Asymptotic behaviour of some anisotropic problems

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Basic notation

Ω is a bounded open set of $\mathbb{R}^n$. For $r > 1$ we set

$$W^{1,r}(\Omega) = \{v \in L^r(\Omega) \mid \partial_{x_i} v \in L^r(\Omega) \quad \mid \forall i = 1, \ldots, n\}. \tag{1}$$

We equip this space with the norm

$$||v||_{1,r,\Omega} = \Big(\int_\Omega |v|^r + \sum_{i=1}^{n} |\partial_{x_i} v|^r dx \Big)^{\frac{1}{r}} \tag{2}$$

and we set

$$W_0^{1,r}(\Omega) = \overline{\mathcal{D}(\Omega)} = \text{ the closure of } \mathcal{D}(\Omega) \text{ in } W^{1,r}(\Omega). \tag{3}$$

($\mathcal{D}(\Omega)$ denotes the space of C^∞-functions with compact support in Ω).

Basic notation

It is well known that $W_0^{1,r}(\Omega)$ is a reflexive Banach space which can be equipped with the equivalent norm

$$||\nabla v||_{r,\Omega} = \Big(\int_\Omega |\nabla v(x)|^r dx \Big)^{\frac{1}{r}}. \tag{4}$$

(∇ denotes the usual gradient and $| \ |$ the euclidean norm, i.e. $|\nabla v(x)| = (\sum_1^n (\partial_{x_i} v)^2)^{\frac{1}{2}}$, $| \ |_{r,\Omega}$ denotes the L^r-norm on Ω). The dual of $W_0^{1,r}(\Omega)$ is denoted by $W^{-1,r'}(\Omega)$, $r' = \frac{r}{r-1}$ and consists in the distributions of the form

$$f = f_0 - \sum_{i=1}^n \partial_{x_i} f_i, \quad f_i \in L^{r'}(\Omega). \tag{5}$$

We use the notation

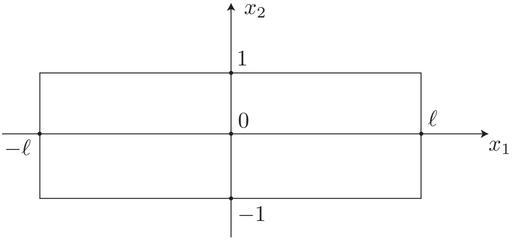
$$\langle f, v \rangle = \int_\Omega f_0 v + \sum_{i=1}^n f_i \partial_{x_i} v dx. \tag{6}$$

A model problem

We denote by Ω_ℓ the open subset of $\mathbb{R}^2$ defined as

$$\Omega_\ell = (-\ell, \ell) \times (-1, 1). \tag{7}$$

We will set $\omega = (-1, 1)$ and $\partial\Omega_\ell$ will denote the boundary of Ω_ℓ.



If $p, q > 1$ we would like to consider u_ℓ solution to

$$\begin{cases} -\partial_{x_1}\left(|\partial_{x_1} u_\ell|^{p-2}\partial_{x_1} u_\ell\right) - \partial_{x_2}\left(|\partial_{x_2} u_\ell|^{q-2}\partial_{x_2} u_\ell\right) = f \quad \text{in } \Omega_\ell, \\ u_\ell = 0 \quad \text{on } \partial\Omega_\ell. \end{cases} \tag{8}$$

A model problem

More precisely we are interested to the asymptotic behaviour of u_ℓ when $\ell \to +\infty$. f is a function or distribution depending only on x_2.
A natural candidate for the limit of the problem is u_∞ solution to

$$\begin{cases} -\partial_{x_2}\left(|\partial_{x_2} u_\infty|^{q-2}\partial_{x_2} u_\infty\right) = f \quad \text{in } \omega, \\ u_\infty = 0 \quad \text{on } \partial\omega, \end{cases} \tag{9}$$

where $\partial\omega = \{-1, 1\}$ is the boundary of ω. Let us recast these problems under their natural weak form.

We can first introduce the weak formulation of (9). If $f \in W^{-1,q'}(\omega)$ is given by

$$f = f(x_2) = f_0(x_2) - \partial_{x_2} f_1(x_2), \tag{10}$$

where $f_0, f_1 \in L^{q'}(\omega)$.

