On the Free Boundary Problem of 3-D Compressible Euler Equations Coupled With a Nonlinear Poisson Equation

Tao Luo
City University of Hong Kong

Joint with Konstantina Trivisa & Huihui Zeng

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The Free Boundary Problem of Euler-Poisson in 3D

$$
\begin{cases}
D_t\rho + \rho \, div v = 0, \ \text{in} \ \mathscr{D}, \\
\rho D_t v + \partial P(\rho, s) = \rho \partial \phi, \ \text{in} \ \mathscr{D}, \\
D_t s = 0, \ \text{in} \ \mathscr{D}, \\
\Delta \phi + e^{-\phi} = \rho, \ \text{in} \ \mathscr{D},
\end{cases}
\tag{1.1}
$$

where $D_t = \partial_t + v^k \partial_k = \partial_t + \partial_v$ (we use the Einstein summation convention thoughout this paper, e.g. $v^k \partial_k = \sum_{k=1}^{3} v^k \partial_k$.)
Assumption on the pressure

$$
P(\rho, s) \in C^6(\mathbb{R}^+ \times \mathbb{R}), P_\rho(\rho, s) > 0, \ \text{for} \ \rho > 0, s \in \mathbb{R}. \tag{1.2}
$$

Given a simply connected bounded domain $\mathscr{D}_0 \subset \mathbb{R}^3$, and initial data (ρ_0, v_0, s_0) on $\mathscr{D}_0$, we want to find a set $\mathscr{D} \subset [0, T] \times \mathbb{R}^3$, a vector field v and scalar functions ρ, s and ϕ solving (1.1) and satisfy the initial conditions:

$$\mathscr{D}_0 = \{x : \ (0, x) \in \mathscr{D}\} \ \text{ and } \ (\rho, v, s) = (\rho_0, v_0, s_0) \ \text{ on } \ \{0\} \times \mathscr{D}_0. \tag{1.3}$$

Let $\mathscr{D}_t = \{x \in \mathbb{R}^3 : \ (t, x) \in \mathscr{D}\}$. We require the boundary conditions on the free surface $\partial \mathscr{D}_t$,

$$P = \bar{P}, \ \ \phi = 0, \ \text{ and } \ v_N = \mathscr{V}(\partial \mathscr{D}_t) \ \text{ on } \ \partial \mathscr{D}_t \tag{1.4}$$

for each t, where $\bar{P}$ is a positive constant, N is the exterior unit normal to $\partial \mathscr{D}_t$, $v_N = N^i v_i$, and $\mathscr{V}(\partial \mathscr{D}_t)$ is the normal velocity of $\partial \mathscr{D}_t$.

Physical Backgroud

System of PDEs in (1.1) can be used to model the motion of a plasma consisting of cold ions and hot electrons. In this context, ρ denotes the density of ions, and $e^{-\phi}$ the density of electrons (the electrons follow the classical Maxwell-Boltzmann relation), ϕ is the electric potential field, v is the velocity of the ions, $P(\rho, s)$ is the pressure and s is the entropy.

Goal

1) Identify suitable stability condition motivated by the Taylor sign condition for the problem of Euler equations,
2) Obtain a priori estimates on the Sobolev norms of the fluid variables and bounds for some geometric quantities of the free surface, such as the second fundamental form and the injectivity radius of the normal exponential map.

Related Results on Fluids Free Boundary Problems

Incompressible Euler Equations

Local Well-posedness in Sobolev Spaces:

1. First by S. Wu for irrotational flow with gravity (water wave problem), 2d (*Invent. Math.* ,1997), 3d (*J. Amer. Soc* ,1999).

