On controllability of the incompressible MHD system

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Let Ω be a bounded domain of $\mathbb{R}^N$ with $N \in \{2,3\}$, and $\Gamma = \partial\Omega$. Consider the incompressible flow governed by the equations

$$\begin{cases} \partial_t \vec{\mathbf{u}} - \nu\Delta\vec{\mathbf{u}} + (\vec{\mathbf{u}}\cdot\nabla)\,\vec{\mathbf{u}} + \nabla p = \vec{\mathbf{f}} & \text{in } \Omega\times(0,T), \\ \text{div } \vec{\mathbf{u}} = 0 & \text{in } \Omega\times(0,T), \\ \text{Boundary condition of } \vec{\mathbf{u}} & \text{on } \Gamma_1\times(0,T) \end{cases} \tag{1.1}$$

Control problem:

For given initial state $\vec{\mathbf{u}}_0$, target state $\vec{\mathbf{u}}_1$, and $T > 0$, can we find

(1) the boundary control of $\vec{\mathbf{u}}$ on $\Gamma \setminus \Gamma_1$, as Γ_1 is a properly sub-boundary of Γ, such that the solution of (1.1) with $\vec{\mathbf{u}}(\cdot,0) = \vec{\mathbf{u}}_0$ at the $T-$moment reaches or approaches $\vec{\mathbf{u}}_1$ approximately, or

(2) the interior control $\vec{\mathbf{f}} = \mathbb{I}_\omega\xi$ in (1.1) for a fixed sub-domain ω of Ω, with $\Gamma_1 = \Gamma$, such that the solution of (1.1) with $\vec{\mathbf{u}}(\cdot,0) = \vec{\mathbf{u}}_0$ at the $T-$moment reaches or approaches $\vec{\mathbf{u}}_1$ approximately?

Notions of controllability

- **Small time global approximate controllability:** For any $T > 0$, any initial and target states $\vec{\mathbf{u}}_0,\ \vec{\mathbf{u}}_1$, and any small number $\delta > 0$, one can find the boundary or interior control for (1.1), such that the solution of (1.1) with $\vec{\mathbf{u}}(\cdot, 0) = \vec{\mathbf{u}}_0$ satisfies $\|\vec{\mathbf{u}}(\cdot, T) - \vec{\mathbf{u}}_1\| < \delta$;
- **Small time global exact controllability:** For any $T > 0$, and any initial and target states $\vec{\mathbf{u}}_0,\ \vec{\mathbf{u}}_1$, one can find the boundary or interior control for (1.1), such that the solution of (1.1) with $\vec{\mathbf{u}}(\cdot, 0) = \vec{\mathbf{u}}_0$ satisfies $\vec{\mathbf{u}}(\cdot, T) = \vec{\mathbf{u}}_1$;
  - **Exact null controllability:** In the above exact controllability definition, when the target state $\vec{\mathbf{u}}_1 \equiv 0$ vanishes, then we call it the exact null controllability;
  - **Local exact controllability:** In the above exact controllability definition, when the target state $\vec{\mathbf{u}}_1$ is only a small perturbation of the initial data $\vec{\mathbf{u}}_0$, then we call it the local exact controllability;
  - **Exact controllability to a trajectory:** In the above exact controllability definition, when the target state $\vec{\mathbf{u}}_1$ is the state at $T-$moment of a trajectory to the uncontrolled problem.

Lions' question (1989)

Problem:

Let $\varepsilon > 0$, $T > 0$, and two states

$$\vec{\mathbf{u}}_0, \vec{\mathbf{u}}_1 \in H_0(\Omega) = \text{Closure in } L^2(\Omega) \text{ of } \{\mathbf{f} \in C_0^\infty(\Omega), \nabla \cdot \mathbf{f} = 0\}$$

be arbitrarily fixed, is there an interior control $\xi \in L^2(\omega \times (0,T))$ and at least one Leray-Hopf weak solution $\vec{\mathbf{u}}$ to the problem (1.1) for the Navier-Stokes system with the no-slip boundary condition $\mathbf{u}|_\Gamma = 0$, which satisfies the terminal constraint

$$\|\vec{\mathbf{u}}(\cdot, T) - \vec{\mathbf{u}}_1\|_{L^2(\Omega)} < \varepsilon.$$

Main difficulties:

- No smallness assumption on $\|\vec{\mathbf{u}}_0 - \vec{\mathbf{u}}_1\|$;
- The nonslip boundary condition $\mathbf{u}|_\Gamma = 0$.

Known results on incompressible Euler flow

Consider the following problem for incompressible Euler equation:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{0}, & \text{in } \Omega \times (0,T), \\ \operatorname{div}(\mathbf{u}) = 0, & \text{in } \Omega \times (0,T), \\ \mathbf{u}\cdot\mathbf{n} = 0, & \text{on } (\Gamma \setminus \Gamma_c) \times (0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, & \text{in } \Omega \\ \mathbf{u}(\cdot,T) = \mathbf{u}_T, & \text{in } \Omega. \end{cases} \tag{1.2}$$

Controllability in this case has been comprehensively studied, for instance

- ⋆ (1993) J.-M. Coron, *C. R. Acad. Sci. Paris Sér. I Math.* (2d case, simply connected)
- ⋆ (1996) J.-M. Coron, *J. Math. Pures Appl.* (2d case, multi-connected)
- ⋆ (2000) O. Glass, *ESAIM:Cocv.* (3d case, multi-connected)
- ⋆ (2016) E. Fernndez-Cara et al., *Math. Control Signals Syst.* (with Boussinesq heat effects, simply connected 2d and 3d)

Known results on Navier-Stokes equations

- J. M. Coron, F. Marbach, F. Sueur: JEMS 2020, Global exact controllability of NS with Navier condition.
- J. M. Coron, F. Marbach, F. Sueur, P. Zhang: Annals of PDE 2019, Exact controllability of NS in a rectangle with non-slip BCs on the top and bottom boundaries with a little help of a force.

