E, \bar B)$ is a *global* smooth solution to the compressible Euler-Maxwell near $(1, 0, \frac32, 0, 0)$ with a small amplitude $\eta_0 > 0$ independent of ε.

Remark:

- The robust $L^2 \cap L^\infty(wdv)$ approach in low-regularity function spaces by Guo seems not applicable in case of the non-relativistic VMB.
- However, we are able to design ε-dependent energy functional $\mathcal{E}_{N,\varepsilon}(t)$ and corresponding dissipation functional $\mathcal{D}_{N,\varepsilon}(t)$ to close the a priori estimate

$$\sup_{0\le t\le \tau}\left[\mathcal{E}_{N,\varepsilon}(t) + c\int_0^t \mathcal{D}_{N,\varepsilon}(s)\,ds\right] \le \frac{1}{2}\varepsilon^2.$$

  L^∞ bound of solutions is a consequence of Sobolev embeddings.
- ε-singularity of $\mathcal{E}_{N,\varepsilon}(t)$ and $\mathcal{D}_{N,\varepsilon}(t)$ occurs to the **highest-order** derivatives.

Boltzmann equation (1872):

- The unknown:

$$F = F(t,x,v) \geq 0,\ \ t > 0, x \in \Omega \subset \mathbb{R}^3, v \in \mathbb{R}^3,$$

the **velocity distribution function** of particles in a rarefied gas.

- Governed by

$$\underbrace{\{\partial_t + v \cdot \nabla_x\}F}_{\text{free transport}} = \underbrace{Q(F,F)}_{\text{binary collision}} \ ,$$

with the Boltzmann collision operator

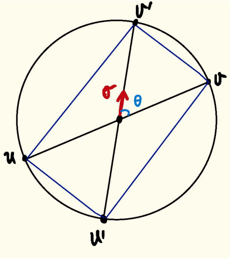
$$Q(G,F)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\sigma)[\underbrace{G(u')F(v')}_{\text{gain}} - \underbrace{G(u)F(v)}_{\text{loss}}]\, d\sigma du,$$

where

$$v' = \frac{v+u}{2} + \frac{|v-u|}{2}\sigma, \quad u' = \frac{v+u}{2} - \frac{|v-u|}{2}\sigma,$$

satisfying

$$v + u = v' + u',$$
$$|v|^2 + |u|^2 = |v'|^2 + |u'|^2.$$



θ: deviation angle

$$\cos\theta = \sigma \cdot \frac{v-u}{|v-u|} = \frac{v'-u'}{|v'-u'|} \cdot \frac{v-u}{|v-u|}$$

Collision kernel:

$$B(v-u,\sigma) = |v-u|^\gamma b(\cos\theta),$$

$$-3 < \gamma \leq 1,$$

- **non-cutoff:**

$$\frac{1}{C_b \theta^{1+2s}} \leq \sin\theta b(\cos\theta) \leq \frac{C_b}{\theta^{1+2s}}, \quad \forall\, \theta \in (0, \frac{\pi}{2}],$$

$$C_b > 0, \quad 0 < s < 1.$$

  A physical example: For potential $U(r) = r^{-\ell}$ $(\ell > 1)$ (inverse power law),

$$\gamma = \frac{\ell - 4}{\ell}, \quad s = \frac{1}{\ell}.$$

- **cutoff** (H. Grad):

$$\int_0^{\pi/2} \sin\theta b(\cos\theta)\, d\theta < \infty.$$

Basic properties of $Q(F,F)$:

- **Collision invariants:**

$$\int_{\mathbb{R}^3} \phi(v) Q(F,F)(v)\, dv = 0 \text{ for } \phi(v) = 1, v, |v|^2.$$

- **Entropy-entropy product:** For a solution $F = F(t,x,v)$ satisfying $\partial_t F + v \cdot \nabla_x F = Q(F,F)$,

$$\partial_t \int_{\mathbb{R}^3} F \ln F\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} vf \ln F\, dv$$
$$= -\int_{\mathbb{R}^3} Q(F,F) \ln F\, dv \leq 0,$$

where $=$ holds iff $Q(F,F) = 0$ holds, iff F is taken as a local Maxwellian:

$$\overline{M} \equiv M_{[\bar\rho,\bar u,\bar\theta]}(t,x,v) := \frac{\bar\rho(t,x)}{\sqrt{(2\pi R\overline{\theta}(t,x))^3}} \exp\Big\{ -\frac{|v-\overline{u}(t,x)|^2}{2R\overline{\theta}(t,x)} \Big\}.$$