2. Several extensions to non-irrotational flows have been obtained by different methods: Christodoulou-Lindblad (*Comm. Pure Appl. Math.* 2000), Lindblad (*Annals of Math.* 2005), Lannes (*J. Amer. Math. Soc.* 2005), Coutand- Shkoller (*J. Amer. Math. Soc.* 2007), Shatah-Zeng (*Comm. Pure Appl. Math.* 2008) , P. Zhang-Z. Zhang (*Comm. Pure Appl. Math.* 2008).... .– M. Ming-C. Wang (Water waves problem with surface tension in a corner domain, SIAM 2020, CPAM 2021)

Global or almost global solutions for irrotational incompressible Euler equations with gravity

irrotational, incompressible with gravity, no surface-tension, (infinite depth):

– Wu (2d: Invent. Math. 2009; 3d, Invent. Math. 2011);

– Germain, Masmoudi and Shatah (3d: Annal. Math.2012);

– Ionescu and Pusateri (2d: Invent. Math.2014);

.....

Very active recently on the global solution (irrotational) : with surface tension, finite depth.....:
Alazard-Delort (Ann. Sci. EC. Norm. Super(2015)), Alvarez-Samaniego- D. Lannes(Invent. Math.), Y. Deng, A.D. Ionescu, B. Pausader, F. Pusateri, (Acta Math. (2017)), P. Germain, N. Masmoudi, J. Shatah (CPAM 2015), A. Ionescu, F. Pusateri (CPAM 2016), A. Ionescu, F. Pusateri (Mem. Amer. Math. Soc, 2018), X. C Wang (Anal. PDE 10 (4) (2017), CPAM 2018, Adv. in Math. 2019).
Long-time solution for incompressible Euler-Poisson
Bieri, Lydia; Miao, Shuang; Shahshahani, Sohrab; Wu, Sijue, Comm. Math. Phys. 355 (2017).

The Free Boundary Problem for Compressible Euler and Related Equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0 \text{ in } \mathfrak{D}_t, \\ \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \nabla_{\mathbf{x}} p(\rho, s) = 0 \text{ in } \mathfrak{D}_t, \\ \partial_t s + v \cdot \partial s = 0, \text{ in } \mathfrak{D}_t, \\ p(\rho, s) = \bar{p} \text{ on } \Gamma(t) := \partial \mathfrak{D}_t, \\ \mathscr{V}(\Gamma(t)) = v \cdot \mathbf{n}, \\ (\rho, v, s) = (\rho_0, v_0, s_0) \text{ on } \mathfrak{D}_0. \end{cases} \tag{1.5}$$

$(\mathbf{x}, t) \in \mathbb{R}^n \times [0, \infty)$ $(n = 1, 2, 3)$: the space and time variable,

ρ: density, v: velocity, s: entropy, p: pressure.

$\mathfrak{D}_t \subset \mathbb{R}^n$: the changing volume occupied by the gas at time t.

$\mathscr{V}(\Gamma(t))$: normal velocity of $\Gamma(t)$,
$\mathbf{n}$: exterior unit normal vector to $\Gamma(t)$.

$$\bar{p} > 0 : \text{ constant.} \tag{1.6}$$

The case of non-vacuum boundary (on $\partial \mathscr{D}_t,\ p = \bar{p} > 0.$)

The full compressible Euler equations with the free boundary being a graph
Trakhinin (CPAM 2009) (local well-posedness): for the case when the free boundary is a graph and the gravity effect is taken into consideration based on the approach of symmetrization of hyperbolic systems. The assumption that the free boundary is a graph in is crucially used to flatten the boundary in Trakhinin (CPAM 2009).

Loss of derivatives in Trakhinin's solution

Local-in-time well-posedness via Nash-Moser iteration. It should be noted that these well-posedness results do not contain full a priori estimates since the iteration schemes based on the linearization lose the regularity on the moving boundary, the linearized problems do not preserve the full estimates of the nonlinear problems of which the full symmetry of the problems provided by the physical laws (e.g. conservation laws) is used. In fact, it is proved in Trakhinin (CPAM 2009) that when the initial data of the fluid variables $(\rho_0, v_0, s_0) \in H^{m+7}$ and $\partial \mathscr{D}_0 \in H^{m+7}$ for $m \geq 6$, there is a local-in-time solution with $(\rho, v, s)(\cdot, t) \in H^m$ and $\partial \mathscr{D}_t \in H^m$ for $t \in (0, T]$ for some $T > 0$. The solution loses 7-derivatives.