Known results on viscous MHD

For the problem of MHD with viscosity and magnetic resistivity, BCs being nonslip for velocity and Navier for magnetic field, with an internal control:

- V. Barbu et al., *Ad. Diff. Equ.* (2005): local exact controllability for a steady state.
- T. Havarneanu et al., *SIAM J. Control Optim.* (2007): local exact controllability for a sufficient regular state.
- M. Badra, *JMFM* (2014): local exact controllability for a general state.

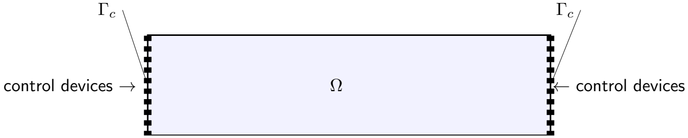
Question: Does the controllability hold globally for the viscous MHD or the ideal MHD, with the control being located on the boundary?

The ideal incompressible MHD system

We consider the following ideal MHD problem:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \mu(\mathbf{H}\cdot\boldsymbol{\nabla})\mathbf{H} + \boldsymbol{\nabla} p = \mathbf{0}, & \text{in } \Omega\times(0,T), \\ \partial_t \mathbf{H} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{H} - (\mathbf{H}\cdot\boldsymbol{\nabla})\mathbf{u} = \mathbf{0}, & \text{in } \Omega\times(0,T), \\ \operatorname{div}(\mathbf{u}) = \operatorname{div}(\mathbf{H}) = 0, & \text{in } \Omega\times(0,T), \\ \mathbf{u}\cdot\mathbf{n} = \mathbf{H}\cdot\mathbf{n} = 0, & \text{on } (\Gamma\setminus\Gamma_c)\times(0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0,\ \mathbf{H}(\cdot,0) = \mathbf{H}_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

where $\mathbf{u} = (u_1, u_2)'$ is the velocity field, $\mathbf{H} = (H_1, H_2)'$ magnetic field, $p \in \mathbb{R}$ pressure, and $\Omega = (0,L)\times(0,W)$, the controlled part of the boundary $\Gamma_c = (\{0\}\times(0,W)) \cup (\{L\}\times(0,W))$.



The controllability goal

Question: Fix the following:

- A time $T > 0$.
- Initial data $\mathbf{u}_0$ and $\mathbf{H}_0$.
- Final data $\mathbf{u}_T$ and $\mathbf{H}_T$.

Do there exist boundary controls $\mathbf{g}(\mathbf{x}, t)$ and $\mathbf{h}(\mathbf{x}, t)$ such that prescribing

$$M_1(\mathbf{u})_{|\Gamma_c} = \mathbf{g} \text{ and } M_2(\mathbf{H})_{|\Gamma_c} = \mathbf{h} \text{ along } \Gamma_c$$

implies that the solution $(\mathbf{u}, \mathbf{H}, p)$ with initial data $\mathbf{u}_0$ and $\mathbf{H}_0$ satisfies

$$\mathbf{u}(\cdot, T) = \mathbf{u}_T(\cdot), \, \mathbf{H}(\cdot, T) = \mathbf{H}_T(\cdot) \tag{2.2}$$

?

The controllability goal

Working directly with specific boundary conditions on Γ_c is too difficult.

$\mathbf{Question}'$: For each fixed choice of $T > 0$ and initial data $(\mathbf{u}_0, \mathbf{H}_0)$, does there exist a solution $(\mathbf{u}, \mathbf{H}, p)$ to the under-determined problem (2.1) such that

$$\mathbf{u}(\cdot, T) = \mathbf{u}_T,\ \mathbf{H}(\cdot, T) = \mathbf{H}_T \text{ in } \Omega \quad ?$$

If the answer is yes: Inspect $\mathbf{u}_{|\Gamma_c}$, $\mathbf{H}_{|\Gamma_c}$ and choose $\mathbf{g} = M_1(\mathbf{u})_{|\Gamma_c}$, $\mathbf{h} = M_2(\mathbf{H})_{|\Gamma_c}$.

A transformed problem for the control prob. (2.1)-(2.2)

Let $\mathbf{z}^+ := \mathbf{u} + \sqrt{\mu}\mathbf{H}$ and $\mathbf{z}^- := \mathbf{u} - \sqrt{\mu}\mathbf{H}$, Elsässer variables, the control problem (2.1)-(2.2) can be transformed into the following one:

$$\begin{cases} \partial_t \mathbf{z}^+ + (\mathbf{z}^- \cdot \boldsymbol{\nabla})\mathbf{z}^+ + \boldsymbol{\nabla} p^+ = \mathbf{0}, & \text{in } \Omega \times (0,T), \\ \partial_t \mathbf{z}^- + (\mathbf{z}^+ \cdot \boldsymbol{\nabla})\mathbf{z}^- + \boldsymbol{\nabla} p^- = \mathbf{0}, & \text{in } \Omega \times (0,T), \\ \mathrm{div}(\mathbf{z}^+) = \mathrm{div}(\mathbf{z}^-) = 0, & \text{in } \Omega \times (0,T), \\ \mathbf{z}^+ \cdot \mathbf{n} = \mathbf{z}^- \cdot \mathbf{n} = 0, & \text{on } (\Gamma \setminus \Gamma_c) \times (0,T), \\ \mathbf{z}^\pm(\cdot,0) = \mathbf{z}_0^\pm := \mathbf{u}_0 \pm \sqrt{\mu}\mathbf{H}_0, & \text{in } \Omega, \\ \mathbf{z}^\pm(\cdot,T) = \mathbf{z}_T^\pm := \mathbf{u}_T \pm \sqrt{\mu}\mathbf{H}_T, & \text{in } \Omega. \end{cases} \tag{2.3}$$

Obviously, (2.3) is equivalent to (2.1)-(2.2) as long as $\boldsymbol{\nabla} p^+ = \boldsymbol{\nabla} p^-$.

