Linear inviscid damping and enhanced dissipation for shear flows

Zhifei Zhang

School of Mathematical Sciences, Peking University

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Classical problem:

Stability of laminar flows at high Reynolds number.

Some classical laminar flows:

- Plane Couette flow: $(y, 0, 0)$
- Plane Poiseuille flow: $(1 - y^2, 0, 0)$
- Pipe Poiseuille flow: $(0, 0, 1 - r^2)$

These are steady solutions of the Navier-Stokes equations:

$$\begin{cases} \partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla P = 0 \quad \text{in} \quad \mathbb{R}_+ \times \Omega, \\ \nabla \cdot v = 0 \quad \text{in} \quad \mathbb{R}_+ \times \Omega, \\ v = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\nu = Re^{-1} \ll 1$ is the viscosity coefficient.

Consider the linearized NS system around the laminar flow:

$$\partial_t u - \mathcal{L}_\nu u = 0.$$

Let U solve the eigenvalue problem: $\mathcal{L}_\nu U = \lambda U$. The system is linearly stable if $\mathrm{Re}\lambda \leq 0$ and unstable if $\mathrm{Re}\lambda > 0$.

- Plane Couette flow: stable for any Reynolds number(*Romanov, Funk. Anal. 1973*);
- Plane Poiseuille flow: stable for Reynolds number less than 5772(*Orszag, JFM 1971*), and unstable for high Reynolds number(*Grenier et al, Adv Math 2016*);
- Pipe Poiseuille flow: stable at high Reynolds number(*Chen-Wei-Zhang, CPAM 2023*).

  Conjecture: *Pipe Poiseuille flow is stable for any Reynolds number*.

Nonlinear stability

Transition threshold problem(*Trefethen et al, Science 1993*): *Given a norm $\|\cdot\|_X$, find a $\beta = \beta(X)$ such that*

$$\|u_0\|_X \ll Re^{-\beta} \Longrightarrow \textit{stability}.$$

3-D Couette flow in $\Omega = \mathbb{T} \times \mathbb{R} \times \mathbb{T}$:

- If X is Gevrey class, then $\beta \le 1$(*Bedrossian-Germain-Masmoudi, Mem AMS 2021*).
- If $X = H^N$, then $\beta \le \frac{3}{2}$(*Bedrossian-Germain-Masmoudi, Ann Math 2017*).
- If $X = H^2$, then $\beta \le 1$(*Wei-Zhang, CPAM 2021*).

3-D Couette flow in $\Omega = \mathbb{T} \times [-1, 1] \times \mathbb{T}$:

- If $X = H^2$, then $\beta \le 1$(*Chen-Wei-Zhang, Mem AMS in press*).

3-D Plane Poiseuille flow in $\Omega = \mathbb{T} \times [-1, 1] \times \mathbb{T}$:

- If $X = H^2$, then $\beta \le \frac{7}{4}$(*Chen-Ding-Lin-Zhang, preprint*).

The linearized NS system

We consider general monotone shear flows $(u(y), 0)$ satisfying

$$\text{(M)} \quad u \in H^3(0,1), \quad u'(y) \geq c_0 \quad \text{for some} \quad c_0 > 0.$$

The linearized 2D NS system around $(u(y), 0)$ in $\Omega = \mathbb{T} \times [0,1]$ takes as follows

$$\partial_t v + A_\nu v = 0,$$

where

$$A_\nu v = \mathbb{P}\Big(-\nu\Delta v + u(y)\partial_x v + (v^2 \partial_y u, 0)\Big)$$

with $D(A_\nu) = H^2(\Omega) \cap H^1_{0,\sigma}(\Omega)$ and $\mathbb{P}$ Leray-Helmhotz projection.

The linearized NS system

Let $\omega = \partial_x v^2 - \partial_y v^1$ be the vorticity and ϕ be the stream function so that

$$\Delta\phi = \omega, \quad v = (-\partial_y\phi, \partial_x\phi).$$

Then the linearized NS system can be written as

$$\begin{cases} \partial_t\omega + \mathcal{L}_\nu\omega = 0, \\ \Delta\phi = \omega, \quad \partial_x\phi|_{y=0,1} = \partial_y\phi|_{y=0,1} = 0, \\ \omega(0,x,y) = \omega_0(x,y), \end{cases} \tag{1}$$

where

$$\mathcal{L}_\nu\omega = \Big(-\nu\Delta + u(y)\partial_x - u''(y)\partial_x\Delta^{-1} \Big)\omega.$$

Main goals

We are concerned with the following three problems:

- Linear stability
- Linear inviscid damping
- Linear enhanced dissipation

The later two mechanisms play a crucial role in the transition threshold problem.