Isentropic Euler equations in general bounded domains

1. H. Lindblad (CMP 2005)
Local-in-time well-posedness via Nash-Moser iteration.
2. H. Lindblad- C. Luo (CPAM 2018): The higher order energy estimates .

Comparison of isentropic and non-isentropic cases

Isentropic case (s is constant): the pressure P is a sole strictly increasing function of density ρ , the enthalpy $(h(\rho) = \int_1^\rho P'(\lambda)\lambda^{-1} d\lambda)$, pressure and density are equivalent. One may take either one of them as an independent thermal dynamical variable. This is an advantage taken in the estimates in Lindblad-Luo (CPAM 2018). Indeed, the enthalpy h is used in in Lindblad-Luo as an independent thermal dynamical variable which satisfies a nice wave equation.
However, this does not hold anymore for a variable entropy s for $P = P(\rho, s)$ for a non-isentropic flow.

Stability Condition

The Taylor sign condition of the pressure, $\partial_N P < 0$ on $\partial \mathscr{D}_t$, plays an important role to the stability in the study of the free boundary problems of inviscid fluids, excluding the Rayleigh-Taylor type instability, without which problems may become ill-posed (see Ebin for the problem of incompressible Euler equations and Hao-Luo (CMP2020) for ideal incompressible MHD).

For the problem of compressible Euler equations coupled with a nonlinear Poisson equation considered in this paper, we find that only the Taylor sign condition for the pressure may not be adequate.

Stability Condition

From the momentum equation:

$$D_t v \cdot N = -\frac{\partial_N P}{\rho} + \partial_N \phi, \text{ on } \partial\mathscr{D}_t. \tag{1.7}$$

The acceleration of the free surface $\partial\mathscr{D}_t$ is due to two parts, $-\frac{\partial_N P}{\rho}$ and $\partial_N \phi$. Therefore, besides the Taylor sign condition $\partial_N P < 0$ on $\partial\mathscr{D}_t$ for pressure P, we also propose another stability condition $\partial_N \phi > 0$ on $\partial\mathscr{D}_t$ so that

$$D_t v \cdot N > 0, \text{ on } \partial\mathscr{D}_t.$$

Stability Condition in isentropic case

$s = constant$, $P = P(\rho, s) =: P(\rho)$, let $h(\rho) = \int_1^\rho P'(\lambda)\lambda^{-1} d\lambda$ be the enthalpy,

$$D_t v + \partial(h(\rho) - \phi) = 0.$$

In this case, the following generalized Taylor sign condition

$$\partial_N(h(\rho) - \phi) < 0, \text{ on } \partial\mathscr{D}_t \tag{1.8}$$

can be used to replace the conditions $-\partial_N P > 0$ and $\partial_N \phi > 0$ on $\partial\mathscr{D}_t$ proposed in this paper for the general variable entropy case.

Stability Condition: non-isentropic vs isentropic

Isentropic: $h(\rho) - \phi = const$ on $\partial \mathscr{D}_t$ so that $\nabla(h(\rho) - \phi) = N\partial_N(h(\rho) - \phi)$ on $\partial \mathscr{D}_t$, where N is the unit outer normal on $\partial \mathscr{D}_t$, and thus

$$\eta D_t \partial^r \mathscr{P} - \partial^r v_m N^m = \eta \partial^r D_t \mathscr{P} + \eta \mathscr{R}_r, \text{ on } \partial \mathscr{D}_t, \tag{1.9}$$

where $\mathscr{P} = h(\rho) - \phi$, $\eta = -\frac{1}{\partial_N \mathscr{P}}$, $\partial_m \mathscr{P} = \partial_N \mathscr{P} N_m$ and $\mathscr{R}_r = [D_t, \partial^r]\mathscr{P} + \partial^r v^m \partial_m \mathscr{P}$.