By taking the divergence in the equation for $\mathbf{H}$ and also by multiplying with $\mathbf{n}$ along Γ, one can obtain that $q := (2\sqrt{\mu})^{-1}(p^+ - p^-)$ is harmonic satisfying

$$\begin{cases} \Delta q = 0, \quad \text{in} \quad \Omega \times (0,T), \\ \partial_{\mathbf{n}} q|_{\Gamma_c} = -\operatorname{sign}(n_1)\Big(\partial_t H_1 + (u_1\partial_1 + u_2\partial_2)H_1 - (H_1\partial_1 + H_2\partial_2)u_1\Big) \\ \partial_{\mathbf{n}} q|_{\Gamma\setminus\Gamma_c} = 0. \end{cases} \tag{2.4}$$

Main results (ESAIM:COCV, 2021)

Theorem 1: Let the integer $m \geq 3$ and the control time $T > 0$ be fixed. Denote by

$$C^{m,\alpha}_{\sigma,\Gamma_c}(\overline{\Omega};\mathbb{R}^2) := \{\mathbf{f} \in C^{m,\alpha}(\overline{\Omega};\mathbb{R}^2) \mid \operatorname{div}(\mathbf{f}) = 0 \text{ in } \Omega, \mathbf{f}\cdot\mathbf{n} = 0 \text{ on } \Gamma\setminus\Gamma_c\}.$$

Then, for all initial- and final data $(\mathbf{z}_0^+, \mathbf{z}_0^-, \mathbf{z}_T^+, \mathbf{z}_T^-) \in C^{m,\alpha}_{\sigma,\Gamma_c}(\overline{\Omega};\mathbb{R}^2)$, there exists a solution $(\mathbf{z}^+, \mathbf{z}^-, p^+, p^-)$ to the control problem (2.3) such that

$$(\mathbf{z}^+, \mathbf{z}^-) \in \left[C^0([0,T]; C^{1,\alpha}(\overline{\Omega};\mathbb{R}^2)) \cap L^\infty([0,T]; C^{m,\alpha}(\overline{\Omega};\mathbb{R}^2))\right]^2.$$

Returning the original system of incompressible ideal MHD (2.1)-(2.2), we have the following small-time global exact controllability result.

Theorem 2: Let the integer $m \geq 3$ and $T > 0$ be fixed. Then, for all initial- and final data $(\mathbf{u}_0, \mathbf{H}_0, \mathbf{u}_T, \mathbf{H}_T) \in C^{m,\alpha}_{\sigma,\Gamma_c}(\overline{\Omega}; \mathbb{R}^2)^4$, the problem

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \mu(\mathbf{H} \cdot \boldsymbol{\nabla})\mathbf{H} + \boldsymbol{\nabla} p = \mathbf{0}, & \text{in } \Omega \times (0,T), \\ \partial_t \mathbf{H} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{H} - (\mathbf{H} \cdot \boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} q = \mathbf{0}, & \text{in } \Omega \times (0,T), \\ \operatorname{div}(\mathbf{u}) = \operatorname{div}(\mathbf{H}) = 0, & \text{in } \Omega \times (0,T), \\ \mathbf{u} \cdot \mathbf{n} = \mathbf{H} \cdot \mathbf{n} = 0, & \text{on } (\Gamma \setminus \Gamma_c) \times (0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0,\ \mathbf{H}(\cdot,0) = \mathbf{H}_0 & \text{in } \Omega \end{cases} \tag{2.5}$$

with

$$\mathbf{u}(\cdot,T) = \mathbf{u}_T,\ \mathbf{H}(\cdot,T) = \mathbf{H}_T \quad \text{in} \quad \Omega, \tag{2.6}$$

has a solution $(\mathbf{u}, \mathbf{H}, p, q)$, with

$$(\mathbf{u}, \mathbf{H}) \in \left[C^0([0,T]; C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2)) \cap L^\infty([0,T]; C^{m,\alpha}(\overline{\Omega}; \mathbb{R}^2))\right]^2$$

and $q(\cdot,t)$ being for each $t \in [0,T]$ a harmonic function given by (2.4).

A local null controllability result

By choosing $T = 1$ and $\mathbf{z}_T^{\pm} = \mathbf{0}$, and applying the curl operator in (2.3), one obtains the following control problem

$$\begin{cases} \partial_t j^+ + (\mathbf{z}^- \cdot \boldsymbol{\nabla}) j^+ = g^+, & \text{in } \Omega \times (0,1), \\ \partial_t j^- + (\mathbf{z}^+ \cdot \boldsymbol{\nabla}) j^- = g^-, & \text{in } \Omega \times (0,1), \\ \operatorname{curl}(\mathbf{z}^{\pm}) = j^{\pm}, \quad \operatorname{div}(\mathbf{z}^{\pm}) = 0, & \text{in } \Omega \times (0,1), \\ \mathbf{z}^+ \cdot \mathbf{n} = \mathbf{z}^- \cdot \mathbf{n} = 0, & \text{on } (\Gamma \setminus \Gamma_c) \times (0,1), \\ j^+(\cdot, 0) = \operatorname{curl}(\mathbf{z}_0^+),\ j^-(\cdot, 0) = \operatorname{curl}(\mathbf{z}_0^-) & \text{in } \Omega, \\ j^+(\cdot, 1) = 0,\ j^-(\cdot, 1) = 0 & \text{in } \Omega \\ \mathbf{z}^+(\cdot, 0) = \mathbf{z}_0^+,\ \mathbf{z}^-(\cdot, 0) = \mathbf{z}_0^-, & \text{in } \Omega, \\ \mathbf{z}^+(\cdot, 1) = \mathbf{0},\ \mathbf{z}^-(\cdot, 1) = \mathbf{0}, & \text{in } \Omega, \end{cases} \tag{2.7}$$

where

$$g^{\pm} := \quad \partial_2 z_1^{\mp} \partial_1 z_1^{\pm} + \partial_2 z_2^{\mp} \partial_2 z_1^{\pm} - \partial_1 z_1^{\mp} \partial_1 z_2^{\pm} - \partial_1 z_2^{\mp} \partial_2 z_2^{\pm}. \tag{2.8}$$

A local null controllability result

The following local null controllability result is the main step for proving Theorems 1 and 2.