Long time dynamics:

It would be expected that the mesocopic motion by

$$\partial_t F + v \cdot \nabla_x F = Q(F, F)$$

is getting in large time close to the dynamics for

$$F(t, x, v) = M_{[\bar{\rho}, \bar{u}, \bar{\theta}]}(t, x, v)$$

governed by the local conservation laws:

$$\partial_t \int_{\mathbb{R}^3} \phi(v) F(t, x, v)\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} v \phi(v) F(t, x, v)\, dv = 0,$$

$$\phi(v) = 1, v, |v|^2$$

and the entropy inequality:

$$\partial_t \int_{\mathbb{R}^3} F \ln F\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} vf \ln F\, dv \leq 0.$$

These are approximately equivalent with the **compressible Euler system:**

$$\begin{cases} \partial_t \bar\rho + \nabla_x \cdot (\bar\rho \bar u) = 0, \\ \partial_t (\bar\rho \bar u) + \nabla_x \cdot (\bar\rho \bar u \otimes \bar u) + \nabla_x \bar p = 0, \\ \partial_t \left[\bar\rho(\bar\theta + \frac{1}{2}|\bar u|^2)\right] + \nabla_x \cdot \left[\bar\rho \bar u(\bar\theta + \frac{1}{2}|\bar u|^2)\right] + \nabla_x \cdot (\bar p \bar u) = 0, \end{cases}$$

$$\bar p = R\bar\rho\bar\theta = \frac{2}{3}\bar\rho\bar\theta,$$

with the entropy inequality

$$\partial_t (\bar\rho \ln \frac{\bar\rho}{\bar\theta^{3/2}}) + \nabla_x \cdot (\bar\rho \bar u \ln \frac{\bar\rho}{\bar\theta^{3/2}}) \le 0.$$

Question

Rigorous justification?

Cf. Chapter 6 of Hydrodynamic Limits of the Boltzmann Equation by Laure Saint-Raymond.

Analytical framework:

For the Boltzmann with cutoff,

- Nishida (1978): abstract Cauchy-Kovalevskaya + spectral analysis of linearized Boltzmann equation
- Ukai-Asano (1983): contraction mapping with time-dependent norm, include initial layer.

Hilbert expansion:

We start from **Boltzmann** equation (**cutoff, hard potentials** $0 \le \gamma \le 1$):

$$\partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon).$$

The solution F^ε is found via the Hilbert expansion:

$$F^\varepsilon = F_0 + \sum_{n=1}^{6} \varepsilon^n F_n + \varepsilon^3 F_R^\varepsilon,$$

where $F_0, \cdots, F_6$ are independent of ε. As a consequence,

$$F_0 \equiv \overline{M} = M_{[\bar\rho, \bar u, \bar T]}(t,x,v) := \frac{\bar\rho(t,x)}{\sqrt{(2\pi \overline{T}(t,x))^3}} \exp\Big\{ -\frac{|v - \overline{u}(t,x)|^2}{2\overline{T}(t,x)} \Big\},$$

where fluid parameters $(\bar\rho, \bar u, \bar T)(t,x)$ are the solutions of the compressible Euler system. Then the remainder F_R^ε satisfies

$$\partial_t F_R^\varepsilon + v \cdot \nabla_x F_R^\varepsilon \underbrace{- \frac{1}{\varepsilon}\{Q(\overline{M}, F_R^\varepsilon) + Q(F_R^\varepsilon, \overline{M})\}}_{\text{linearization around a \textbf{given} Euler flow}} = \varepsilon^2 Q(F_R^\varepsilon, F_R^\varepsilon) + \cdots .$$

Theorem (Caflisch 1980)

Let $\Omega = \mathbb{T}$, $[\bar{\rho}, \bar{u}, \bar{T}](t, x_1)$ *be a smooth solution without vacuum to the Euler system over* $[0, \tau]$ *and* $\overline{M} = M_{[\bar{\rho}, \bar{u}, \bar{T}]}(t, x_1, v)$. *There is* $\varepsilon_0 > 0$ *such that for each* $0 < \varepsilon \le \varepsilon_0$, *a smooth solution* F^ε *to the cutoff Boltzmann equation* $\partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon)$ *with* $0 \le \gamma \le 1$ *exists for* $0 \le t \le \tau$ *with*

$$\sup_{0 \le t \le \tau} \|F^\varepsilon - \overline{M}\|_{L^2_{x_1, v}} \le C_\tau \varepsilon,$$

where C_τ *is independent of* ε.

