Continued gravitational collapse for gaseous star and pressureless Euler-Poisson system

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Partial Differential Equations in Fluid Dynamics, Hangzhou, August 8th, 2023

Background

The 3-d compressible Euler-Poisson (EP) system

$$\begin{aligned} \partial_t \rho + div(\rho \vec{u}) &= 0, \\ \rho(\partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u}) + \nabla P(\rho) + \rho \nabla \Phi &= 0, \\ \triangle \Phi = 4\pi\rho, \quad \lim_{|x| \to \infty} \Phi(t,x) &= 0, \end{aligned} \tag{1}$$

where ρ, $\vec{u}$, $P(\rho)$ and Φ denote the density, velocity, pressure, and the gravitational potential respectively. Here $P(\rho) = \rho^\gamma, \gamma > 1$.

- $1 < \gamma < \frac{4}{3}$, supercritical-mass;
- $\gamma = \frac{4}{3}$, critical-mass;
- $\gamma > \frac{4}{3}$, subcritical-mass.

Figure: Gaseous Star, download from Baidu

Lane-Emden solution as $\gamma \in (\frac{6}{5}, 2)$

- $\frac{4}{3} < \gamma < 2$, conditional stable, Rein, ARMA, 2003;
- $\gamma = \frac{4}{3}$, instable, Deng-Liu-Yang-Yao, ARMA, 2002;
- $\frac{6}{5} \leq \gamma < \frac{4}{3}$, instable, Jang, ARMA 2008, CPAM 2014.

The nonlinear stability of Lane-Emden stars is still open!

Expanding solution

- affine solution for Euler system, Sideris, ARMA 2017;
- affine solution for EP system as $\gamma = 1 + \frac{1}{n}$ or $\gamma \in (1, \frac{14}{13})$, Hadžić-Jang, CMP 2019;
- stability of affine solution for $\gamma = \frac{4}{3}$, Hadžić-Jang, CPAM 2018.

Collapse solution

The collapsing solution describes the gravitational collapse of gaseous star.

- no collapse for $\gamma > \frac{4}{3}$, Deng-Liu-Yang-Yao, ARMA 2002;
- homologous collapse for $\gamma = \frac{4}{3}$, Goldreich-Weber, Astro-phys. J. 1980; Makino, Transp. Theory Stat. Phys. 1992;
- continued collapse for $1 < \gamma < \frac{4}{3}$, Guo-Hadžić-Jang, ARMA, 2021.

It is noted that Guo-Hadžić-Jang's collapsing solution is based on a special collapse solution of the pressureless EP system

$$\begin{aligned} \partial_t \rho + div(\rho \vec{u}) &= 0, \\ \rho(\partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u}) + \rho \nabla \Phi &= 0, \\ \triangle \Phi &= 4\pi\rho. \end{aligned} \tag{2}$$

Main results

In this talk, we classify all spherically symmetric solutions of pressureless EP system into the cases of escape and collapse.

$\exists! v^*(r) > 0$ such that

1) escape: if the initial velocity $\chi_1(r) \geq v^*(r)$, then the dust escapes away from the gravitational force forever;

2) collapse: if $\chi_1(r) < v^*(r)$, then the dust collapses at the origin in a finite time $t^*(r)$ even it may expand initially, i.e., $\chi_1(r) > 0$.

Moreover, we prove that there exist a class of spherically symmetric solutions of the original EP system (1), which formulate a continued gravitational collapse in finite time, based on the pressureless EP solutions if $\chi_1(r) < v^*(r)$.

Problem

Consider the gaseous star surrounded by vacuum. Denote $\Omega(t)$ as the support of $\rho(t,\cdot)$ with a boundary $\partial\Omega(t)$, $\mathcal{V}_{\partial\Omega(t)}$ as the normal velocity of $\partial\Omega(t)$, and $\vec{n}(t)$ as the outward unit normal vector to $\partial\Omega(t)$.

