GLIMM'S METHOD, CONVEX INTEGRATION, AND DENSITY OF WILD DATA FOR THE EULER SYSTEM OF GAS DYNAMICS

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Euler system of gas dynamics

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \ \mathbf{m} = \varrho \mathbf{u}$$

Momentum equation – Newton's second law

$$\partial_t \mathbf{m} + \mathrm{div}_x \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + \nabla_x p(\varrho, \vartheta) = 0$$

Energy balance – First law of thermodynamics

$$\partial_t E + \mathrm{div}_x \left[(E + p(\varrho, \vartheta)) \frac{\mathbf{m}}{\varrho} \right] = 0, \ \ E = \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)$$

Constitutive relations

$$p(\varrho, \vartheta) = \varrho \vartheta, \ \ e(\varrho, \vartheta) = c_v \vartheta, \ \ c_v > 1$$

Periodic boundary conditions

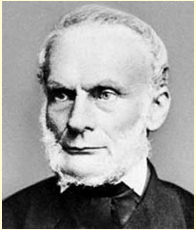
$$\Omega = \mathbb{T}^2 = \left\{ (x_1, x_2) \ \middle| \ x_1 \in [0,1]|_{\{0;1\}}, \ \ x_2 \in [0,1]|_{\{0;1\}} \right\}$$

Admissibility

Leonhard Paul
Euler
1707–1783

Entropy

$$s(\varrho, \vartheta) = c_v \log \vartheta - \log \varrho$$



Rudolf
Clausius
1822–1888

Entropy inequality – admissibility criterion for weak solutions

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) \geq 0$$

Local well posedness

Theorem (Local existence for smooth data)

Consider the initial data

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

in the class

$$\varrho_0 \in W^{k,2}(\mathbb{T}^d),\ \inf_{\mathbb{T}^d} \varrho_0 > 0,$$

$$\vartheta_0 \in W^{k,2}(\mathbb{T}^d),\ \inf_{\mathbb{T}^d} \vartheta_0 > 0,$$

$$\mathbf{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d),\ k > \frac{d}{2} + 1.$$

Then there exists $T_{\rm max} > 0$ *such that the Euler system admits a classical solution* $(\varrho, \vartheta, \mathbf{u})$ *unique in the class*

$$\varrho \in C([0,T]; W^{k,2}(\mathbb{T}^d)),\ \vartheta \in C([0,T]; W^{k,2}(\mathbb{T}^d)),$$

$$\mathbf{u} \in C([0,T]; W^{k,2}(\mathbb{T}^d; R^d))$$

for any $0 < T < T_{\rm max}$.

Admissible (entropy) weak solutions

Field equations

$$\left[\int_\Omega \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x\mathrm{d}t$$

$$\text{for all } \varphi \in C^1([0,T] \times \overline{\Omega})$$

$$\left[\int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$$\text{for all } \boldsymbol{\varphi} \in C^1([0,T] \times \overline{\Omega}; R^3), \ \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\left[\int_\Omega E \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[E \partial_t \varphi + \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] \cdot \nabla_x \varphi \right] \ \mathrm{d}x\mathrm{d}t$$

$$\text{for all } \ \varphi \in C^1([0,T] \times \overline{\Omega})$$

Entropy inequality

$$\left[\int_\Omega \varrho s \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} \boxed{\geq} \int_0^\tau \int_\Omega \left[\varrho s \partial_t \varphi + s \mathbf{m} \cdot \nabla_x \varphi \right] \ \mathrm{d}x\mathrm{d}t$$

$$\text{for all } \varphi \in C^1([0,T] \times \overline{\Omega}), \ \ \varphi \geq 0$$

Comparison with the isentropic Euler system

Leonhard Paul Euler 1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathbf{m} = 0$$

Momentum equation – Newton's second law

$$\partial_t \mathbf{m} + \mathrm{div}_x \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + \nabla_x p(\varrho) = 0, \ \ p(\varrho) = a\varrho^\gamma$$

Impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0, \ \ \Omega \subset R^d, \ \ d = 2, 3$$

Initial state (data)

$$\varrho(0, \cdot) = \varrho_0, \ \ (\varrho \mathbf{u})(0, \cdot) = \varrho_0 \mathbf{u}_0$$