Proof:

- Construct smooth profiles F_i $(1 \leq 6)$ iteratively:

$$F_i(t, x_1, v) \leq C|\xi|^{3i}\overline{M},$$

  in particular, F_1 cubic growth in large v due to $v_1\partial_{x_1}F_0 = v_1\partial_{x_1}\overline{M}$.

- Cutoff assumption is essential, so can use Grad's splitting $L = -\nu + K$. To overcome large-velocity growth, develop a decomposition:

$$F_R = \underbrace{\sqrt{\overline{M}}g}_{\text{low } v \text{ part}} + \underbrace{\sqrt{\mu_m}h}_{\text{high } v \text{ part}}$$

  where $\mu_m = \frac{1}{\sqrt{(2\pi T_m)^3}} \exp\left\{-\frac{|v|^2}{2T_m}\right\}$ with $T_m > \max_{t,x} T(t,x)$ so that $\mu_m \geq c\overline{M}$. Split K correspondingly as

$$Kh = \chi_{|v|\leq M}Kh + \chi_{|v|>M}Kh.$$

- Show contraction in $H^1_{x_1}L^\infty_\beta$. Choice for initial data: $g(0) = h(0) \equiv 0$, so $F_R(0) \equiv 0$. **Loss of positivity of ID and hence solutions.**

Instead of using Caflisch's decomposition, **Guo-Jang-Jiang (2010)** applied the L^2-L^∞ approach:

Let $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. Write $F_R^\varepsilon = \sqrt{\overline{M}} f^\varepsilon$, then

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon - \frac{1}{\varepsilon}\{Q(\overline{M}, \sqrt{\overline{M}} f^\varepsilon) + Q(\sqrt{\overline{M}} f^\varepsilon, \overline{M})\}$$

$$= \underbrace{-\frac{\{\partial_t + v \cdot \nabla_x\}\sqrt{\overline{M}}}{\sqrt{\overline{M}}} f^\varepsilon}_{(*) \sim (\partial_t, \partial_x)\bar{u}|v|^3 f^\varepsilon} + \cdots .$$

L^2 estimate on f^ε meets an obstacle.

Idea: Let

$$F_R^\varepsilon = (1+|v|^2)^{-\beta}\sqrt{\mu_m} h^\varepsilon = \frac{1}{w(v)}\sqrt{\mu_m} h^\varepsilon,$$

$$\mu_m = (2\pi T_m)^{-\frac{3}{2}} \exp\left(-\frac{|v|^2}{2T_m}\right), \quad T_m < \max_{t,x} \bar{T}(t,x) < 2T_m,$$

$$\boxed{\int (*) f^\epsilon \sim \|(\partial_t, \partial_x)\bar{u}\|_{L^2} \|h^\epsilon\|_{L^\infty} \|f^\epsilon\|_{L^2}.}$$

Then h^ε satisfies

$$\partial_t h^\varepsilon + v \cdot \nabla_x h^\varepsilon + \frac{1}{\varepsilon}\nu(\overline{M})h^\varepsilon + \frac{1}{\varepsilon}K_w h^\varepsilon = \cdots,$$

where $K_w g = wK(\frac{g}{w})$ and

$$-\frac{1}{\sqrt{\mu_m}}\{Q(\overline{M}, \sqrt{\mu_m}g) + Q(\sqrt{\mu_m}g, \overline{M})\} = (\nu(\overline{M}) + K)g.$$

Strategy of estimates:

- Use L^2 norm of f^ε to control the low-order velocity part and L^∞ norm of h^ε for the large velocity part.
- Obtain L^∞ estimate for $\varepsilon^{3/2}h^\varepsilon$ along the trajectory in terms of L^2 norm of f^ε, close the estimates in L^2 and apply the Gronwall argument over $[0, \tau]$.