Proposition 3: Let $m \geq 3$ be fixed. There exists a constant $\tilde{s} > 0$, such that if the initial data $(\mathbf{z}_0^+, \mathbf{z}_0^-) \in C^{m,\alpha}_{\sigma,\Gamma_c}(\overline{\Omega}; \mathbb{R}^2)^2$ satisfy the constraint

$$\max\{\|\mathbf{z}_0^+\|_{m,\alpha,\Omega}, \|\mathbf{z}_0^-\|_{m,\alpha,\Omega}\} < \tilde{s},$$

then the problem (2.7) admits a solution $(\mathbf{z}^+, \mathbf{z}^-, j^+, j^-)$ of regularity

$$\begin{aligned}(\mathbf{z}^+, \mathbf{z}^-, j^+, j^-) \in \quad & \left[C^0([0,1]; C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2)) \cap L^\infty([0,1]; C^{m,\alpha}(\overline{\Omega}; \mathbb{R}^2))\right]^2 \\ & \times \left[C^0([0,1]; C^{0,\alpha}(\overline{\Omega})) \cap L^\infty([0,1]; C^{m-1,\alpha}(\overline{\Omega}))\right]^2,\end{aligned}$$

with $\mathbf{z}^+(\mathbf{x}, 1) = \mathbf{z}^-(\mathbf{x}, 1) = \mathbf{0}$ for all $\mathbf{x} \in \Omega$.

Proof of Theorem 2

(1) Assuming that Proposition 3 is true, we show now how to deduce Theorems 1 and 2 with the help of a scaling and gluing argument, as for instance in [Coron 1993, 1995] for the Euler equation.
Note that if $(\mathbf{u}, \mathbf{H}, p, q)$ solve (2.5), then this is also true for $(\hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{p}, \hat{q})$ defined by

$$\begin{aligned} \hat{\mathbf{u}}(\mathbf{x}, t) &:= -\mathbf{u}(\mathbf{x}, T-t), \\ \hat{\mathbf{H}}(\mathbf{x}, t) &:= -\mathbf{H}(\mathbf{x}, T-t), \\ \hat{p}(\mathbf{x}, t) &:= p(\mathbf{x}, T-t), \\ \hat{q}(\mathbf{x}, t) &:= q(\mathbf{x}, T-t). \end{aligned} \tag{2.9}$$

(2) Next, split $[0, T]$ into $[0, T/2] \cap [T/2, 1]$, choose $0 < \epsilon < T/2$ small such that $\tilde{\mathbf{u}}_0 := \epsilon \mathbf{u}_0$, $\tilde{\mathbf{H}}_0 := \epsilon \mathbf{H}_0$ and $\tilde{\mathbf{u}}_T := \epsilon \mathbf{u}_T$, $\tilde{\mathbf{H}}_T := \epsilon \mathbf{H}_T$ satisfy

$$\begin{cases} \max\left\{ \|\tilde{\mathbf{u}}_0 + \sqrt{\mu}\tilde{\mathbf{H}}_0\|_{m,\alpha,\Omega},\ \|\tilde{\mathbf{u}}_0 - \sqrt{\mu}\tilde{\mathbf{H}}_0\|_{m,\alpha,\Omega} \right\} < \tilde{s} \\ \max\left\{ \|\tilde{\mathbf{u}}_T + \sqrt{\mu}\tilde{\mathbf{H}}_T\|_{m,\alpha,\Omega},\ \|\tilde{\mathbf{u}}_T - \sqrt{\mu}\tilde{\mathbf{H}}_T\|_{m,\alpha,\Omega} \right\} < \tilde{s}, \end{cases}$$

where $\tilde{s} > 0$ is the small constant given in Proposition 3.

Proof of Theorem 2 (cont.)

By applying Proposition 3 with $T = 1$, we get solutions $(\mathbf{u}^*, \mathbf{H}^*, p^*, q^*)$ and $(\mathbf{u}^{**}, \mathbf{H}^{**}, p^{**}, q^{**})$ of (2.5), obeying

$$\begin{cases} (\mathbf{u}^*, \mathbf{H}^*)(\cdot, 0) & = (\mathbf{u}_0(\cdot), \mathbf{H}_0(\cdot)), \\ (\mathbf{u}^*, \mathbf{H}^*, p^*, q^*)(\cdot, 1) & \equiv (\mathbf{0}, \mathbf{0}, 0, 0), \\ (\mathbf{u}^{**}, \mathbf{H}^{**})(\cdot, 0) & = -(\mathbf{u}_T(\cdot), H_T(\cdot)), \\ (\mathbf{u}^{**}, \mathbf{H}^{**}, p^{**}, q^{**})(\cdot, 1) & \equiv (\mathbf{0}, \mathbf{0}, 0, 0). \end{cases}$$

(3) Now, define

$$\begin{cases} (\mathbf{u}^a, \mathbf{H}^a, p^a, q^a)(\mathbf{x}, t) := \left(\epsilon^{-1}\mathbf{u}^*, \epsilon^{-1}\mathbf{H}^*, \epsilon^{-2}p^*, \epsilon^{-2}q^*\right)(\mathbf{x}, \epsilon^{-1}t), & \Omega \times [0, \epsilon], \\ (\mathbf{u}^a, \mathbf{H}^a, p^a, q^a)(\mathbf{x}, t) := (\mathbf{0}, \mathbf{0}, 0, 0), & \Omega \times [\epsilon, T/2], \end{cases}$$

as well as

$$\begin{cases} \left(\mathbf{u}^b, \mathbf{H}^b\right)(\mathbf{x}, t) := -\left(\epsilon^{-1}\mathbf{u}^{**}, \epsilon^{-1}\mathbf{H}^{**}\right)(\mathbf{x}, \epsilon^{-1}(T-t)), & \Omega \times [T-\epsilon, T], \\ \left(p^b, q^b\right)(\mathbf{x}, t) := \left(\epsilon^{-2}p^{**}, \epsilon^{-2}q^{**}\right)(\mathbf{x}, \epsilon^{-1}(T-t)), & \Omega \times [T-\epsilon, T], \\ \left(\mathbf{u}^b, \mathbf{H}^b, p^b, q^b\right)(\mathbf{x}, t) := (\mathbf{0}, \mathbf{0}, 0, 0), & \Omega \times [T/2, T-\epsilon], \end{cases}$$

Proof of Theorem 2 (cont.)