Linear inviscid damping

The linearized 2-D Euler equation around Couette flow:

$$\omega_t + y\partial_x\omega = 0 \Rightarrow \omega(t,x,y) = \omega_0(x-ty,y).$$

In 1907, Orr found that

$$\|V_{\neq}(t)\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

This is so called **inviscid damping**, which is an analogue of **Landau damping** in plasma physics.

Linear enhanced dissipation

The linearized 2-D NS around Couette flow:

$$\partial_t \omega - \nu \Delta \omega + y \partial_x \omega = 0, \quad \omega(0) = \omega_0.$$

It holds that

$$\|\omega_{\neq}(t)\|_{L^2} \leq C e^{-\nu^{\frac{1}{3}} t} \|\omega_0\|_{L^2}.$$

This decay rate $\nu^{\frac{1}{3}}$ is much faster than the diffusion rate ν. This is so called **enhanced dissipation**.

We define

$$m(\nu) = \inf\left\{\mathrm{Re}\lambda : A_\nu v = \lambda v, v \in D(A_\nu)\right\},$$

$$m_e(\nu) = \inf\left\{\mathrm{Re}\lambda : A_\nu v = \lambda v, v \in D(A_\nu), \int_{\mathbb{T}} v \mathrm{d}x = 0\right\}.$$

Theorem 1. (*Chen-Wei-Zhang, CMP 2023*)
Assume that $\mathcal{L}_E = u(y)\partial_x - u''(y)\partial_x\Delta^{-1}$ has no embedding eigenvalues or eigenvalues. There exist $0 < \nu_1 \le 1$ and $c > 0$ independent of ν so that if $0 < \nu \le \nu_1$, then it holds that

$$m(\nu) \ge c\nu, \quad m_e(\nu) \ge c\nu^{\frac{1}{3}}.$$

Some remarks.

- Linear stability for concave monotone shear flows(*Almog-Helffer, ARMA 2021*).
- If the flow is monotone and concave, then $\mathcal{L}_E$ has no embedding eigenvalues or eigenvalues.
- The spectral gap estimate $m_e(\nu) \geq c\nu^{\frac{1}{3}}$ corresponds to the enhanced dissipation phenomenon.

Main results: enhanced dissipation

Theorem 2. (*Chen-Wei-Zhang, CMP 2023*)
Assume that $\mathcal{L}_E = u(y)\partial_x - u''(y)\partial_x\Delta^{-1}$ *has no embedding eigenvalues or eigenvalues. Let* ω *solve* (1) *with the initial data* $\omega_0 \in H^1_y H^{-1}_x$ *and* $\int_{\mathbb{T}} \omega_0(x,y)\mathrm{d}x = 0$. *There exist* $0 < \nu_1 \le 1, 0 < \epsilon_0 \le 1$ *so that if* $0 < \nu \le \nu_1$, *then it holds that*

$$\left\|\mathrm{e}^{\epsilon_0\nu^{1/3}t}v\right\|_{L^\infty L^2} + \left\|\mathrm{e}^{\epsilon_0\nu^{1/3}t}v\right\|_{L^2L^2} \le C\nu^{\frac{1}{3}}\|\omega_0\|_{H^1_yH^{-1}_x} + C\|\omega_0\|_{L^2_yH^{-1}_x}.$$

Remark. *Similar estimates have been established for the linearized NS system around Couette flow* (Chen-Li-Wei-Zhang, ARMA 2020). *These estimates play a crucial role in nonlinear stability of Couette flow.*

Main results: inviscid damping

Taking the Fourier transform in x, the linearized NS system is reduced to

$$\begin{cases} \partial_t \omega - \nu(\partial_y^2 - \alpha^2)\omega + \mathrm{i}\alpha\Big(u(y)\omega - u''\phi\Big) = 0, \\ \omega = (\partial_y^2 - \alpha^2)\phi, \quad \phi|_{y=0,1} = \partial_y \phi|_{y=0,1} = 0, \\ \omega(0, y) = \omega_0(y). \end{cases} \tag{2}$$