A model problem

Then, the weak formulation to (9) corresponding to f reads

$$\begin{cases} u_\infty \in W_0^{1,q}(\omega), \\ \int_\omega |\partial_{x_2} u_\infty|^{q-2}\ \partial_{x_2} u_\infty \partial_{x_2} v\ dx_2 = \langle f, v\rangle \\ \qquad\qquad\qquad = \int_\omega f_0 v + f_1 \partial_{x_2} v dx_2 \quad \forall v \in W_0^{1,q}(\omega). \end{cases} \tag{11}$$

To arrive to a weak formulation for (8) one introduces

$$W^{1,p,q}(\Omega_\ell) =$$

$$\{v \in L^p(\Omega_\ell) \cap L^q(\Omega_\ell) \mid \partial_{x_1} v \in L^p(\Omega_\ell),\ \partial_{x_2} v \in L^q(\Omega_\ell)\}. \tag{12}$$

It is a reflexive Banach space when equipped with the norm

$$||v||_{1,p,q,\Omega_\ell} = |v|_{p,\Omega_\ell} + |v|_{q,\Omega_\ell} + |\partial_{x_1} v|_{p,\Omega_\ell} + |\partial_{x_2} v|_{q,\Omega_\ell}. \tag{13}$$

A model problem

Then we define

$$W_0^{1,p,q}(\Omega_\ell) = \overline{\mathcal{D}(\Omega_\ell)} = \text{ the closure of } \mathcal{D}(\Omega_\ell) \text{ in } W^{1,p,q}(\Omega_\ell). \quad (14)$$

If f is defined by (10) if follows easily that there exists a unique u_ℓ weak solution to (8) i.e. satisfying

$$\begin{cases} u_\ell \in W_0^{1,p,q}(\Omega_\ell), \\ \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-2}\partial_{x_1} u_\ell \partial_{x_1} v + |\partial_{x_2} u_\ell|^{q-2}\partial_{x_2} u_\ell \partial_{x_2} v \quad dx_1 dx_2 \\ = \langle f, v \rangle = \int_{\Omega_\ell} f_0 v + f_1 \partial_{x_2} v \; dx_1 dx_2 \quad \forall v \in W_0^{1,p,q}(\Omega_\ell). \end{cases} \quad (15)$$

We are interested in showing that $u_\ell \to u_\infty$ when $\ell \to \infty$. The operators defined by (8), (9) are strictly monotone, hemicontinuous, coercive from $W_0^{1,p,q}(\Omega_\ell)$, $W_0^{1,q}(\omega)$ into their duals. Existence and uniqueness of a solution for (15), (11) follows from classical arguments

Let us first prove the following lemma.

Lemma

Suppose that f is given by (10). If u_ℓ is the solution to (15) there exists a constant C independent of ℓ such that

$$\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + |\partial_{x_2} u_\ell|^q \, dx \leq C\ell. \tag{16}$$

Proof : Taking $v = u_\ell$ in (15) we get

$$\begin{aligned}\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + |\partial_{x_2} u_\ell|^q \, dx = \langle f, u_\ell \rangle = \int_{\Omega_\ell} f_0 u_\ell + f_1 \partial_{x_2} u_\ell \, dx \\ \leq |f_0|_{q',\Omega_\ell} |u_\ell|_{q,\Omega_\ell} + |f_1|_{q',\Omega_\ell} |\partial_{x_2} u_\ell|_{q,\Omega_\ell} \\ \leq \Big(C|f_0|_{q',\Omega_\ell} + |f_1|_{q',\Omega_\ell} \Big) |\partial_{x_2} u_\ell|_{q,\Omega_\ell}\end{aligned} \tag{17}$$

this by the Hölder and the Poincaré inequality.

Preliminaries

Then let us notice that for $i = 0, 1$ one has

$$|f_i|_{q',\Omega_\ell} = \Big(\int_{-\ell}^{\ell} \int_\omega |f_i(x_2)|^{q'} dx_2 dx_1 \Big)^{\frac{1}{q'}} = (2\ell)^{\frac{1}{q'}} |f_i|_{q',\omega}.$$

Thus from (17) we derive for some constant $C = C(q, f)$

$$|\partial_{x_2} u_\ell|_{q,\Omega_\ell}^q \leq C\ell^{\frac{1}{q'}} |\partial_{x_2} u_\ell|_{q,\Omega_\ell}$$

Since $q' = \frac{q}{q-1}$ this is equivalent for some new constant to

$$|\partial_{x_2} u_\ell|_{q,\Omega_\ell} \leq C\ell^{\frac{1}{q}}.$$

Going back to (17), the result follows.

Preliminaries

Somehow one can ignore f thanks to the following remark.