Weak vs. strong continuity

$$\mathbf{U} = [\varrho, \mathbf{m}], \ \mathbf{m} = \varrho \mathbf{u}$$

Weak continuity

$$\mathbf{U} \in C_{\rm weak}([0,T]; L^p(\Omega; R^d)), \ t \mapsto \int_\Omega \mathbf{U} \cdot \boldsymbol{\varphi} \ {\rm d}x \in C[0,T]$$

$$\boldsymbol{\varphi} \in L^{p'}(\Omega; R^d)$$

Strong continuity

$$\tau \in [0,T], \ \|\mathbf{U}(t,\cdot) - \mathbf{U}(\tau,\cdot)\|_{L^p(\Omega;R^d)} \text{ whenever } t \to \tau$$

Strong vs. weak

$$\text{strong } \Rightarrow \text{weak}, \ \text{weak} \not\Rightarrow \text{strong}$$

General ill–posedness in the isentropic case

Theorem (A.Abbatiello, EF 2021)

Anna Abbatiello (Roma La Sapienza)

Let $d = 2, 3$. Let $\mathcal{R}$ denote the set of bounded Riemann integrable functions. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.

Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]\ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i$$

Good/bad news - complete Euler system

Theorem (Local ill posedness) (EF, C.Klingenberg, O. Kreml, S. Markfelder) [2020]
Let $\varrho_0 > 0$, $\vartheta_0 > 0$ be piecewise constant, arbitrary.

Then there exist (infinitely many) $\mathbf{u}_0 \in L^\infty$ such that the Euler system admits infinitely many global in time admissible solutions.

Wild data

Initial state

The initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$ are **wild** if there exists $T > 0$ such that the Euler system admits infinitely many (weak) *admissible* solutions on any time interval $[0, \tau]$, $0 < \tau < T$

Results on density of wild data

- **Incompressible Euler system.** Székelyhidi–Wiedemann, Daneri–Székelyhidy
- **Isentropic Euler system.** Ming, Vasseur, and Yu. Energy dissipating solutions global in time
- **Barotropic Euler system.** E.Chiodaroli, EF Admissible weak solutions local in time

W (wild) convergence

Data space

$$\mathrm{L}^1_{+,s_0}(\Omega; R^{N+2})$$
$$= \left\{ [\varrho, \mathbf{m}, E] \in L^1(\Omega; R^{N+2}) \ \Big| \ \varrho \geq 0, \ E \geq 0, \ s(\varrho, \mathbf{m}, E) \geq s_0 > -\infty \right\}.$$

W–convergence $[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}] \to [\varrho_0, \mathbf{m}_0, E_0]$

- $$\varrho_{0,n} > 0, \quad s(\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}) \geq s_0 > -\infty$$
- $$[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}] \to [\varrho_0, \mathbf{m}_0, E_0] \quad \text{in } L^1(\Omega; R^{N+2})$$
- the initial data $[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}]$ give rise to a sequence of admissible weak solutions $[\varrho_n, \mathbf{m}_n, E_n]$ satisfying

$$\int_0^T \int_\Omega \left(\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \frac{1}{N} \frac{|\mathbf{m}_n|^2}{\varrho_n} \mathbb{I} \right) : \nabla_x^2 \varphi \ \mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } n \to \infty$$

for any $\varphi \in C_c^\infty((0, T) \times \Omega)$

The last condition is automatically satisfied for available convex integration solutions!

Non–existence of wild data?

Reachable set

We say that a trio $[\varrho_0, \mathbf{m}_0, E_0]$ is *reachable* if there exists a sequence of initial data $\{\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}\}_{n=1}^{\infty}$ such that

$$[\varrho_{0,n}, \mathbf{m}_{0,n}, E_{0,n}] \stackrel{(W)}{\to} [\varrho_0, \mathbf{m}_0, E_0].$$

Theorem (EF, Klingenberg, Markfelder Calc. Variations PDE 2020)

Let $s_0 \in R$ be given. Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded smooth domain. Then the complement of the set of reachable data is an open dense set in $\mathrm{L}^1_{+,s_0}(\Omega; R^{N+2})$.