Non-isentropic case: it is impossible to write $\frac{\partial P(p,s)}{\rho} - \partial \phi$ as a gradient of a scalar function. This is a big difference between the isentropic and non-isentropic flow.

One may attempt to try the following stability condition for the non-isentropic flow,

$$\frac{\partial_N P(\rho, s)}{\rho} - \partial_N \phi < 0, \text{ on } \partial \mathscr{D}_t, \tag{1.10}$$

motivated by (1.7). However this does not work since one cannot write $\frac{\partial_N P(\rho,s)}{\rho} - \partial_N \phi$ as $\partial_N \mathscr{P}$ for some scaler function $\mathscr{P}$ as in the isentropic case.

Geometric Quantities on the Free Surface

Orthogonal projection Π to the tangent space of the boundary:
For a $(0, r)$ tensor $\alpha(t, x)$,

$$(\Pi\alpha)_{i_1 \cdots i_r} = \Pi_{i_1}^{j_1} \cdots \Pi_{i_r}^{j_r} \alpha_{j_1 \cdots j_r}, \quad \text{where} \quad \Pi_i^j = \delta_i^j - \mathcal{N}_i \mathcal{N}^j. \tag{2.11}$$

The tangential derivative of the boundary:

$$\bar{\partial}_i = \Pi_i^j \partial_j,$$

The second fundamental form of the boundary:

$$\theta_{ij} = \bar{\partial}_i \mathcal{N}_j.$$

Injectivity radius

The injectivity radius of the normal exponential map of the boundary $\partial\mathfrak{D}_t$, ι_0, is the largest number such that the map

$$\begin{aligned} \partial\mathfrak{D}_t \times (-\iota_0, \iota_0) \;\to\; \{x \in \mathbb{R}^n :\; \mathrm{dist}(x, \partial\mathfrak{D}_t) < \iota_0\} : \\ (\bar{x}, \iota) \;\mapsto\; x = \bar{x} + \iota\mathcal{N}(\bar{x}) \end{aligned}$$

is an injection.

Another Geometric quantity

Def: Let $0 < \varepsilon_1 \leq 1/2$ be a fixed number, and let $\iota_1 = \iota_1(\varepsilon_1)$ be the largest number such that

$$|\mathscr{N}(\bar{x}_1) - \mathscr{N}(\bar{x}_2)| \leq \varepsilon_1 \ \text{ whenever } \ |\bar{x}_1 - \bar{x}_2| \leq \iota_1 \ \ \bar{x}_1, \bar{x}_2 \in \partial \mathscr{D}_t.$$

It is easier to control the time evolution of ι_1 than ι_0.

Lemma (CL00) Suppose that $|\theta| \leq \mathscr{K}$. Then

$$\iota_0 \geq \min\{\iota_1/2,\ 1/\mathscr{K}\} \ \text{ and } \ \iota_1 \geq \min\{2\iota_0,\ \varepsilon_1/\mathscr{K}\}. \tag{2.12}$$

Quadratic form Q

Q is a positive definite quadratic form, such that for $(0, r)$ tensors α and β

1) $Q(\alpha, \beta) = < \Pi\alpha, \Pi\beta >$, is the inner product of the tangential components when it is restricted on the boundary,

2) in the interior $Q(\alpha, \alpha)$ increases to the norm $|\alpha|^2$ in the interior.

Here Π is the orthogonal projection to the tangent space of the boundary.