For $\alpha \neq 0$, the Rayleigh operator $\mathcal{R}_\alpha$ is defined by

$$\mathcal{R}_\alpha = (\partial_y^2 - \alpha^2)^{-1}\Big(u(y)(\partial_y^2 - \alpha^2) - u''(y)\Big).$$

Theorem 3. (*Chen-Wei-Zhang, CMP 2023*)

Let $|\alpha| \geq 1$. *Assume that* $\mathcal{R}_\alpha$ *has no embedding eigenvalues or eigenvalues. Let* $(\omega, \ \phi)$ *solve* (2) *with* $\omega_0 \in H^1$, $\langle \omega_0, \mathrm{e}^{\pm\alpha y} \rangle = 0$. *There exist* $0 < \nu_1 \leq 1$, $0 < \epsilon_1 \leq 1/2$ *such that if* $0 < \nu \leq \nu_1$, *then it holds that*

$$\|(\partial_y \phi, \alpha\phi)(t)\|_{L^2} \leq C|\alpha|^{-2}(1+t)^{-1}\mathrm{e}^{-\epsilon_1(\nu\alpha^2)^{1/3}t}\Big(\|\partial_y\omega_0\|_{L^2} + |\alpha|\|\omega_0\|_{L^2}\Big).$$

Remark.

- When $\nu = 0$, the inviscid damping was proved in (*Wei-Zhang-Zhao, CPAM 2018*);
- The decay estimate of $\|\alpha\phi\|_{L^2}$ should not be optimal;
- The result is new even for the linearized NS system with Navier-slip boundary condition.

Orr-Sommerfeld equation

The key ingredient of the proof is to solve the Orr-Sommerfeld(OS) equation:

$$\begin{cases} -\nu(\partial_y^2-\alpha^2)w+\mathrm{i}\alpha\Big((u(y)-\lambda)w-u''\psi\Big)=F, \\ w=(\partial_y^2-\alpha^2)\psi, \\ \psi|_{y=0,1}=\partial_y\psi|_{y=0,1}=0. \end{cases}$$

Here $\lambda=\lambda_r+i\lambda_i\in\mathbb{C}$.

Resolvent estimates

If $\alpha\lambda_i \geq -\epsilon_0(\nu\alpha^2)^{\frac{1}{3}}$, then we have

$$\|(\partial_y\psi, \alpha\psi)\|_{L^2} \leq C\nu^{-\frac{1}{2}}|\alpha|^{-1}\|F\|_{\tilde{H}^{-1}},$$
$$\|(\partial_y\psi, \alpha\psi)\|_{L^2} \leq C\nu^{-\frac{1}{6}}|\alpha|^{-\frac{4}{3}}\|F\|_{L^2},$$
$$\|(\partial_y\psi, \alpha\psi)\|_{L^2} \leq C|\alpha|^{-2}\|(\partial_y F, \alpha F)\|_{L^2}.$$

In particular, when $F = 0$, the OS equation has only trivial solution, which implies the linear stability and

$$m_e(\nu) \geq c\nu^{\frac{1}{3}}.$$

Resolvent estimates

In our work(*Chen-Li-Wei-Zhang, ARMA 2020*), a key idea is that we first solve the OS equation with Navier-slip boundary condition:

$$\begin{cases} -\nu(\partial_y^2-\alpha^2)w+\mathrm{i}\alpha\Big((u-\lambda)w-u''\psi\Big)=F,\\ (\partial_y^2-\alpha^2)\psi=w,\ \ w|_{y=0,1}=\psi|_{y=0,1}=0, \end{cases}$$

and then match the boundary condition via constructing the boundary correctors.

Main advantages:

- Energy method due to favorable boundary conditions, especially in the case when nonlocal term $u''\psi=0$.
- Boundary correctors via solving the Airy equation, which has the explicit solution.

Resolvent estimates with Navier-slip BC:

- $F \in H^{-1}$:

$$\nu^{\frac{1}{6}}|\alpha|^{\frac{4}{3}}\|(\psi',\alpha\psi)\|_{L^2} + (\nu\alpha^2)^{\frac{1}{3}}\|w\|_{L^2} \le C\nu^{-\frac{1}{3}}|\alpha|^{\frac{1}{3}}\|F\|_{H^{-1}}.$$

- $F \in L^2$:

$$\nu^{\frac{1}{6}}|\alpha|^{\frac{4}{3}}\|(\psi',\alpha\psi)\|_{L^2} + (\nu\alpha^2)^{\frac{1}{3}}\|w\|_{L^2} \le C\|F\|_{L^2}.$$