Theorem (Guo-Jang-Jiang 2010)

$$\sup_{0\le t\le\tau}(\varepsilon^{3/2}\|h^\varepsilon(t)\|_{L^\infty_{x,v}} + \|f^\varepsilon(t)\|_{L^2_{x,v}}) \le C_\tau(\varepsilon^{3/2}\|h_0^\varepsilon\|_{L^\infty_{x,v}}, \|f_0^\varepsilon\|_{L^2_{x,v}}).$$

Guo-Jang (2010) further obtained the global *higher-order* Hilbert expansion

$$F^{\varepsilon} = \sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^{\varepsilon}$$

to the Vlasov-Poisson-Boltzmann system.

Theorem (Guo-Jang 2010)

There exists a solution $F^{\varepsilon}(t,x,v)$ to the VPB system in the Euler scaling:

$$\begin{aligned} \partial_t F^{\varepsilon} + v \cdot \nabla_x F^{\varepsilon} + \nabla_x \phi^{\varepsilon} \cdot \nabla_v F^{\varepsilon} &= \frac{1}{\varepsilon} Q(F^{\varepsilon}, F^{\varepsilon}), \\ \Delta_x \phi^{\varepsilon} &= \int F^{\varepsilon}\, dv - 1, \end{aligned}$$

such that

$$\sup_{0 \le t \le \varepsilon^{-m}} \| F^{\varepsilon}(t,\cdot\cdot) - M_{[\bar\rho,\bar u,\bar T]}(t,\cdot,\cdot) \| = O(\varepsilon), \quad 0 < m \le \frac{1}{2}\frac{2k-3}{2k-2},\ k \ge 6.$$

Here $[\bar\rho, \bar u, \bar T](t,x)$ is the smooth solution around constant equilibrium for the hydrodynamic compressible Euler-Poisson system with $\bar T = C\bar\rho^{\frac{2}{3}}$.

Problems left:

- What happens to the **non-cutoff** Boltzmann or Landau equation for which the Grad's splitting is no longer available?

  Still possible to obtain L^∞ estimates using the De Giorgi argument instead of the direct L^2-L^∞ interplay: Alonso-Morimoto-Sun-Yang (arXiv 2020), Guo-Hwang-Jang-Ouyang (ARMA 2020), Kim-Guo-Hwang (PMJ 2020),...but so far unknown to employ them for the fluid limit, as need to obtain estimate uniform in ε.

- How to extend Guo-Jang's work to the VMB system where the self-consistent electromagnetic field satisfying the Maxwell equations is included?

  Again, L^2-L^∞ interplay fails for the fluid limit, as one loses the Glassey-Strauss representation, although it works for the relativistic case; see a recent work by Guo-Xiao (CMP 2021).

Our strategy:

- **Derive an ε-dependent high-order energy estimates on basis of the macro-micro decomposition of Liu-Yu (CMP,2002) and Liu-Yang-Yu (Phys D 2004)**

VMB system for dynamics of electrons in $\mathbb{R}^3_x$:

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - (E + v \times B) \cdot \nabla_v F = \frac{1}{\varepsilon} Q(F, F), \\ \partial_t E - \nabla_x \times B = \int_{\mathbb{R}^3} vF \, dv, \\ \partial_t B + \nabla_x \times E = 0, \\ \nabla_x \cdot E = n_b - \int_{\mathbb{R}^3} F \, dv, \quad \nabla_x \cdot B = 0. \end{cases}$$

- $E = E(t, x) = (E_1, E_2, E_3)(t, x)$: self-consistent electric field
- $B = B(t, x) = (B_1, B_2, B_3)(t, x)$: self-consistent magnetic field
- $n_b > 0$ is assumed to be a constant denoting the spatially uniform density of the ionic background. Take $n_b = 1$ without loss of generality.

For brevity we focus on the hard sphere model for the Boltzmann collision operator:

$$Q(F_1, F_2)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v - v_*)\cdot\omega|\{F_1(v')F_2(v'_*) - F_1(v)F_2(v_*)\}\, d\omega\, dv_*,$$

where $\omega \in \mathbb{S}^2$ is a unit vector in $\mathbb{R}^3$, and the velocity pairs (v, v_*) before collisions and (v', v'_*) after collisions are given by

$$v' = v - [(v - v_*)\cdot\omega]\omega, \quad v'_* = v_* + [(v - v_*)\cdot\omega]\omega,$$

in terms of the conservations of momentum and kinetic energy:

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

VMB system for the hard sphere model, global classical solutions near global Maxwellians:

- $\mathbb{T}^3$: Guo (2003)
- $\mathbb{R}^3$: Strain (2006), D.-Strain (2011)

Corresponding to VMB, the hydrodynamic description for the motion of electrons at the fluid level is also given by the following compressible Euler-Maxwell system which is an important fluid model in plasma physics:

$$
\begin{cases}
\partial_t \bar{\rho} + \nabla_x \cdot (\bar{\rho}\bar{u}) = 0, \\
\partial_t(\bar{\rho}\bar{u}) + \nabla_x \cdot (\bar{\rho}\bar{u} \otimes \bar{u}) + \nabla_x \bar{p} = -\bar{\rho}(\bar{E} + \bar{u} \times \bar{B}), \\
\partial_t \bar{E} - \nabla_x \times \bar{B} = \bar{\rho}\bar{u}, \\
\partial_t \bar{B} + \nabla_x \times \bar{E} = 0, \\
\nabla_x \cdot \bar{E} = n_b - \bar{\rho}, \quad \nabla_x \cdot \bar{B} = 0.
\end{cases}
$$

Here the unknowns are the electron density $\bar{\rho} = \bar{\rho}(t,x) > 0$, the electron velocity $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)(t,x)$, and the electromagnetic field $(\bar{E}, \bar{B}) = (\bar{E}, \bar{B})(t,x)$. Moreover, $\bar{p} = K\bar{\rho}^{5/3}$ is the pressure satisfying the power law with the adiabatic exponent $\gamma = \frac{5}{3}$. We take the physical constant $K = 1$ without loss of generality.

Remark: It can be formally derived from the VMB system in the isentropic case for the macro fluid system:

$$
\frac{\bar{\rho}}{\bar{\theta}^{3/2}} \equiv 1
$$

Euler-Maxwell system in $\mathbb{R}^d$, global classical solutions near constant equilibrium:

- Germain-Masmoudi (2014), Ionescu-Pausader (2014): $d = 3$, electrons dynamics, method of space-time resonance
- Guo-Ionescu-Pausader (2016): $d = 3$, two-fluid model for electrons and ions, can be relativistic
- Deng (2017): $d = 2$, electrons dynamics
- Many others for Euler-Poisson and results in $\mathbb{T}^3$ or $\mathbb{T}^2$

Proposition (Ionescu-Pausader, JEMS 2014)

Let $(\bar{\rho}, \bar{u}, \bar{E}, \bar{B})(t, x)$ *be a global-in-time smooth solution to the compressible Euler-Maxwell system, and let* $\bar{\theta}(t, x) = \frac{3}{2}\bar{\rho}^{2/3}(t, x)$*, then the following estimate holds for all* $t \geq 0$*:*

$$\|(\bar{\rho} - 1, \bar{u}, \bar{\theta} - \frac{3}{2}, \bar{E}, \bar{B})\|_{W^{N_0,2}}$$
$$+ (1 + t)^{\vartheta}\{\|(\bar{\rho} - 1, \bar{\theta} - \frac{3}{2}, \bar{B})\|_{W^{N,\infty}} + \|(\bar{u}, \bar{E})\|_{W^{N+1,\infty}}\} \leq C\eta_0.$$

Here $\vartheta = 101/100$*,* $\eta_0 > 0$ *is a sufficiently small constant and* $N_0 > 0$ *is a large integer, where integer* N *satisfies* $3 \leq N < N_0$*.*

Macro-micro decomposition:

For a solution (F, E, B) to $(\text{VMB})_\varepsilon$ system, we define

$$F = M_{[\rho,u,\theta]} + G,$$

with

$$\begin{cases} \rho(t,x) \equiv \int_{\mathbb{R}^3} \psi_0(v) F(t,x,v)\, dv, \\ \rho(t,x) u_i(t,x) \equiv \int_{\mathbb{R}^3} \psi_i(v) F(t,x,v)\, dv, \quad \text{for } i = 1,2,3, \\ \rho(t,x)\big[e(t,x) + \frac{1}{2}|u(t,x)|^2\big] \equiv \int_{\mathbb{R}^3} \psi_4(v) F(t,x,v)\, dv. \end{cases}$$