Assume a physical vacuum condition on the initial data, that is,

$$-\infty < \nabla\left(\frac{dP}{d\rho}(\rho)\right) \cdot \vec{n}\Big|_{\partial\Omega(t)} < 0. \tag{3}$$

Reformulated problem

Scaling transformation

$$\rho = \bar{\varepsilon}^{-3}\tilde{\rho}(s,y), \quad \vec{u} = \bar{\varepsilon}^{-\frac{1}{2}}\tilde{\vec{u}}(s,y), \quad \Phi = \bar{\varepsilon}^{-1}\tilde{\Phi}(s,y), \tag{4}$$

where

$$s = \bar{\varepsilon}^{-\frac{3}{2}}t, \quad y = \bar{\varepsilon}^{-1}x.$$

Then the rescaled variables $(\tilde{\rho}, \tilde{\vec{u}}, \tilde{\Phi})$ satisfy

$$\begin{aligned} \partial_s\tilde{\rho} + div(\tilde{\rho}\tilde{\vec{u}}) = 0,\\ \tilde{\rho}(\partial_s\tilde{\vec{u}} + (\tilde{\vec{u}}\cdot\nabla)\tilde{\vec{u}}) + \varepsilon\nabla P(\tilde{\rho}) + \tilde{\rho}\nabla\tilde{\Phi} = 0,\\ \triangle\tilde{\Phi} = 4\pi\tilde{\rho}, \quad \lim_{|x|\to\infty}\tilde{\Phi}(t,x) = 0, \end{aligned} \tag{5}$$

where $\varepsilon = \bar{\varepsilon}^{4-3\gamma}$ will be chosen small later, $\tilde{\Omega}(s) = \varepsilon^{-\frac{1}{4-3\gamma}}\Omega(t)$.

Lagrangian coordinate

Assume $\tilde{\Omega}$ is the unit ball $\{y \in \mathbb{R}^3 : |y| \leq 1\}$. Let $\eta : \tilde{\Omega} \to \tilde{\Omega}(s)$ be the solution of

$$\begin{aligned} \partial_s \eta(s,y) &= \tilde{\vec{u}}(s, \eta(s,y)), \\ \eta(0,y) &= \eta_0(y). \end{aligned} \tag{6}$$

Introduce the ansatz:

$$\eta(s,y) = \chi(s,r)y, \quad r = |y|, \quad r \in [0,1] \tag{7}$$

which leads to

$$\chi_{ss} + \frac{G(r)}{\chi^2} + \varepsilon P[\chi] = 0, \tag{8}$$

$$P[\chi] := \frac{\chi^2}{\omega^\alpha r^2}(r\partial_r)(\omega^{1+\alpha}\mathscr{F}[\chi]^{-\gamma}), \tag{9}$$

Continuation ⋯

where $\alpha \triangleq \frac{1}{\gamma-1}$,

$$\mathscr{F}[\chi] = \chi^2(\chi + r\partial_r\chi) \tag{10}$$

is the Jacobian determinant of $D\eta$, and $\omega(r)$ is the enthalpy defined by

$$\omega(r)^\alpha = \tilde{\rho}(\chi_0(r)r)\mathscr{F}[\chi_0](r),\ \chi_0(r) = \chi(0, r). \tag{11}$$

From the continuity equation $(5)_1$,

$$\frac{d}{ds}(\tilde{\rho}(s, \chi(s, r)y)\mathscr{F}[\chi](s, r)) = 0, \tag{12}$$

which gives

$$\tilde{\rho}(s, \chi(s, r)y) = \omega(r)^\alpha \mathscr{F}[\chi]^{-1}. \tag{13}$$

Denote the mean density of the gas by

$$G(r) \triangleq \frac{1}{r^3}\int_0^r 4\pi\omega^\alpha s^2 ds. \tag{14}$$

Pressureless EP system

Consider the pressureless equation

$$\chi_{ss} + \frac{G(r)}{\chi^2} = 0, \tag{15}$$

with initial conditions

$$\chi(0, r) = \chi_0(r) = 1, \quad \chi_s(0, r) = \chi_1(r). \tag{16}$$

The total energy $E(s) = \frac{1}{2}\chi_s^2 - \frac{G(r)}{\chi}$ is conserved, i.e.,

$$\chi_s^2 = \chi_1^2 + 2G(r)(\frac{1}{\chi} - 1). \tag{17}$$

Let $v^*(r) := \sqrt{2G}$.