Then, the functions

$$\mathbf{u}(\mathbf{x},t) := \begin{cases} \mathbf{u}^a(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [0,T/2], \\ \mathbf{u}^b(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [T/2,T], \end{cases}$$

$$\mathbf{H}(\mathbf{x},t) := \begin{cases} \mathbf{H}^a(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [0,T/2], \\ \mathbf{H}^b(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [T/2,T], \end{cases}$$

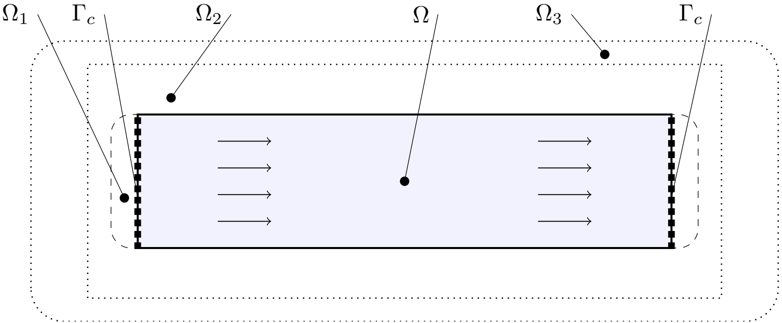
$$p(\mathbf{x},t) := \begin{cases} p^a(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [0,T/2], \\ p^b(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [T/2,T], \end{cases}$$

$$q(\mathbf{x},t) := \begin{cases} q^a(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [0,T/2], \\ q^b(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [T/2,T], \end{cases}$$

are solutions of the control problem (2.5)-(2.6).

Proof of Proposition 3

Similar to [Coron, 1993] for the Euler equation, we introduce three extensions of the domain Ω as



where

$$\Omega_2 := (-l, L+l) \times (-l, W+l),$$

for a positive constant $l > 0$, and $\overline{\Omega} \subseteq \overline{\Omega}_1 \subseteq \Omega_2$, $\overline{\Omega}_2 \subseteq \Omega_3$, with $\Omega_1 \subseteq \{\mathbf{x} = (x_1, x_2)' \in \mathbb{R}^2 \,|\, 0 \leq x_2 \leq W\}$.

Construction of a special flow

- Define
$$\mathbf{y}^*(\mathbf{x},t) = \begin{pmatrix} \gamma(t)\chi(\mathbf{x}) \\ 0 \end{pmatrix}, \quad (\mathbf{x},t) \in \overline{\Omega}_3 \times [0,1],$$
with $\gamma \in C_0^\infty(0,1)$ being non-negative and $\gamma(t) > M$ as $t \in [\frac{1}{4}, \frac{3}{4}]$, for a large $M > 0$, and $\chi \in C_0^\infty(\overline{\Omega_3})$ a cutoff function satisfying $\chi(\mathbf{x}) = 1$ for $\mathbf{x} \in \overline{\Omega}_2$.
- The functions $(\overline{\mathbf{y}}, \overline{\mathbf{H}}, \overline{p}, \overline{q})$ defined by
$$\begin{cases} \overline{\mathbf{y}}(\mathbf{x},t) := \mathbf{y}^*(x,t), & (\mathbf{x},t) \in \Omega \times [0,1], \\ \overline{\mathbf{H}}(\mathbf{x},t) := \mathbf{0}, & (\mathbf{x},t) \in \Omega \times [0,1], \\ \overline{p}(\mathbf{x},t) := -x_1 \frac{d}{dt}\gamma(t), & (\mathbf{x},t) \in \Omega \times [0,1], \\ \overline{q}(\mathbf{x},t) := 0, & (\mathbf{x},t) \in \Omega \times [0,1], \end{cases} \tag{2.10}$$
with $\mathbf{x} = (x_1, x_2)'$, are solutions of (2.1) with $T = 1$ and the data $\mathbf{u}_0 = \mathbf{u}_T = \mathbf{H}_0 = \mathbf{H}_T = \mathbf{0}$.

For $(\mathbf{x}, s, t) \in \overline{\Omega}_3 \times [0,1] \times [0,1]$, denote by $\boldsymbol{\mathcal{Y}}(\mathbf{x}, s, t)$ the flow defined by

$$\begin{cases} \frac{d}{dt}\boldsymbol{\mathcal{Y}}(\mathbf{x}, s, t) = \mathbf{y}^*(\boldsymbol{\mathcal{Y}}(\mathbf{x}, s, t), t), \\ \boldsymbol{\mathcal{Y}}(\mathbf{x}, s, s) = \mathbf{x}. \end{cases}$$

It is easy to have

Lemma 1: The constant $M > 0$ can be chosen large enough, such that $\boldsymbol{\mathcal{Y}}(\mathbf{x}, 0, 1) \notin \overline{\Omega}_2$ for all $\mathbf{x} \in \overline{\Omega}_2$.

Lemma 2: For any given $\mathbf{z} \in C^0(\overline{\Omega} \times [0,1]; \mathbb{R}^2)$, denote by the extension $\mathfrak{z} := \mathbf{y}^* + \pi(\mathbf{z} - \overline{\mathbf{y}})$, with π the continuous extension operator $C^0(\overline{\Omega}) \to C^0(\overline{\Omega}_3)$, $\operatorname{supp}(\pi(\mathbf{f})) \subseteq \Omega_2$, let $\boldsymbol{\mathcal{Z}}(\mathbf{x}, s, t)$ be defined by

$$\begin{cases} \frac{d}{dt}\boldsymbol{\mathcal{Z}}(\mathbf{x}, s, t) = \mathfrak{z}(\boldsymbol{\mathcal{Z}}(\mathbf{x}, s, t), t), \\ \boldsymbol{\mathcal{Z}}(\mathbf{x}, s, s) = \mathbf{x}. \end{cases}$$

Then, there exists a small $\nu > 0$, such that $\|\mathbf{z} - \overline{\mathbf{y}}\|_{C^0(\overline{\Omega} \times [0,1])} < \nu$ implies $\boldsymbol{\mathcal{Z}}(\mathbf{x}, 0, 1) \notin \overline{\Omega}_2$ for all $\mathbf{x} \in \overline{\Omega}_2$.