- $F \in H^1$:

$$\nu^{\frac{1}{6}}|\alpha|^{\frac{4}{3}}\|(\psi',\alpha\psi)\|_{L^2} + (\nu\alpha^2)^{\frac{1}{3}}\|w\|_{L^2} \le C\nu^{\frac{1}{6}}|\alpha|^{-\frac{2}{3}}\|(F',\alpha F)\|_{L^2}.$$

Formal prediction via scaling analysis:

Consider the OS equation in a boundary layer of order δ. In this inner layer, we have

$$\nu\partial_y^2 w \sim \nu\delta^{-2} w, \quad \alpha(u-\lambda)w \sim \alpha\delta w \quad \text{if } \lambda \sim \delta.$$

These two terms should have the same scale, which gives $\delta = (\nu/\alpha)^{1/3}$. Thus, in the inner layer, the solution w behaves as follows

$$\alpha\delta w = \alpha(\nu/\alpha)^{1/3} w = (\nu\alpha^2)^{1/3} w \sim F.$$

This shows that

$$(\nu\alpha^2)^{\frac{1}{3}}\|w\|_{L^2} \le C\|F\|_{L^2}.$$

Compactness method

The proof of the following cases are relatively easy:

- $\nu\alpha^2 \geq 1$: viscous term is dominant;
- $\alpha \gg 1$: nonlocal term could be viewed as a perturbation;
- λ_r is far away from the range of u.

The most difficult case is that

$$\alpha \leq M, \quad 0 \leq \alpha\lambda_i \leq \varepsilon, \quad \lambda_r \in [u(0) - 2M_2, u(1) + 2M_2],$$

where $M_2 = \|u\|_{H^3} + \|u''\|_{L^\infty}$. Our goal is to show that

$$\|\psi\|_{H^1} \leq C\|F\|_{H^1}.$$

If the conclusion is not true, then for any $\varepsilon_n \to 0$, there exists $\psi_n \in H^3(0,1) \cap H_0^1(0,1)$, $w_n \in H_0^1(0,1)$, and $0 < \nu_n, \alpha_n\lambda_{i,n} \leq \varepsilon_n$, $\lambda_{r,n} \in [u(0) - 2M_2, u(1) + 2M_2]$, $\alpha_n \in [1, M]$ such that

$$-\nu_n(\partial_y^2 - \alpha_n^2)w_n + \mathrm{i}\alpha_n\Big((u - \lambda_n)w_n - u''\psi_n\Big) = F_n,$$
$$(\partial_y^2 - \alpha_n^2)\psi_n = w_n, \quad \lambda_n = \lambda_{r,n} + \mathrm{i}\lambda_{i,n}$$

with

$$\|\psi_n\|_{H^1} = 1, \quad \|F_n\|_{H^1} \leq 1/n.$$

We may take a subsequence(still denote by $\{\nu_n, \alpha_n, \lambda_n, \psi_n, w_n\}$), such that ν_n, $\lambda_{i,n} \to 0$ with $\lambda_{i,n} \geq 0$, and $\lambda_{r,n} \to \lambda$, $\alpha_n \to \alpha$, and $\|F_n\|_{H^1} \to 0$, and $\psi_n \rightharpoonup \psi$ weakly in $H^1(0,1)$.

Let $u_n = u - \lambda_{r,n}$, $\varsigma_n = \lambda_{i,n}$. Then we have

$$-\nu_n(\partial_y^2 - \alpha_n^2)w_n + \mathrm{i}\alpha_n\Big((u_n - \mathrm{i}\varsigma_n)w_n - u_n''\psi_n\Big) = F_n,$$
$$\psi_n'' - \alpha_n^2\psi_n = w_n, \quad \mathbf{Im}\ u_n = 0$$

with

$$\psi_n \rightharpoonup \psi,\ F_n \to 0 \quad \text{in} \quad H^1(0,1).$$
$$u_n \to u - \lambda \quad \text{in} \quad H^3(0,1), \quad \varsigma_n \geq 0,\ \varsigma_n \to 0.$$