Lemma

If u_ℓ is the solution to (15) and u_∞ solution to (11) one has

$$\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \partial_{x_1} v + \int_{\Omega_\ell} \Big\{ |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \Big\} \partial_{x_2} v \ dx = 0 \quad \forall v \ \in W_0^{1,p,q}(\Omega_\ell). \tag{18}$$

Proof : First by (15) if $v \ \in W_0^{1,p,q}(\Omega_\ell)$ one has

$$\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \ \partial_{x_1} v + |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell \ \partial_{x_2} v = \int_{\Omega_\ell} f_0 v + f_1 \partial_{x_2} v \ dx \tag{19}$$

Preliminaries

If $v \in W_0^{1,p,q}(\Omega_\ell)$ one has for almost every x_1

$$v(x_1, \cdot) \in W_0^{1,q}(\omega).$$

Thus by (11)

$$\int_\omega |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \ \partial_{x_2} v(x_1, x_2) \ dx_2 = \int_\omega f_0 v + f_1 \partial_{x_2} v dx_2.$$

Integrating in x_1 it comes

$$\int_{\Omega_\ell} |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \ \partial_{x_2} v \ dx = \int_{\Omega_\ell} f_0 v + f_1 \partial_{x_2} v dx. \qquad (20)$$

Subtracting from (19), (18) follows.

Preliminaries

Let us recall the following result which garanties also the strict monotonicity of the operators at hand.

Lemma

For any $q > 1$ there exist positive constants c_q, C_q such that

$$||\xi|^{q-2}\xi - |\eta|^{q-2}\eta| \leq C_q|\xi - \eta|(|\xi| + |\eta|)^{q-2} \quad \forall \xi, \eta \in \mathbb{R}^n, \tag{21}$$

$$(|\xi|^{q-2}\xi - |\eta|^{q-2}\eta) \cdot (\xi - \eta) \geq c_q|\xi - \eta|^2(|\xi| + |\eta|)^{q-2} \quad \forall \xi, \eta \in \mathbb{R}^n. \tag{22}$$

Then we have some monotonicity results.

Preliminaries

Lemma

Let $u_\ell = u_\ell(f)$ *be the solution to* (15) *and* $u_\infty = u_\infty(f)$ *be the solution to* (11). *Suppose that* $f \geq \tilde{f}$, $f \geq 0$ *then one has*

$$u_\ell(\tilde{f}) \leq u_\ell(f) \quad , \quad 0 \leq u_\ell(f) \leq u_\infty(f). \tag{23}$$

(If f *is not a function,* $f \geq 0$ *means* $\langle f, v \rangle \geq 0 \ \forall v \in W_0^{1,q}(\omega)$, $v \geq 0$*).*

The proof uses standard argument using as test functions $(u_\ell(\tilde{f}) - u_\ell(f))^+$...

The results coming next could be different following the case where $f = f_0(x_2) - \partial_{x_2} f_1(x_2)$ is a function (i.e. $= f_0$) or a distribution.

Also p and q do not have a symmetric role. The value 2 is another threshold for these problems.

Preliminaries

We can now show :

Lemma

If u_ℓ is the solution to (15) and u_∞ solution to (11) one has for every smooth functin $\varphi = \varphi(x_1)$ vanishing at $\{-\ell.\ell\}$

$$\int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + \left(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2}(u_\ell - u_\infty) \Big\} \varphi \, dx \leq \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-1} |\partial_{x_1} \varphi| |u_\ell - u_\infty| \, dx. \tag{24}$$

Preliminaries

Proof : Taking $v = (u_\ell - u_\infty)\varphi$ in (18) one gets

$$\int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + \Big(|\partial_{x_2} u_\ell|^{q-2}\partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2}\partial_{x_2} u_\infty \Big)\ \partial_{x_2}(u_\ell - u_\infty) \Big\}\varphi\ dx = - \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-2}\partial_{x_1} u_\ell\ \partial_{x_1}\varphi\ (u_\ell - u_\infty)\ dx. \tag{25}$$

(Recall that u_∞ is independent of x_1). Then (24) follows easily.

Convergence results

Denote by $\rho = \rho(x_1)$ a smooth function such that

$$0 \le \rho \le 1,\ \ \rho = 1 \text{ on } (-\frac{1}{2}, \frac{1}{2}),\ \ \rho = 0 \text{ near } \{-1, 1\},\ \ |\partial_{x_1}\rho| \le C. \tag{26}$$

and set for $\alpha > 0$

$$\varphi = \rho^\alpha = \rho^\alpha(\frac{x_1}{\ell}),$$

Lemma

Let $f = f_0 \in L^{q'}(\omega)$ and u_ℓ, u_∞ be the solutions to (15), (11). Then it holds for some constant C independent of ℓ

$$\int_{\Omega_\ell} \Big\{|\partial_{x_1} u_\ell|^p + \Big(|\partial_{x_2} u_\ell|^{q-2}\partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2}\partial_{x_2} u_\infty\Big)\ \partial_{x_2}(u_\ell - u_\infty)\Big\}\rho^\alpha\ dx \le \frac{C}{\ell^{p-1}} \quad (27).$$