Proof of Proposition 3: The local controllability result given in Proposition 3 is proved by using a fixed point argument for problem (2.7) in a small neighborhood of the special flow (2.10).

Remark:

(1) I.Kukavica, M. Novack, V. Vicol (JDE 2022) continued to study the same problem for investigating when the extra-force ∇q vanishes in the magnetic equation of the control problem (2.5)-(2.6).

(2) Recently, M. Rissel (arXiv: 2306.03712) extends this study to a general simply-connected domain in 2D.

The viscous incompressible MHD system

Next, we consider the following problem in $\Omega \subset \mathbb{R}^2$, with viscosity $\nu_1 > 0$ and resistivity $\nu_2 > 0$:

$$\begin{cases} \partial_t \mathbf{u} - \nu_1 \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mu(\mathbf{B} \cdot \nabla)\mathbf{B} + \nabla p = \mathbf{0} & \text{in } \Omega \times (0, T_{\mathrm{ctrl}}), \\ \partial_t \mathbf{B} - \nu_2 \Delta \mathbf{B} + (\mathbf{u} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{u} = \mathbf{0} & \text{in } \Omega \times (0, T_{\mathrm{ctrl}}), \\ \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0 & \text{in } \Omega \times (0, T_{\mathrm{ctrl}}), \\ \mathbf{u} \cdot \vec{\mathbf{n}} = 0, \; (\nabla \times \mathbf{u}) \times \vec{\mathbf{n}} = \left[M_1 \vec{\mathbf{u}} + L_1 \vec{\mathbf{B}} \right]_\tau, & \text{on } (\Gamma \setminus \Gamma_{\mathrm{c}}) \times (0, T_{\mathrm{ctrl}}), \\ \mathbf{B} \cdot \vec{\mathbf{n}} = 0, \; (\nabla \times \mathbf{B}) \times \vec{\mathbf{n}} = \left[M_2 \vec{\mathbf{u}} + L_2 \vec{\mathbf{B}} \right]_\tau, & \text{on } (\Gamma \setminus \Gamma_{\mathrm{c}}) \times (0, T_{\mathrm{ctrl}}), \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0, \, \mathbf{B}(\cdot, 0) = \mathbf{B}_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

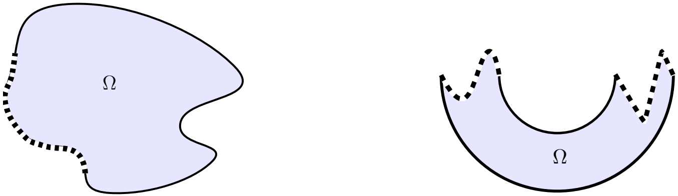
where $\vec{\mathbf{n}}$ is the unit outward normal vector on Γ, and the notation $[\mathbf{h}]_\tau = \mathbf{h} - (\mathbf{h} \cdot \vec{\mathbf{n}})\,\vec{\mathbf{n}}$ and friction matrices

$$L_1, L_2, M_1, M_2 \in C^\infty(\Gamma \setminus \Gamma_{\mathrm{c}}; \mathbb{R}^{2\times 2}).$$

Theorem 3 (Manuel & W.: arXiv 2203:10758): Assume one of the following configurations

(a) Ω is a bounded simply-connected subdomain of $\mathbb{R}^2$, and the open subset $\Gamma_c \subseteq \Gamma$ is connected. The friction coefficient matrices satisfy $L_1, L_2, M_1 \in C^\infty(\Gamma \setminus \Gamma_c; \mathbb{R}^{2\times 2})$ and $M_2 \equiv 0$.

(b) For $r_2 > r_1 > 0$ and $D_r = \{x \in \mathbb{R}^2 \,|\, |x| < r\}$, the domain $\Omega \subseteq A_{r_1}^{r_2} = D_{r_2} \setminus \overline{D_{r_1}}$ is simply-connected and bounded by a closed Lipschitz curve Γ, while the controlled part is $\Gamma_c = \Gamma \setminus \partial A_{r_1}^{r_2}$. The friction coefficient matrices $M_1, L_1, L_2 \in C^\infty$ are arbitrary and $M_2 = \rho I_{2\times 2}$, for $\rho \in \mathbb{R}$.



(a) A planar simply-connected domain with Γ_c being connected.

(b) An annulus section where the controls act along the cuts.

Figure: The controls act along the dashed boundaries which represent Γ_c.

Then, for any given initial states $\mathbf{u}_0, \mathbf{B}_0 \in L^2_{\mathrm{c}}(\Omega)$, target states $\mathbf{u}_1, \mathbf{B}_1 \in Ł^2_{\mathrm{c}}(\Omega)$, $T_{\mathrm{ctrl}} > 0$, and $\delta > 0$, there exists at least one weak controlled trajectory

$$(\mathbf{u}, \mathbf{B}) \in \left[C^0_w([0, T_{\mathrm{ctrl}}]; L^2_{\mathrm{c}}(\Omega)) \cap L^2((0, T_{\mathrm{ctrl}}); H^1(\Omega))\right]^2$$

to the MHD equations (3.1) which obeys the terminal condition

$$\|\mathbf{u}(\cdot, T_{\mathrm{ctrl}}) - \mathbf{u}_1\|_{L^2(\Omega)} + \|\mathbf{B}(\cdot, T_{\mathrm{ctrl}}) - \mathbf{B}_1\|_{L^2(\Omega)} < \delta, \qquad (3.2)$$

where $L^2_c(\Omega) = \left\{\mathbf{f} \in L^2(\Omega) : \ \nabla \cdot \mathbf{f} = 0 \text{ in } \Omega, \mathbf{f} \cdot \vec{\mathbf{n}} = 0 \text{ on } \Gamma \setminus \Gamma_{\mathrm{c}}\right\}$.