For a $(0, r)$ tensor $\alpha(t, x)$,

$$(\Pi\alpha)_{i_1 \cdots i_r} = \Pi_{i_1}^{j_1} \cdots \Pi_{i_r}^{j_r} \alpha_{j_1 \cdots j_r}, \quad \text{where} \quad \Pi_i^j = \delta_i^j - \mathcal{N}_i \mathcal{N}^j. \tag{2.13}$$

The construction is from Chistodoulou and Lindblad's work on the problem for incompressible Euler equations.

Construction of Quadratic Form Q

$$Q(\alpha,\beta) =: \chi^{i_1 j_1} \cdots \chi^{i_r j_r} \alpha_{i_1 \cdots i_r} \beta_{i_1 \cdots i_r},$$

$$\chi^{ij}(t,x) = \delta^{ij} - \eta(d(t,x))^2 \mathcal{N}^i(t,x)\mathcal{N}^j(t,x) \ \text{ in } \ \mathfrak{D}_t,$$

$$\mathcal{N}^j(t,x) = \delta^{ij}\mathcal{N}_i(t,x), \ \ \mathcal{N}_i(t,x) = \partial_i d(t,x) \quad d(t,x) = \mathrm{dist}(x, \partial\mathfrak{D}_t),$$

$$\eta(s) = \begin{cases} 1, & 0 \le s \le \iota_0/4, \\ 0, & \iota_0/2 \le s \le \iota_0, \end{cases}$$

$0 \le \eta(s) \le 1$ and $|\eta'(s)| \le 8$ for $s \in [0, \iota_0]$.

ι_0 is the injectivity radius of the normal exponential map of the boundary $\partial\mathfrak{D}_t$.

Hodge-type inequality

For a vector field v ,

$$|\partial^r v|^2 \leq C(\delta^{mn} Q(\partial^r v_m, \partial^r v_n) + |\partial^{r-1} divv|^2 + |\partial^{r-1} curlv|^2). \tag{2.14}$$

Higher order energy functional

Let

$$i(\rho, s) = \rho \int_1^\rho \frac{P(\eta, s)}{\eta^2} d\eta, \ \rho > 0, \tag{2.15}$$

and

$$j(\rho, s) = i(\rho, s) - i_\rho(1, s)(\rho - 1), \ \rho > 0. \tag{2.16}$$

Define

$$E_0(t) = \int_{\mathscr{D}_t} \left(\frac{1}{2}\rho|v|^2 + j(\rho, s) + \bar{P} \right) dx. \tag{2.17}$$

Set

$$E_r(t) = \int_{\mathscr{D}_t} (\delta^{mn}\rho Q(\partial^r v_m, \partial^r v_n) + F(P,s)Q(\partial^r P, \partial^r P))dx$$
$$+ \int_{\mathscr{D}_t} \rho(|\partial^{r-1}\mathrm{curl}\, v|^2 + |\partial^{r-1} divv|^2 + |\partial^r s|^2)dx$$
$$+ \int_{\partial\mathscr{D}_t} |(\Pi\partial^r P|^2 \nu + |\Pi\partial^r \phi|^2 \mu)d\sigma,\ 1 \le r \le 4, \qquad (2.18)$$

where $\nu = -(\partial_N P)^{-1}$, $\mu = (\partial_N \phi)^{-1}$, $F(P,s) = \frac{\rho_P(P,s)}{\rho(P,s))} > 0$,

$$P_r(t) = \int_{\mathscr{D}_t} (|D_t^{r+1}P|^2 + P_\rho |\partial D_t^r P|^2)dx,\ 0 \le r \le 4, \qquad (2.19)$$

Let

$$\mathscr{E}_r(t) = E_r(t) + P_r(t) + \int_{\mathscr{D}_t} |\partial^r P|^2 dx + \int_{\mathscr{D}_t} |\partial^r \phi|^2 dx,\ 0 \le r \le 4. \qquad (2.20)$$