Pressureless solutions

Theorem 3.1

Let $\chi_{dust}(s,r)$ be the solution of (15).
Escape case:
(1) If $\chi_1(r) > v^*(r)$, then $\chi_{dust}(s,r) > 0$ for all $s > 0$. The asymptotic behavior is

$$\chi_{dust}(s,r) \sim \sqrt{k_0}s, \quad as \quad s \to +\infty, \tag{18}$$

where $k_0 = \chi_1^2 - 2G$ is the initial energy.
(2) If $\chi_1(r) = v^*(r)$, then $\chi_{dust}(s,r) = (1 + 3\sqrt{\frac{G}{2}}s)^{\frac{2}{3}}$.
Collapse case: *If $\chi_1(r) < v^*(r)$, there exists a unique $t^*(r) > 0$ satisfying $\chi_{dust}(t^*(r), r) = 0$ such that the asymptotic behavior of the trajectory $\chi_{dust}(s,r)$ is*

$$\chi_{dust}(s,r) \sim (\frac{9G}{2})^{\frac{1}{3}}(t^*(r) - s)^{\frac{2}{3}}, \quad as \quad s \to t^*(r). \tag{19}$$

Remark 1

In the collapse case, $\chi_1(r)$ could be positive. That is, the trajectory may expand initially, but finally collapse to the center in a finite time.

Remark 2

Guo-Hadzic-Jang (ARMA 2021) constructed a special solution $\chi_{dust}(s, r) = (1 - 3\sqrt{\frac{G}{2}}s)^{\frac{2}{3}}$ in the case $\chi_1(r) = -v^*(r) < 0$. That is the gaseous star collapses initially.

Since the solution consists of trajectories $\chi_{dust}(s, r)$, all smooth solutions of pressureless Euler-Poisson system can be classified into four cases.

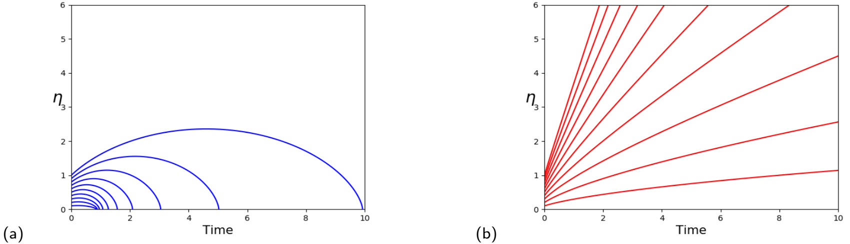

Figure: (a) Whole collapse. (b) Linear expansion.

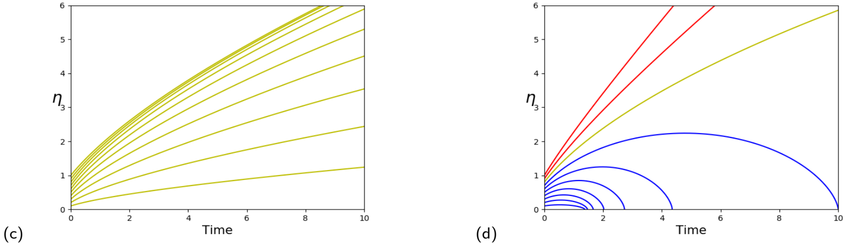

Figure: (c) Expansion with rate $\frac{2}{3}$. (d) Partial collapse and partial expansion.

We focus on the case (a), i.e., the whole collapse. In this case, all trajectories satisfy (19) and

$$\chi_1'(r) > 0 \tag{20}$$

which guarantee the adjacent trajectories do not collide before collapse. Assume that χ_1 and the enthalpy ω^α satisfy

$$\chi_1(r) = \chi_1(0) + c_1 r^n + o(r^n), \quad \omega^\alpha(r) = 1 - c_2 r^n + o(r^n) \tag{21}$$

in a neighbourhood of the center $r = 0$. The exponent $n \in \mathbb{N}$ represents the degree of flatness of the star near the center. We also assume

$$\frac{d\omega^\alpha}{dr} < 0 \tag{22}$$

for any $r \in (0, 1]$.