Remark: For the general case that Ω is a multi-connected domain of $\mathbb{R}^N$ with $N = 2, 3$, to have the similar small time global approximate controllability as stated in the above theorem, one needs to modify the second equation of (3.1) as

$$\partial_t \mathbf{B} - \nu_2 \Delta \mathbf{B} + (\mathbf{u} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{u} = \nabla q + \xi, \quad \text{in } \Omega \times (0, T_{\mathrm{ctrl}}),$$

with $\xi = 0$ when the friction matrix $M_2 = 0$ in (3.1).

Idea of the proof (domain extension)

For simplicity: Consider 2-D simply-connected case with $\mathbf{u}_1 = \mathbf{B}_1 = \mathbf{0}$ (null controllability)

Let $\mathcal{E} \subseteq \mathbb{R}^N$ be a smoothly bounded domain with

$$\Omega \subseteq \mathcal{E}, \quad \Gamma_c \subseteq \overline{\mathcal{E}}, \quad \Gamma_c \cap \mathcal{E} \neq \emptyset, \quad \Gamma \setminus \Gamma_c \subseteq \partial\mathcal{E}$$

Also introduce

- extended initial defined in $\mathcal{E}$,
- extended friction coefficient matrices $M_1, M_2, L_1, L_2 \in C^\infty(\overline{\mathcal{E}}, \mathbb{R}^{N\times N})$

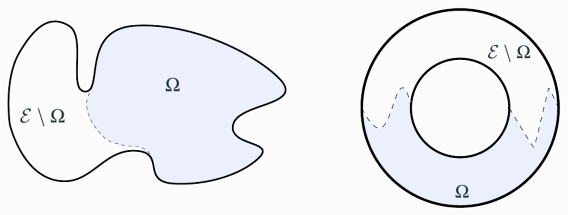


Figure: Domain extensions.

Idea of the proof (weak controlled trajectories)

A weak controlled trajectory is any pair

$$(\mathbf{u}, \mathbf{B}) \in \left[C^0_w([0,T]; L^2_c(\Omega)) \cap L^2((0,T); H^1(\Omega))\right]^2,$$

which is the restriction to Ω of a Leray-Hopf weak solution to

$$\begin{cases} \partial_t \mathbf{u} - \nu_1 \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mu(\mathbf{B}\cdot\nabla)\mathbf{B} + \nabla p = \xi & \text{in } \mathcal{E}\times(0,T), \\ \partial_t \mathbf{B} - \nu_2 \Delta \mathbf{B} + (\mathbf{u}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{u} + \nabla q = \eta & \text{in } \mathcal{E}\times(0,T), \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{B} = 0 & \text{in } \mathcal{E}\times(0,T), \\ (\nabla\times\mathbf{u})\times\vec{\mathbf{n}} = [M_1\mathbf{u} + L_1\mathbf{B}]_\tau,\ \ \mathbf{u}\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial\mathcal{E}\times(0,T), \\ (\nabla\times\mathbf{B})\times\vec{\mathbf{n}} = [M_2\mathbf{u} + L_2\mathbf{B}]_\tau,\ \ \mathbf{B}\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial\mathcal{E}\times(0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0,\ \mathbf{B}(\cdot,0) = \mathbf{B}_0 & \text{in } \mathcal{E}. \end{cases}$$

with external forces ξ, η supported in $\overline{\mathcal{E}} \setminus \overline{\Omega}$.

Idea of the proofs (asymptotic expansions)

Apply the transformations

$$\mathbf{z}^{\pm} = \mathbf{u} \pm \sqrt{\mu}\mathbf{B}, \quad p^{\pm} = p \pm \sqrt{\mu}q, \quad \lambda^{\pm} = \frac{\nu_1 \pm \nu_2}{2}, \quad \xi^{\pm} = \xi \pm \sqrt{\mu}\eta.$$

Scaling: for any positive $\epsilon \lll 1$ define

$$\mathbf{z}^{\pm,\epsilon}(x,t) = \epsilon \mathbf{z}^{\pm}(x,\epsilon t), \quad p^{\pm,\epsilon}(x,t) = \epsilon^2 p^{\pm}(x,\epsilon t), \quad \xi^{\pm,\epsilon}(x,t) = \epsilon^2 \xi^{\pm}(x,\epsilon t),$$

which satisfy in $\mathcal{E}_{\frac{T}{\epsilon}} = \mathcal{E} \times (0, \frac{T}{\epsilon})$ a problem of the form

$$\begin{cases} \partial_t \mathbf{z}^{\pm,\epsilon} - \epsilon \Delta(\lambda^{\pm}\mathbf{z}^{+,\epsilon} + \lambda^{\mp}\mathbf{z}^{-,\epsilon}) + (\mathbf{z}^{\mp,\epsilon}\cdot\nabla)\mathbf{z}^{\pm,\epsilon} + \nabla p^{\pm,\epsilon} = \xi^{\pm,\epsilon} & \text{in } \mathcal{E}_{\frac{T}{\epsilon}}, \\ \nabla \cdot \mathbf{z}^{\pm,\epsilon} = 0 & \text{in } \mathcal{E}_{\frac{T}{\epsilon}}, \\ \mathbf{z}^{\pm,\epsilon}\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial\mathcal{E}_{\frac{T}{\epsilon}}, \\ (\nabla \times \mathbf{z}^{\pm,\epsilon}) \times \mathbf{n} = [M^{\pm}\mathbf{z}^{+,\epsilon} + L^{\pm}\mathbf{z}^{-,\epsilon}]_{\tau} & \text{on } \partial\mathcal{E}_{\frac{T}{\epsilon}}, \\ \mathbf{z}^{\pm,\epsilon}(\cdot,0) = \epsilon \mathbf{z}_0^{\pm} = \epsilon(\mathbf{u}_0 \pm \sqrt{\mu}\mathbf{B}_0) & \text{in } \mathcal{E}, \end{cases}$$

where $M^{\pm}, L^{\pm}$ are determined from M_1, M_2, L_1, L_2.