Finally, we set

$$\mathscr{E}(t) = \sum_{r=0}^{4} \mathscr{E}_r(t). \qquad (2.21)$$

Main Theorem

Theorem (L. -Trivisa-Zeng) Suppose that the initial data satisfy that

$$c_1 \leq \rho_0(x) \leq c_2,\ x \in \mathscr{D}_0, \tag{2.22}$$

$$-\partial_N P(x,0) \geq \varepsilon_0, \partial_N \phi(x,0) \geq \varepsilon_1,\ x \in \partial \mathscr{D}_0, \tag{2.23}$$

$$K(0) =: \max_{\partial \mathscr{D}_0}(|\theta| + |\frac{1}{\iota_0}|) < \infty, \tag{2.24}$$

for some positive constants $c_1,\ c_2,\ \varepsilon_0, \varepsilon_1,$, and the pressure $P(\rho, s)$ satisfies (1.2). Then there exists a continuous function

$$\mathscr{T}\left(K(0), \mathscr{E}(0), \varepsilon_0^{-1}, \varepsilon_1^{-1} c_1, c_2, vol \mathscr{D}_0\right) > 0$$

such that if

$$T \leq \mathscr{T}\left(K(0), \mathscr{E}(0), \varepsilon_0^{-1}, \varepsilon_1^{-1} c_1, c_2, vol \mathscr{D}_0\right), \tag{2.25}$$

then any smooth solution of the free boundary problem (1.1)-(1.4) for $0 \leq t \leq T$ satisfies

$$0 \leq t \leq \mathscr{T}\left(K(0), \mathscr{E}(0), \varepsilon_0^{-1}, \varepsilon_1^{-1}, c_1, c_2, vol\mathscr{D}_0\right)$$

the following statements hold,

$$\mathscr{E}(t) \leq 2\mathscr{E}(0),\ 0 \leq t \leq T, \tag{2.26}$$

$$\frac{2}{3}\min_{x\in\mathscr{D}_0}\rho_0(x) \leq \rho(x,t) \leq \frac{3}{2}\max_{x\in\mathscr{D}_0}\rho_(x), \tag{2.27}$$

$$\mathscr{W}(t) \leq 2\mathscr{W}(0),\ \mathscr{U}(t) \leq 2\mathscr{U}(0), \tag{2.28}$$

$$K(t) =: \max_{\partial\mathscr{D}_t}(|\theta| + |\frac{1}{\iota_0}|) \leq C(\iota_1(0))\mathscr{W}(0)\sqrt{\mathscr{E}(0)+1}, \tag{2.29}$$

where

$$\mathscr{W}(t) = \| -\frac{1}{\partial_N P}\|_{L^\infty(\partial\mathscr{D}_t)},\ \mathscr{U}(t) = \|\frac{1}{\partial_N \phi}\|_{L^\infty(\partial\mathscr{D}_t)}.$$

Isentropic case

$s = constant$, $P = P(\rho, s) =: P(\rho)$, let $h(\rho) = \int_1^\rho P'(\lambda)\lambda^{-1} d\lambda$ be the enthalpy. Under the following generalized Taylor sign condition

$$\partial_N(h(\rho) - \phi) < -\varepsilon_0 < 0, \text{ on } \partial\mathscr{D}_0, \tag{2.30}$$

one may still obtain the related a priori estimates. However, since one does not have the estimates of $\int_{\partial\mathscr{D}_t} |\Pi\partial^r\phi|^2$ and one does not have the lower positive bound for $\partial_N\phi$ on $\partial\mathscr{D}_t$ in this case , one cannot use the projection formula $\Pi\partial^2\phi = \theta\partial_N\phi$ on $\partial\mathscr{D}_t$ to obtain the $L^\infty(\partial\mathscr{D}_t)$ bound for the second fundamental form θ. One may attempt to use the projection formula $\Pi\partial^2(h(\rho) - \phi) = \theta\partial_N(h(\rho) - \phi)$ on $\partial\mathscr{D}_t$. However, the estimate on $\partial^2(h(\rho) - \phi)$ involves the bound for θ, so it does not work. Instead, we use the evolution equation for θ to obtain the bound for θ, which requires one more derivative.