To conclude a contradiction, we need to prove that

- $\psi_n \to \psi$ in $H^1(0,1)$: similar to the proof of limiting absorption principle in [*Wei-Zhang-Zhao, Ann PDE 2019*].
- λ is an embedding eigenvalue of $\mathcal{R}_\alpha$: for any $\varphi \in H_0^1(0,1)$,

$$\alpha\int_0^1(\psi'\varphi'+\alpha^2\psi\varphi)\mathrm{d}y+p.v.\int_0^1\frac{\alpha u''\psi\varphi}{u-\lambda}\mathrm{d}y \\ +\sum_{\{y_\lambda:u(y_\lambda)=\lambda\}}\mathrm{i}\pi\frac{(\alpha u''\psi\varphi)(y_\lambda)}{u'(y_\lambda)}=0.$$

We rewrite the equation of w_n as follows

$$\begin{aligned}-\nu_n\partial_y^2 w_n + \mathrm{i}\alpha_n(u_n - \mathrm{i}\varsigma_n)w_n &= F_n + \mathrm{i}\alpha_n u_n''\psi_n - \nu_n\alpha_n^2 w_n \\ &= g_n - \nu_n\alpha_n^2 w_n.\end{aligned}$$

A natural idea is to test the function $\varphi/(u_n - \mathrm{i}\varsigma_n)$ to the above equation. However, the integration by parts will lead to many singular terms near the point where $u = \lambda$.

Our key idea is to consider

$$\int_0^1 \Big(-\nu_n\partial_y^2 w_n + \mathrm{i}\alpha_n(u_n - \mathrm{i}\varsigma_n)w_n\Big)(U_n\varphi)\mathrm{d}y$$

where

$$\Big(-\nu_n\partial_y^2 + \mathrm{i}\alpha_n(u_n - \mathrm{i}\varsigma_n)\Big)U_n = \alpha_n + o(1).$$

Let J_n solve $-\nu\partial_y^2 J_n + i\alpha_n y J_n = \alpha_n$, where

$$J_n(y) = \int_0^\infty \mathrm{e}^{-\mathrm{i}ty-\nu_n t^3/(3\alpha_n)}\mathrm{d}t.$$

Construct a function V_n so that

$$(V_n')^2 V_n = u_n - \mathrm{i}\varsigma_n, \quad |V_n'| \geq \tilde{c}_0.$$

Then $H_n(y) = J_n(V_n)$ solve

$$-\nu_n\partial_y^2 H_n + \mathrm{i}\alpha_n(u_n - \mathrm{i}\varsigma_n)H_n + \nu_n\partial_y H_n V_n''/V_n' = (V_n')^2\alpha_n.$$

So, $U_n = H_n/(V_n')^2$.

Compactness method

Then we have

$$\int_0^1 \Big(-\nu_n\partial_y^2 w_n + \mathrm{i}\alpha_n(u_n - \mathrm{i}\varsigma_n)w_n\Big)\frac{H_n\varphi}{(V_n')^2}\mathrm{d}y$$
$$= \int_0^1 w_n \frac{\Big(-\nu_n\partial_y^2 H_n + \mathrm{i}\alpha_n(u_n - \mathrm{i}\varsigma_n)H_n + \nu_n\partial_y H_n V_n''/V_n'\Big)\varphi}{(V_n')^2}\mathrm{d}y$$
$$- \int_0^1 \nu_n w_n \frac{\partial_y H_n}{V_n'}\Big(\frac{\varphi}{V_n'}\Big)'\mathrm{d}y + \int_0^1 \nu_n \partial_y w_n H_n \Big(\frac{\varphi}{(V_n')^2}\Big)'\mathrm{d}y.$$

It is easy to show that

$$\lim_{n\to+\infty}\int_0^1 \Big(-\nu_n\partial_y^2 w_n + \mathrm{i}\alpha_n(u_n - \mathrm{i}\varsigma_n)w_n\Big)\frac{H_n\varphi}{(V_n')^2}\mathrm{d}y$$
$$= -\alpha\int_0^1 (\psi'\varphi' + \alpha^2\psi\varphi)\mathrm{d}y.$$

Compactness method

Let $G_n(y) = \dfrac{(g_n - \nu_n \alpha_n^2 w_n)\varphi}{(V_n')^2}$. The Fubini theorem gives

$$\int_0^1 (g_n - \nu_n \alpha_n^2 w_n) \frac{H_n \varphi}{(V_n')^2} \mathrm{d}y = \int_0^1 G_n(y) H_n(y) \mathrm{d}y$$
$$= \int_0^{+\infty} \left(\int_0^1 G_n(y) \mathrm{e}^{-\mathrm{i}tV_n(y)} \mathrm{d}y \right) \mathrm{e}^{-\nu_n t^3/(3\alpha_n)} \mathrm{d}t.$$