Idea of the proofs (asymptotic expansions)

Goal: choose $\xi^{\pm,\epsilon}$ such that

$$\|\mathbf{z}^{\pm,\epsilon}(\cdot, T/\epsilon)\|_{L^2(\mathcal{E})} = O(\epsilon^{\frac{9}{8}}), \text{ as } \epsilon \longrightarrow 0. \tag{3.3}$$

Then: for $\epsilon = \epsilon(\delta) > 0$ sufficiently small one has

$$\|\mathbf{u}(\cdot, T)\|_{L^2(\mathcal{E})} + \|\mathbf{B}(\cdot, T)\|_{L^2(\mathcal{E})} = O(\epsilon^{\frac{1}{8}}) < \delta.$$

Ansatz: ($d(x) = \mathrm{dist}(x, \partial\mathcal{E})$)

$$\begin{cases} \mathbf{z}^{\pm,\epsilon} = \mathbf{y}^* + \sqrt{\epsilon}\mathbf{v}^{\pm}(x, t, d(x)/\sqrt{\epsilon}) + \epsilon\mathbf{z}^{\pm,1} + \text{technical profiles} + \epsilon\mathbf{r}^{\pm,\epsilon}, \\ p^{\pm,\epsilon} = p^* + \epsilon p^{\pm,1} + \text{technical profiles} + \epsilon\pi^{\pm,\epsilon}, \\ \xi^{\pm,\epsilon} = \xi^* + \sqrt{\epsilon}\mu^{\pm}(x, t; d(x)/\sqrt{\epsilon}) + \epsilon\xi^{\pm,1} + \epsilon\widetilde{\zeta}^{\pm,\epsilon}. \end{cases}$$

Main ingredients:

- flushing profile $(\mathbf{y}^*, p^*, \xi^*)$ solving controlled incompressible Euler problem,
- $(\mathbf{z}^{\pm,1}, p^{\pm,1}, \xi^{\pm,1})$ solving linearized ideal MHD type controllability problems,
- $\mathbf{v}^{\pm}$ solving linearized Prandtl type problem with controls $\mu^{\pm}$.

The zero order profiles $(\mathbf{y}^*, p^*, \xi^*)$ are chosen for $t \in [0, T]$ as a special solution to the controlled Euler system

$$\begin{cases} \partial_t \mathbf{y}^* + (\mathbf{y}^* \cdot \nabla)\mathbf{y}^* + \nabla p^* = \xi^* & \text{in } \mathcal{E}_T, \\ \nabla \cdot \mathbf{y}^* = \sigma^* & \text{in } \mathcal{E}_T, \\ \mathbf{y}^* \cdot \vec{\mathbf{n}} = 0 & \text{on } \Sigma_T, \\ \mathbf{y}^*(\cdot, 0) = \mathbf{y}^*(\cdot, T) = 0 & \text{in } \mathcal{E}, \end{cases} \tag{3.4}$$

which can be solved by using the return method, with

$$\operatorname{supp}(\xi^*) \subseteq (\overline{\mathcal{E}} \setminus \overline{\Omega}) \times (0, T), \quad \operatorname{supp}(\sigma^*) \subseteq (\overline{\mathcal{E}} \setminus \overline{\Omega}) \times (0, T).$$

Boundary layer profile problem

The boundary layer profiles $(\mathbf{v}^+, \mathbf{v}^-)(x,t;z)$ in the expansion satisfy the following problem

$$\begin{cases} \partial_t \mathbf{v}^\pm - \partial_z^2(\lambda^\pm \mathbf{v}^+ + \lambda^\mp \mathbf{v}^-) + [(\mathbf{y}^* \cdot \nabla)\mathbf{v}^\pm + (\mathbf{v}^\mp \cdot \nabla)\mathbf{y}^*]_\tau + \mathfrak{f} z \partial_z \mathbf{v}^\pm = \mu^\pm \\ \partial_z \mathbf{v}^\pm(x,t;0) = \mathfrak{g}^\pm(x,t), \quad x \in \overline{\mathcal{E}}, t \in \mathbb{R}_+, \\ \mathbf{v}^\pm(x,t,z) \longrightarrow 0 \quad (z \to +\infty), \quad x \in \overline{\mathcal{E}}, t \in \mathbb{R}_+, \\ \mathbf{v}^\pm(x,0;z) = 0, \quad x \in \overline{\mathcal{E}}, z \in \mathbb{R}_+, \end{cases} \tag{3.5}$$

where $\mathfrak{f}(x,t) = -\frac{\mathbf{y}^*(x,t)\cdot\vec{\mathbf{n}}(x)}{d(x)}$, $\mathfrak{g}^\pm(x,t) = \chi_{\partial\mathcal{E}}(x)\mathcal{N}^\pm(\mathbf{y}^*, \mathbf{y}^*)(x,t)$ being given by the Navier condition.

For the above problem (3.5), one can find controls $\mu^\pm$ satisfying $\operatorname{supp}(\mu^\pm(\cdot,t;z)) \subseteq \overline{\mathcal{E}} \setminus \overline{\Omega}$, such that

$$\left\| \mathbf{v}^\pm(\cdot, \frac{T}{\epsilon}; \frac{d(\cdot)}{\epsilon}) \right\|_{L^2(\mathcal{E})} \leq C\epsilon^{\frac{5}{8}}.$$

Thus, we get the conclusion $\|\mathbf{z}^{\pm,\epsilon}(\cdot, T/\epsilon)\|_{L^2(\mathcal{E})} = O(\epsilon^{\frac{9}{8}})$.

Thank You!