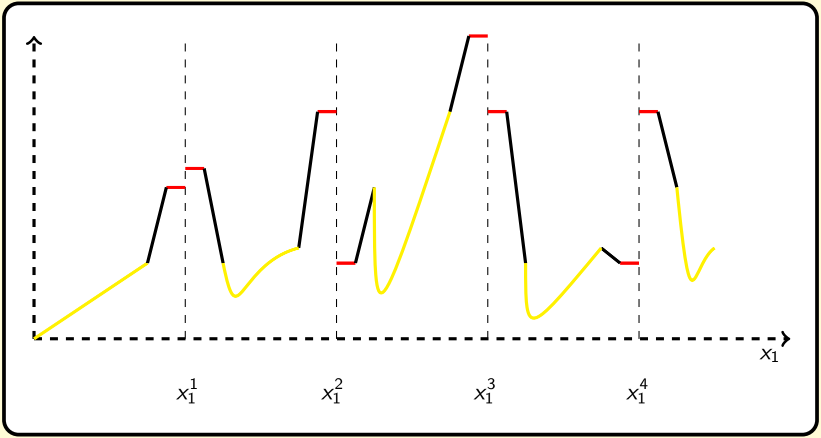
Glimm's method revisited, I

E. Chiodaroli (Pisa)

Strategy:

- Smooth data are dense, fix smooth data
- Euler system is locally solvable for smooth data, fix the smooth solution
- Choose N points and solve the associated Riemann problem locally, use convex integration to obtain infinitely many admissible weak solutions to the Riemann problem
- Paste the two solutions together

Glimm's method revisited, II



Riemann problem

Riemann data

$$\mathbb{R} \times \mathbb{T}^1 = \left\{ (x_1, x_2) \;\middle|\; x_1 \in \mathbb{R},\ x_2 \in [0,1]|_{\{0;1\}} \right\}$$

$$(\varrho_\ell, \vartheta_\ell, \mathbf{u}_\ell),\ (\varrho_r, \vartheta_r, \mathbf{u}_r) \in (0,\infty) \times (0,\infty) \times \mathbb{R}^2$$

$$(\varrho_0, \vartheta_0, \mathbf{u}_0) = \begin{cases} (\varrho_\ell, \vartheta_\ell, \mathbf{u}_\ell),\ x_1 \leq 0, \\ (\varrho_r, \vartheta_r, \mathbf{u}_r),\ x_1 > 0 \end{cases}$$

Theorem

(H. Al Baba, C.Klingenberg, O.Kreml, V. Mácha, S.Markfelder) [2020]

There exist Riemann data $(\varrho_\ell, \vartheta_\ell, \mathbf{u}_\ell)$, $(\varrho_r, \vartheta_r, \mathbf{u}_r)$ such that the Riemann problem for the Euler system admits infinitely many solutions $(\varrho_R, \vartheta_R, \mathbf{u}_R)$ in $[0,T) \times (\mathbb{R} \times \mathbb{T}^1)$, $T > 0$ arbitrary. Moreover, they admit the same (finite) speed of propagation λ. In particular, they coincide with the constant Riemann data if $x_1 < -\lambda t$ or $x_1 > \lambda t$.

Density of wild data – exact statement

Smooth data ansatz

$$\varrho_0 \in W^{k,2}(\mathbb{T}^2),\ \inf_{\mathbb{T}^2} \varrho_0 > 0,\ \vartheta_0 \in W^{k,2}(\mathbb{T}^2),\ \inf_{\mathbb{T}^2} \vartheta_0 > 0$$

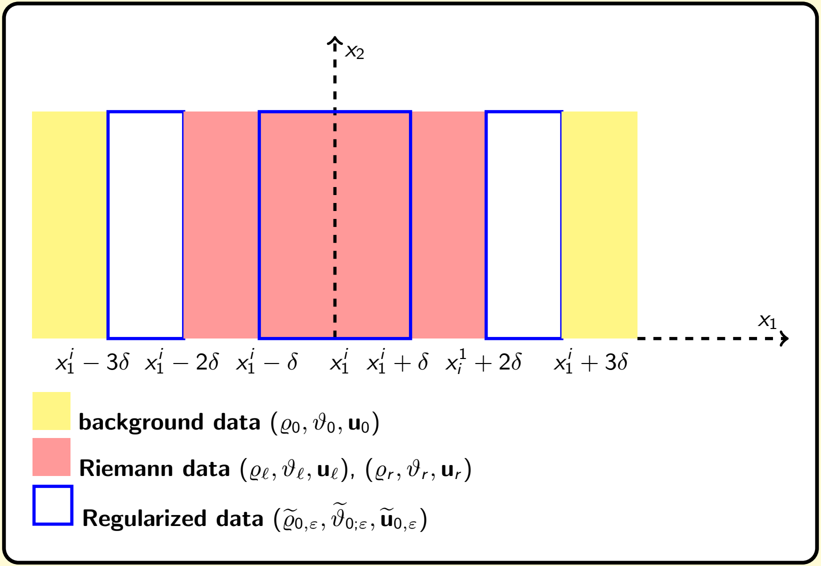
$$\mathbf{u}_0 \in W^{k,2}(\mathbb{T}^2;\mathbb{R}^2),\ k > 2,\ q > 1 \text{ given}$$