Stability condition for Isentropic case

Condition (2.30)

$$\partial_N(h(\rho) - \phi) < -\varepsilon_0 < 0, \text{ on } \partial\mathscr{D}_t, \tag{2.31}$$

means that, with the electric filed, the pressure does not have to satisfy the following Taylor sign condition

$$\partial_N P(\rho) < 0, \text{ on } \partial\mathscr{D}_0. \tag{2.32}$$

Remark

It should be noted that the a priori estimates involving H^5-norms for (ρ, u, s) still hold true for the full compressible Euler equations of gas dynamics without coupling with the electric filed. In this case, we only need the stability condition

$$-\partial_N P(x,0) \geq \varepsilon_0 > 0, \; x \in \partial \mathscr{D}_0. \tag{2.33}$$

Comparison with the solution obtained in Trakhinin (CPAM 2009)

For the free boundary being a graph, it is proved in Trakhinin (CPAM 2009) that when the initial data of the fluid variables $(\rho_0, v_0, s_0) \in H^{m+7}$ and $\partial\mathscr{D}_0 \in H^{m+7}$ for $m \geq 6$, there is a local-in-time solution with $(\rho, v, s)(\cdot, t) \in H^m$ and $\partial\mathscr{D}_t \in H^m$ for $t \in (0, T]$ for some $T > 0$. The solution loses 7-derivatives.

In our a priori estimates, we only require $(\rho_0, v_0, s_0) \in H^5$ and $\partial\mathscr{D}_0 \in H^5$ for the full compressible Euler equations satisfying the Taylor sign condition $\partial_N P(\rho, s) < 0$ on $\partial\mathscr{D}_0$, and this regularity propagates for some time without losing derivatives.

Ideas in the proof of the main theorem

The key idea of the proof of the theorem is to prove

$$\begin{aligned}&\frac{d}{dt}\mathscr{E}(t)\\&\leq C(\varepsilon^{-1},K,K_1,K_2,M,\bar{M},L,\bar{L})\\&\cdot(1+\|\mu\|_{L^\infty(\partial\mathscr{D}_t)}+\|D_t\mu\|_{L^\infty(\partial\mathscr{D}_t)})\mathscr{P}(\mathscr{E}(t)),\end{aligned} \tag{2.34}$$

for some polynomial $\mathscr{P}$, where

$$\mu=\frac{1}{\partial_N\phi},\ \text{on } \partial\mathscr{D}_t,$$

under the following a priori assumptions: For $0 < t \leq T$,

In $\mathscr{D}_t$

$$|\partial v| + |\partial P| + |\partial \phi| \leq M, \tag{2.35}$$

$$|D_t\phi| + |\partial^2\phi| + |\partial D_t\phi| + |D_t\partial\phi| \leq \bar{M}, \tag{2.36}$$

and on $\partial\mathscr{D}_t$,

$$-\partial_{\mathcal{N}} P \geq \varepsilon > 0, \tag{2.37}$$

$$\partial_{\mathcal{N}} \phi \geq \bar{\varepsilon} > 0, \tag{2.38}$$

$$|\theta| + \frac{1}{\iota_0} \leq K,\ \frac{1}{\iota_1} \leq K_1, \tag{2.39}$$

$$|\partial_N D_t P| + |\partial^2 P| \leq L, |\partial_N D_t \phi| \leq \bar{L}, \tag{2.40}$$

$$\|\bar{\partial}\theta\|_{L^2(\partial\mathscr{D}_t)} \leq K_2. \tag{2.41}$$

Then we justify the above a priori assumptions by

1) the Sobolev inequalities in which the Sobolev constants depend on K_1 (the boundary geometry),

2) the projection formula for θ by using the fact that the free surface is a level surface for P and ϕ,

3) maximal principle for ϕ,

4) the evolution equation for ι_1 and etc.

THANK YOU