It holds that

$$\lim_{n \to +\infty} \int_0^{+\infty} \left(\int_0^1 G_n(y) \mathrm{e}^{-\mathrm{i}tV_n(y)} \mathrm{d}y \right) \mathrm{e}^{-\nu_n t^3/(3\alpha_n)} \mathrm{d}t$$
$$= \sum_{\{y_0 : u(y_0) - \lambda = 0\}} \pi \frac{g_0(y_0)\varphi(y_0)}{u'(y_0)} - p.v. \int_0^1 \frac{\mathrm{i} g_0(y)\varphi(y)}{u(y) - \lambda} \mathrm{d}y.$$

Boundary layer corrector

To match the boundary conditions, we construct the boundary layer corrector by solving the homogeneous OS equation

$$\begin{cases} -\nu(\partial_y^2-\alpha^2)w_i + \mathrm{i}\alpha\Big((u-\lambda)w_i - u''\psi_i\Big) = 0, \quad i\in\{1,2\}, \\ (\partial_y^2-\alpha^2)\psi_i = w_i, \quad \psi_i|_{y=0,1} = 0, \quad i\in\{1,2\}, \\ \partial_y\psi_1|_{y=0} = 1,\ \partial_y\psi_1|_{y=1} = 0, \quad \partial_y\psi_2|_{y=0} = 0,\ \partial_y\psi_2|_{y=1} = 1. \end{cases}$$

Then we can decompose w as

$$w(y) = w_{Na}(y) + c_1 w_1(y) + c_2 w_2(y),$$

where c_1 and c_2 are determined by

$$c_1 = \int_0^1 w_{Na}\frac{\sinh(\alpha(1-y))}{\sinh(\alpha)}\mathrm{d}y,$$

$$c_2 = -\int_0^1 w_{Na}\frac{\sinh(\alpha y)}{\sinh(\alpha)}\mathrm{d}y.$$

The key ingredient is to find two linearly independent solutions of the homogeneous OS equation:

$$-\nu(\partial_y^2-\alpha^2)W_j+\mathrm{i}\alpha\big((u-\lambda)W_j-u''\Psi_j\big)=0.$$

To this end, we first find two approximate solutions

$$W_{a,1}(y)=Ai(\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}L_0(y+d_0)),\quad W_{a,2}(y)=Ai(\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}L_1(y+d_1)),$$

where

$$L_0=|\alpha u'(0)/\nu|^{\frac{1}{3}},\quad d_0=(u(0)-\lambda-\mathrm{i}\nu\alpha)/(u'(0)),$$
$$L_1=|\alpha u'(1)/\nu|^{\frac{1}{3}},\quad d_1=(u(1)-u'(1)-\lambda-\mathrm{i}\nu\alpha)/(u'(1)).$$

Boundary layer corrector

We introduce the decomposition:

$$W_1 = W_{a,1} + W_{e,1}, \quad W_2 = W_{a,2} + W_{e,2},$$

where the error $(W_{e,1}, W_{e,2})$ solves

$$\begin{cases} -\nu(\partial_y^2 - \alpha^2)W_{e,1} + \mathrm{i}\alpha\Big((u-\lambda)W_{e,1} - u''\Psi_{e,1}\Big) \\ \qquad\qquad = -\mathrm{i}\alpha\Big((u - u(0) - u'(0)y)W_{a,1} - u''\Psi_{a,1}\Big), \\ -\nu(\partial_y^2 - \alpha^2)W_{e,2} + \mathrm{i}\alpha\Big((u-\lambda)W_{e,2} - u''\Psi_{e,2}\Big) \\ \qquad\qquad = -\mathrm{i}\alpha\Big((u - u(1) - u'(1)(y-1))W_{a,2} - u''\Psi_{a,2}\Big), \\ (\partial_y^2 - \alpha^2)\Psi_{e,1} = W_{e,1}, \quad (\partial_y^2 - \alpha^2)\Psi_{e,2} = W_{e,2}, \\ W_{e,1}|_{y=0,1} = \Psi_{e,1}|_{y=0,1} = 0, \quad W_{e,2}|_{y=0,1} = \Psi_{e,2}|_{y=0,1} = 0. \end{cases}$$

Thanks a lot for your attention!