Proof : From (24) one derives

$$I = \int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + \left(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2}(u_\ell - u_\infty) \Big\} \rho^\alpha \; dx \leq \frac{\alpha C}{\ell} \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-1} |u_\ell - u_\infty| \rho^{\alpha-1} \; dx. \tag{28}$$

Noting that $\rho^{\alpha-1} = \rho^{\frac{\alpha}{p'}} \rho^{\frac{\alpha}{p}-1}$ and using Hölder's inequality it comes

$$\int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + \left(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2}(u_\ell - u_\infty) \Big\} \rho^\alpha \leq \frac{\alpha C}{\ell} \left(\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p \rho^\alpha \; dx \right)^{\frac{1}{p'}} \left(\int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha-p} \; dx \right)^{\frac{1}{p}}. \tag{29}$$

Thus it follows that

Convergence results

$$I \leq \left(\frac{\alpha C}{\ell}\right)^p \int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha - p}\ dx \leq \left(\frac{\alpha C}{\ell}\right)^p \int_{\Omega_\ell} |u_\ell - u_\infty|^p\ dx, \tag{30}$$

provided we chose $\alpha > p$. From the lemma 2.4 one has

$$u_\ell(f) \leq u_\ell(f^+) \leq u_\infty(f^+) \ \ , \ \ u_\infty(-f^-) \leq u_\ell(-f^-) \leq u_\ell(f),$$

(notice that $u_\ell(-f) = -u_\ell(f)$). Then one derives

$$|u_\ell - u_\infty| \leq |u_\ell| + |u_\infty| \leq \max\{u_\infty(f^+), u_\infty(f^-)\} + |u_\infty(f)|.$$

Since this last function is independent of x_1 one derives from (30)

$$\int_{\Omega_\ell} \left\{ |\partial_{x_1} u_\ell|^p + \left(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2}(u_\ell - u_\infty) \right\} \rho^\alpha\ dx$$

$$\leq \frac{C}{\ell^{p-1}}$$

for some new constant C. This is (27).

Convergence results

Due to the definition of ρ we have obtained

$$\int_{\Omega_{\frac{\ell}{2}}} \left\{ |\partial_{x_1} u_\ell|^p + \left(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2} (u_\ell - u_\infty) \right\} dx$$

$$\leq \frac{C}{\ell^{p-1}}$$

It follows, if ℓ_0 is fixed less than $\frac{\ell}{2}$, that

$$\partial_{x_1} u_\ell \to 0 \text{ in } L^p(\Omega_{\ell_0}) \quad , \quad \partial_{x_2} u_\ell \to \partial_{x_2} u_\infty \text{ in } L^q(\Omega_{\ell_0}).$$

One can estimate the convergence rate in some situations. Indeed one has :

Convergence results

Theorem

Suppose that $p < q$. One has

$$\int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \, dx \leq \frac{C}{\ell^{\frac{pq}{q-p}-1}} \tag{31}$$

Theorem

Suppose that $p \geq q$, $q < 2$, $f \in L^1(\omega)$. It holds for some positive constants C

$$\int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \, dx \leq \frac{C}{\ell^{\frac{pq}{2-q}-1}}. \tag{32}$$

Convergence results

Theorem

Suppose that $p \geq q \geq 2$, $f \in L^1(\omega)$. It holds for some positive constants C, α

$$\int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \, dx \leq Ce^{-\alpha\ell}. \qquad (33)$$

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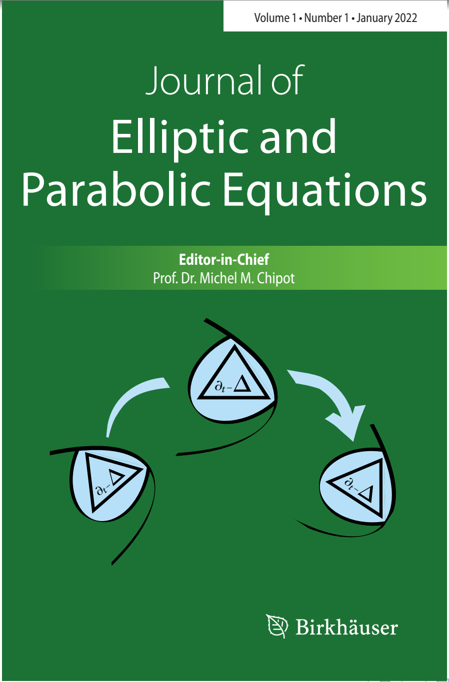

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