Here $\psi_i(v)$ are given by collision invariants

$$\psi_0(v) = 1, \quad \psi_i(v) = v_i \ (i = 1,2,3), \quad \psi_4(v) = \frac{1}{2}|v|^2.$$

Zero-order fluid-type (compressible Euler-Maxwell) system:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p + \rho(E + u \times B) \\ \qquad\qquad = -\int_{\mathbb{R}^3} v \otimes v \cdot \nabla_x G \, dv, \\ \partial_t[\rho(\theta + \frac{1}{2}|u|^2)] + \nabla_x \cdot [\rho u(\theta + \frac{1}{2}|u|^2) + pu] + \rho u \cdot E \\ \qquad\qquad = -\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 v \cdot \nabla_x G \, dv, \end{cases}$$

coupled to

$$\begin{cases} \partial_t E - \nabla_x \times B = \rho u, \quad \partial_t B + \nabla_x \times E = 0, \\ \qquad\qquad \nabla_x \cdot E = 1 - \rho, \quad \nabla_x \cdot B = 0, \end{cases}$$

where the pressure $p = R\rho\theta = \frac{2}{3}\rho\theta$.

From

$$\partial_t G + P_1(v\cdot\nabla_x G) + P_1(v\cdot\nabla_x M) - (E + v\times B)\cdot\nabla_v G = \frac{1}{\varepsilon} L_M G + \frac{1}{\varepsilon} Q(G,G),$$

we write

$$G = \varepsilon L_M^{-1}[P_1(v\cdot\nabla_x M)] + L_M^{-1}\Theta,$$

$$\Theta := \varepsilon\partial_t G + \varepsilon P_1(v\cdot\nabla_x G) - \varepsilon(E + v\times B)\cdot\nabla_v G - Q(G,G).$$

First-order fluid-type (compressible Navier-Stokes-Maxwell) system:

$$\begin{cases} \partial_t\rho + \nabla_x\cdot(\rho u) = 0, \\ \partial_t(\rho u_i) + \nabla_x\cdot(\rho u_i u) + \partial_{x_i} p + \rho(E + u\times B)_i \\ \qquad = \varepsilon\sum_{j=1}^3 \partial_{x_j}(\mu(\theta)D_{ij}) - \int_{\mathbb{R}^3} v_i(v\cdot\nabla_x L_M^{-1}\Theta)\,dv, \quad i = 1,2,3, \\ \partial_t[\rho(\theta + \frac{1}{2}|u|^2)] + \nabla_x\cdot[\rho u(\theta + \frac{1}{2}|u|^2) + pu] + \rho u\cdot E \\ \qquad = \varepsilon\sum_{j=1}^3 \partial_{x_j}(\kappa(\theta)\partial_{x_j}\theta) + \varepsilon\sum_{i,j=1}^3 \partial_{x_j}(\mu(\theta)u_i D_{ij}) \\ \qquad\quad - \int_{\mathbb{R}^3} \frac{1}{2}|v|^2 v\cdot\nabla_x L_M^{-1}\Theta\,dv, \end{cases}$$

coupled to

$$\begin{cases} \partial_t E - \nabla_x\times B = \rho u, \quad \partial_t B + \nabla_x\times E = 0, \\ \qquad \nabla_x\cdot E = 1 - \rho, \quad \nabla_x\cdot B = 0. \end{cases}$$

Here, $D_{ij} = \partial_{x_j} u_i + \partial_{x_i} u_j - \frac{2}{3}\delta_{ij}\nabla_x\cdot u$.

Macro perturbation:

$$(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta}, \widetilde{E}, \widetilde{B})(t,x) = (\rho - \bar{\rho}, u - \bar{u}, \theta - \bar{\theta}, E - \overline{E}, B - \bar{B})(t,x).$$

Micro perturbation:

$$\sqrt{\mu} f(t,x,v) = G(t,x,v) - \overline{G}(t,x,v),$$

where $\overline{G}(t,x,v)$ is given by

$$\overline{G}(t,x,v) \equiv \varepsilon L_M^{-1} P_1 \big\{ v \cdot \big(\frac{|v-u|^2 \nabla_x \bar{\theta}}{2R\theta^2} + \frac{(v-u) \cdot \nabla_x \bar{u}}{R\theta} \big) M \big\}.$$

Note: It's the linearisation of the Chapman-Enskog part $\varepsilon L_M^{-1}[P_1(v \cdot \nabla_x M)]$ around Euler-Maxwell solutions.