Wild data

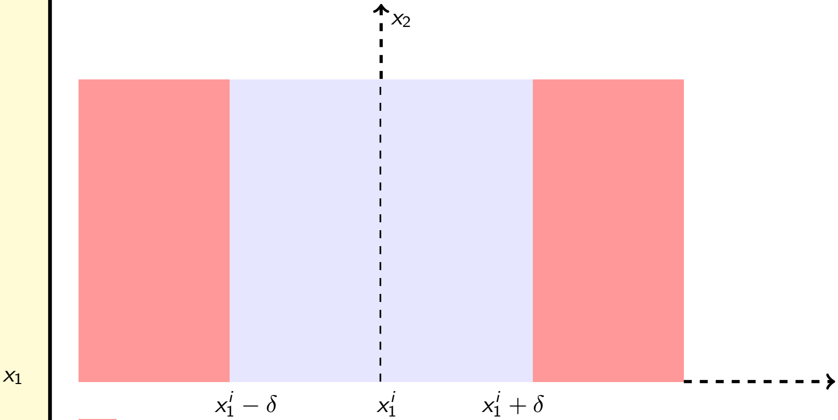
For any $\varepsilon > 0$, there exist initial data $(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})$ enjoying the following properties:

- $(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})$ are smooth out of finitely many curves
- $$\left\| \Big(\varrho_{0,\varepsilon} - \varrho_0; \vartheta_{0,\varepsilon} - \vartheta_0; \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \Big) \right\|_{L^q(\mathbb{T}^2;\mathbb{R}^4)} \leq \varepsilon$$
- There exists $T > 0$ such that the Euler system admits infinitely many admissible weak solutions $(\varrho^n, \vartheta^n, \mathbf{u}^n)_{n \in N}$ in $L^\infty([0,T) \times \mathbb{T}^2; \mathbb{R}^4)$ emanating from the initial data $(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})$ such that
$$(\varrho^n, \vartheta^n, \mathbf{u}^n)|_{[0,\tau)\times B_\varepsilon} \not\equiv (\varrho^m, \vartheta^m, \mathbf{u}^m)|_{[0,\tau)\times B_\varepsilon}, \forall\ m \neq n$$
whenever $0 < \tau < T$ and $B_\varepsilon \subset \mathbb{T}^2$ is a ball of radius ε
- The solutions are smooth out of a set of measure ε in $\mathbb{T}^2$

Magnified initial data



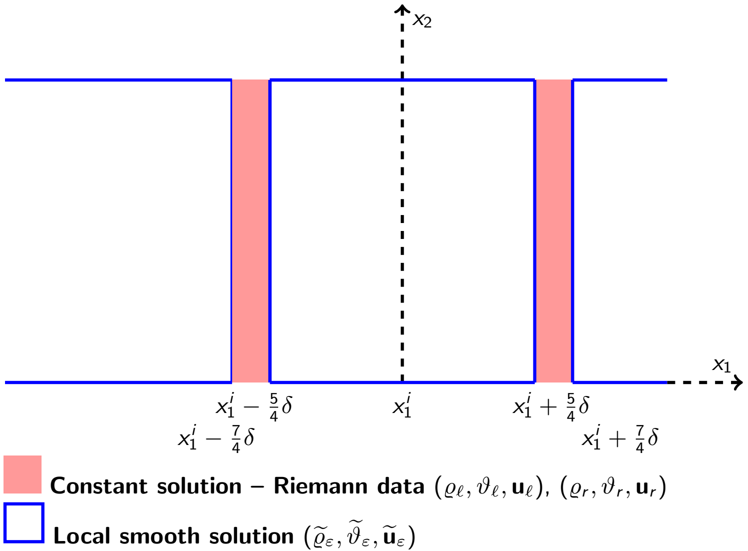
Convex integration solutions



Constant solution – Riemann data $(\varrho_\ell, \vartheta_\ell, \mathbf{u}_\ell)$, $(\varrho_r, \vartheta_r, \mathbf{u}_r)$

Solution fan obtained via convex integration $(\varrho_R, \vartheta_R, \mathbf{u}_R)$

Local smooth solutions



General domains