Theorem 3.2

For any $\gamma \in (1, \frac{4}{3})$*, there exist classical solutions* $\chi(s, r)$ *of* (8) *defined in* $\Xi = \{(s, r) | 1 - \frac{1}{t^*(r)} s > 0\}$*. The solution* $\chi(s, r)$ *behaves qualitatively like the collapsing dust solution* χ_{dust}*, i.e.,*

$$1 \lesssim |\frac{\chi}{\chi_{dust}}| \lesssim 1, \quad 1 \lesssim |\frac{\mathscr{F}[\chi]}{\mathscr{F}[\chi_{dust}]}| \lesssim 1. \tag{23}$$

Moreover, it holds that for any $r \in [0, 1]$*,*

$$\lim_{s \to t^*(r)} \frac{\chi}{\chi_{dust}} = \lim_{s \to t^*(r)} \frac{\mathscr{F}[\chi]}{\mathscr{F}[\chi_{dust}]} = 1. \tag{24}$$

Remark 3

The case (d) in Figure 2 is extremely interesting.

Outline of proof

Let

$$\tau = 1 - \frac{s}{t^*(r)} \tag{25}$$

and use the new coordinate (τ, r) instead of the original one (s, r). The operator $r\partial r$ in the new coordinate (τ, r) is denoted by Λ, and

$$\Lambda = M_g \partial_\tau + r\partial_r, \tag{26}$$

where

$$M_g(\tau, r) := (\tau - 1) r\partial_r \log(\frac{1}{t^*(r)}). \tag{27}$$

Denote $\phi(\tau, r) := \chi(s, r)$, then

$$\phi_{\tau\tau} + \frac{G(r) t^*(r)^2}{\phi^2} + \varepsilon P[\phi] = 0, \tag{28}$$

where

$$P[\phi] := \frac{\phi^2 t^*(r)^2}{\omega^\alpha r^2} \Lambda(\omega^{1+\alpha}[\phi^2(\phi + \Lambda\phi)]^{-\gamma}). \tag{29}$$

The formula of $\chi_{dust} := \phi_0$ can be rewritten as follows,

$$\phi_0 = \tau^{\frac{2}{3}} t^*(r)^{\frac{2}{3}} C(\tau, r), \tag{30}$$

and

$$C(\tau, r) \to (\frac{9G}{2})^{\frac{1}{3}}, \qquad as \quad \tau \to 0. \tag{31}$$

ϕ_0 satisfies

$$\partial_{\tau\tau}(\phi_0) + \frac{G(r)t^*(r)^2}{{\phi_0}^2} = 0. \tag{32}$$

Asymptotic form

We seek the solution ϕ of (28) in the asymptotic form

$$\phi = \phi_{app} + \theta := \sum_{j=0}^{M} \varepsilon^j \phi_j + \theta, \tag{33}$$

where M will be identified later. We expect

$$S(\phi_{app}) = -\partial_\tau^2 \phi_{app} - \frac{G(r)t^*(r)^2}{\phi_{app}^2} - \varepsilon P[\phi_{app}] = o(\varepsilon^M). \tag{34}$$

$$\partial_{\tau\tau}\phi_j - \frac{2G(r)}{C^3(\tau, r)\tau^2}\phi_j = f_j, \quad j \in \{1, \cdots, M\}, \tag{35}$$

where $f_1 = -P[\phi_0]$ and f_j depends only on $\phi_0, \phi_1, \cdots, \phi_{j-1}$.