We define the instant energy as

$$\mathcal{E}_N(t) \equiv \sum_{|\alpha|\leq N-1} \{\|\partial^\alpha(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta}, \widetilde{E}, \widetilde{B})(t)\|^2 + \|\partial^\alpha f(t)\|^2\}$$
$$+ \sum_{|\alpha|+|\beta|\leq N, |\beta|\geq 1} \|\partial^\alpha_\beta f(t)\|^2$$
$$+ \varepsilon^2 \sum_{|\alpha|=N} \{\|\partial^\alpha(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta}, \widetilde{E}, \widetilde{B})(t)\|^2 + \|\partial^\alpha f(t)\|^2\},$$

and the dissipation rate as

$$\mathcal{D}_N(t) \equiv \varepsilon \sum_{1\leq|\alpha|\leq N} \|\partial^\alpha(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta})(t)\|^2 + \varepsilon \sum_{|\alpha|=N} \|\partial^\alpha f(t)\|_\nu^2$$
$$+ \frac{1}{\varepsilon} \sum_{|\alpha|\leq N-1} \|\partial^\alpha f(t)\|_\nu^2 + \frac{1}{\varepsilon} \sum_{|\alpha|+|\beta|\leq N, |\beta|\geq 1} \|\partial^\alpha_\beta f(t)\|_\nu^2.$$

Theorem (D.-Yang-Yu, M3AS 23)

Let $(\bar{\rho}, \bar{u}, \bar{\theta}, \bar{E}, \bar{B})(t,x)$ *be a global smooth solution to the compressible Euler-Maxwell system given in Proposition. Construct a local Maxwellian* $M_{[\bar{\rho},\bar{u},\bar{\theta}]}(t,x,v)$*. Then there exists a small constant* $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0, \varepsilon_0]$*, the Cauchy problem on the Vlasov-Maxwell-Boltzmann system with well prepared initial data*

$$F^\varepsilon(0,x,v) \equiv M_{[\bar{\rho},\bar{u},\bar{\theta}]}(0,x,v) \geq 0, \quad (E^\varepsilon, B^\varepsilon)(0,x) \equiv (\bar{E}, \bar{B})(0,x),$$

admits a unique smooth solution $(F^\varepsilon(t,x,v), E^\varepsilon(t,x), B^\varepsilon(t,x))$ *for all* $t \in [0, T_\varepsilon]$ *with*

$$T_\varepsilon = \frac{1}{4C_1} \frac{1}{\eta_0 \varepsilon^a + \varepsilon^{\frac{1}{2}-a}}, \quad \text{for} \quad a \in [0, \frac{1}{2}),$$

where generic constant $C_1 > 1$ *and small constant* $\eta_0 > 0$ *are independent of* ε*. Moreover, it holds that* $F^\varepsilon(t,x,v) \geq 0$ *and*

$$\mathcal{E}_N(t) + \frac{1}{2} \int_0^t \mathcal{D}_N(s)\, ds \leq \frac{1}{2} \varepsilon^{2-2a},$$

for any $t \in [0, T_\varepsilon]$*.*

Theorem (Conti)

In particular, there exists a constant $C > 0$ independent of ε and T_{max} such that

$$\begin{aligned}\sup_{t\in[0,T_\varepsilon]} \Big\{ &\Big\| \frac{F^\varepsilon(t,x,v) - M_{[\bar\rho,\bar u,\bar\theta]}(t,x,v)}{\sqrt{\mu}} \Big\|_{L^2_x L^2_v} \\ &+ \Big\| \frac{F^\varepsilon(t,x,v) - M_{[\bar\rho,\bar u,\bar\theta]}(t,x,v)}{\sqrt{\mu}} \Big\|_{L^\infty_x L^2_v} \\ &+ \|(E^\varepsilon - \bar E, B^\varepsilon - \bar B)(t,x)\|_{L^2_x} \\ &+ \|(E^\varepsilon - \bar E, B^\varepsilon - \bar B)(t,x)\|_{L^\infty_x} \Big\} \\ \leq C\varepsilon^{1-a}.\end{aligned}$$

Note: For $a = \frac{1}{4}$, we get the distance in $L^2_x \cap L^\infty_x \sim \varepsilon^{\frac{3}{4}}$ uniformly in the time interval $[0, T_\varepsilon]$ with $T_\varepsilon \sim \varepsilon^{-1/4}$ that can be almost global.