Estimates on ϕ_j

Theorem 3.3

It holds that for non-negative integer l,

$$|\partial_\tau^m (r\partial_r)^l \phi_0| \lesssim \begin{cases} \tau^{\frac{2}{3}-m}, & l = 0, \\ \tau^{\frac{2}{3}-m} r^n, & l \geq 1. \end{cases} \quad (36)$$

Theorem 3.4

There exists a sequence $\{\phi_j\}_{j\in\{0,\cdots,M\}}$ of solutions to (35) such that for $j \in \{1, \cdots, M\}$ and $l, m \in \{0, 1, \cdots, K\}$ with large K, it holds that

$$|\partial_\tau^m (r\partial r)^l \phi_j| \leq C_{jkm} \tau^{\frac{2}{3}+j\delta-m} P_{\lambda,-\frac{2}{n}}(\frac{r^n}{\tau}), \quad (37)$$

where $P_{\mu,\nu}(x) := \frac{x^{\mu+\nu}}{(1+x)^\mu}$, $\mu, \nu \in \mathbb{R}$, $x \geq 0$, the constants C_{jkm} depend on K and M, $\delta = \delta(n) := 2(\frac{4}{3} - \gamma - \frac{1}{n}) > 0$ for large n, and $\lambda > \frac{2}{n}$.

Remainder equation

Set

$$\phi = \phi_{app} + \frac{\tau^m}{r} H, \tag{38}$$

then

$$\partial_\tau^2 H + 2\frac{g^{01}}{g^{00}}\partial_r\partial_\tau H + \frac{2m}{g^{00}}\frac{\partial_\tau H}{\tau} + \frac{d^2}{g^{00}}\frac{H}{\tau^2} - \varepsilon\gamma\frac{c[\phi]}{g^{00}}\frac{1}{\omega^\alpha}\partial_r(\omega^{1+\alpha}\frac{1}{r^2}\partial_r[r^2 H]) + \varepsilon\frac{\mathscr{N}_0[H]}{g^{00}} = \frac{1}{g^{00}}(\mathscr{S}(\phi_{app}) - \varepsilon\mathscr{L}_{low}H + \mathscr{N}[H]) \tag{39}$$

Theorem 3.5

Let $\gamma \in (1, \frac{4}{3})$ *and* m *be sufficiently large integer. Set* $N = N(\gamma) = \lfloor \frac{1}{\gamma - 1} \rfloor + 6$*. If* (21) *holds for a sufficiently large* $n = n(\gamma) \in \mathbb{Z}_{>0}$*, there exist* $\sigma_*, \varepsilon_* > 0$*,* $M = M(m, \gamma, n) \gg 1$ *and* $C_0 > 0$*, such that for any* $0 < \sigma < \sigma_*$ *and* $0 < \varepsilon < \varepsilon_*$ *the following is true: for any* $\kappa \in (0, 1)$ *and any initial data* $(H_0^\kappa, H_1^\kappa]$ *satisfying*

$$S_\kappa^N(H_0^\kappa, H_1^\kappa)(\tau = \kappa) \le \sigma^2, \tag{40}$$

there exists a unique solution $\tau \mapsto H^\kappa(\tau, \cdot)$ *to* (39) *on* $[\kappa, 1]$ *satisfying*

$$S_\kappa^N(H^\kappa, H_\tau^\kappa)(\tau) \le C_0(\sigma^2 + \varepsilon^{2M+1}), \qquad \tau \in [\kappa, 1]. \tag{41}$$

Thanks to Theorem 3.5, we can construct a solution H on $\tau \in (0,1]$ by letting $\kappa \to 0$.

Then the classical solution to (28) can be established by

$$\phi(\tau,r) = \phi_{app}(\tau,r) + \tau^m \frac{H(\tau,r)}{r} = \phi_0 + \sum_{j=1}^{M} \varepsilon^j \phi_j(\tau,r) + \tau^m \frac{H(\tau,r)}{r}$$

on the space-time domain $(\tau,r) \in (0,1] \times [0,1]$. From Theorem 3.5 and Theorem 3.4, it is easy to check that

$$1 \lesssim \left|\frac{\phi}{\phi_0}\right| \lesssim 1, \qquad 1 \lesssim \left|\frac{\mathscr{F}[\phi]}{\mathscr{F}[\phi_0]}\right| \lesssim 1$$

Moreover, for any $r \in [0,1]$,

$$\lim_{\tau\to 0} \frac{\phi}{\phi_0} = \lim_{\tau\to 0} \frac{\mathscr{F}[\phi]}{\mathscr{F}[\phi_0]} = 1. \tag{42}$$

Thank you !