Remark:

Although the $L^2 - L^\infty$ approach works well for the Boltzmann with cutoff potentials, in particular, for the hard-sphere model, it cannot be applicable to the VMB case for the hard-sphere model, since one loses the Glassey-Strauss representation for the electric-magnetic fields E and B that is true in the relativistic case, for instance,

$$4\pi E(t,x) = -\int_{|y-x|\le t}\int_{\mathbb{R}^3} \frac{(\omega+\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}\cdot\omega)^2} F(t-|y-x|, y, v)\, dv \frac{dy}{|y-x|^2} + \text{other terms},$$

with $\hat{v} = \frac{v}{\sqrt{1+|v|^2}}$ and $\omega = \frac{y-x}{|y-x|}$. The relativistic velocity $\hat{v}$ is bounded, so the expression $1+\hat{v}\cdot\omega$ is bounded away from 0, Guo-Xiao (CMP 2021).

One point of the proof:

We use the bootstrap argument. Assume

$$\sup_{0\le t\le T} \mathcal{E}_N(t) \le \varepsilon^{2-2a}, \quad a \in [0, \frac{1}{2}).$$

We are devoted to showing

$$\mathcal{E}_N(t) + \frac{1}{2}\int_0^t \mathcal{D}_N(s)\,ds \le \frac{1}{2}\varepsilon^{2-2a}.$$

Indeed, one can prove

$$\begin{aligned}\mathcal{E}_N(t) + \int_0^t \mathcal{D}_N(s)\,ds \le & C_1(\eta_0 + \varepsilon^{\frac{1}{2}-a})\int_0^t \mathcal{D}_N(s)\,ds \\ & + C_1[\eta_0 + \varepsilon^{\frac{1}{2}} + (\eta_0\varepsilon^a + \varepsilon^{\frac{1}{2}-a})t]\varepsilon^{2-2a}.\end{aligned}$$

We therefore require that

$$C_1(\eta_0 + \varepsilon^{\frac{1}{2}-a}) \le \frac{1}{2}, \quad C_1[\eta_0 + \varepsilon^{\frac{1}{2}} + (\eta_0\varepsilon^a + \varepsilon^{\frac{1}{2}-a})t] \le \frac{1}{2},$$

yielding

$$a \in [0, \frac{1}{2}), \quad \text{and} \quad t \le T_{max} = \frac{1}{4C_1}\frac{1}{\eta_0\varepsilon^a + \varepsilon^{\frac{1}{2}-a}}.$$

The key is obtain the estimate

$$\varepsilon^2 \times \sum_{|\alpha|=N} \left\{ \|\partial^\alpha(\widetilde{\rho}, \widetilde{u}, \widetilde{\theta}, \widetilde{E}, \widetilde{B})(t)\|^2 + \|\partial^\alpha f(t)\|^2 + \frac{1}{\varepsilon}\int_0^t \|\partial^\alpha f(s)\|_\nu^2 \, ds \right\}$$
$$\leq C(\eta_0 + \varepsilon^{\frac{1}{2}-a}) \int_0^t \mathcal{D}_N(s)\, ds + C[\eta_0 + \varepsilon^{\frac{1}{2}} + (\eta_0 \varepsilon^{2a} + \varepsilon^{\frac{1}{2}-a})t]\varepsilon^{2-2a}.$$

from

$$\frac{\partial_t F}{\sqrt{\mu}} + \frac{v \cdot \nabla_x F}{\sqrt{\mu}} - \frac{(E + v \times B) \cdot \nabla_v F}{\sqrt{\mu}} = \frac{1}{\varepsilon}\mathcal{L} f + \frac{1}{\varepsilon}\Gamma(\frac{M-\mu}{\sqrt{\mu}}, f)$$
$$+ \frac{1}{\varepsilon}\Gamma(f, \frac{M-\mu}{\sqrt{\mu}}) + \frac{1}{\varepsilon}\Gamma(\frac{G}{\sqrt{\mu}}, \frac{G}{\sqrt{\mu}}) + \frac{1}{\varepsilon}\frac{L_M \overline{G}}{\sqrt{\mu}}.$$

Thank you!