Well-posedness for local and nonlocal quasilinear evolution equations in fluids and geometry

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Nonlocal nonlinear evolution systems

We consider nonlocal nonlinear evolution systems of the form

$$\begin{cases} \partial_t u + \mathcal{A}_s(t,x,u,...,\nabla^l u)(u) = \mathcal{N}(t,x,u), & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0) = u_0 & \text{in } \mathbb{R}^d \end{cases} \tag{1}$$

where $d, l \in \mathbb{N}^+$, $u = u(t,x) : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}^m$, and $\mathcal{A}_s(t,x_0,u(t,x_0),...,\nabla^l u(t,x_0))(.)$ is a linear differential operator of order s with $s > l$ i.e

$$\mathcal{F}[\mathcal{A}_s(t,x_0,u(t,x_0),...,\nabla^l u(t,x_0))(u)](\xi) = A_s(\xi)\mathcal{F}(u(t))(\xi)$$

with $A_s(\xi) \sim |\xi|^s$ (e.g. $\mathcal{A}_s = -(1+u^2)\Delta u$ with $s = 2$) and $\mathcal{N}(t,x,u)$ is a nonlinear differential operator of u, with its derivative strictly less than s.

- Quasilinear systems : it is linear in highest order derivatives
- Local: $\mathcal{A}(t,x,u,...,\nabla^l u)(u)(t,x)$ is determined at x_0 if one needs to know only the values of the function $u(t)$ in an arbitrarily small neighborhood of x_0. Nonlocal case is a relation for which the opposite happens.

Some examples

- The Navier-Stokes equation: $u = (u_1(t,x), ..., u_d(t,x)) : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$

$$\partial_t u - \Delta u + \nabla p + \sum_{i=1}^{d} u_i \partial_{x_i} u = 0, \quad \sum_{i=1}^{d} \partial_{x_i} u_i = 0.$$

It can be rewritten by the following form

$$\partial_t u - \Delta u = \mathcal{N}(u).$$

A critical space is $B^{-1}_{\infty,\infty}$.

- The 2D Muskat equation with surface tension: $f = f(t,x) : (0,\infty) \times \mathbb{R} \to \mathbb{R}$

$$\partial_t f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1 + \partial_x f(x) \Delta_\alpha f(x)}{\langle \Delta_\alpha f(x) \rangle^2} \partial_x \left(\frac{\partial_x^2 f(x-\alpha)}{\langle \partial_x f(x-\alpha) \rangle^3} \right) \frac{d\alpha}{\alpha}, \quad \Delta_\alpha f(x) = \frac{f(x) - f(x-\alpha)}{\alpha}$$

It can be rewritten by

$$\partial_t f + \frac{|D|^3 f}{(1 + (\partial_x f)^2)^{3/2}} = \mathcal{N}(f).$$

Where $\mathcal{F}(|D|^3 f)(\xi) = |\xi|^3 \mathcal{F}(f)(\xi)$.
This is a nonlocal quasilinear evolution equation. A critical space is $W^{1,\infty}(\mathbb{R})$.

- The thin flim equation:

$$\partial_t u = \Delta(e^{-\Delta u}) \quad \Delta = \sum_{i=1}^{d} \partial_{x_i}^2.$$

It can be rewritten by

$$\partial_t u + e^{-\Delta u} \Delta^2 u = \mathcal{N}(u).$$

This is a local quasilinear evolution system.

- The Mean Curvature Flow of entire graph (Wang 2004):
$f = (f^1, ..., f^m) : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^m$

$$\partial_t f^j = A(\nabla f) \nabla^2 f^j \quad j = 1, ..., m$$

where

$$A(\nabla f) = \left[Id + \sum_{i=1}^{m} \nabla f^i \otimes \nabla f^i \right]^{-1}, \quad \frac{Id}{1 + |\nabla f|^2} \leq A(\nabla f) \leq Id.$$

This is a local quasilinear evolution system. A critical space is $W^{2,\infty}(\mathbb{R})$.

Some difficulties:

- A priori controls from conservation laws such as energy, momentum, ... are often too weak even for the existence of weak solutions
- Lack of maximal principles

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Tools:

- De Giorgi-Nash-Moser/ Krylov-Safonov theorem for holder estimates for uniformly elliptic or parabolic equations of second order with rough coefficients
  In general case, these ideas do not work for degenerate parabolic equations, systems, equations with higher order derivatives
- The Schauder estimates: one needs to require regular enough coefficients
- Energy method: one needs to keep singular weights that are suitable for the degeneracy of the problem.

Schauder-type estimates for evolution equations and their applications[1]

[1] Joint work with Ke Chen and Ruilin Hu.

Schauder-type estimates

We consider the elliptic equation

$$\mathrm{div}(A(x)\nabla u) = \mathrm{div}(F)$$

where the coefficient $A : \mathbb{R}^d \to M^{d\times d}$ satisfies $C^{-1}\mathbb{I} \leq A(x) \leq C\mathbb{I}$.
When A is a constant matrix, we have the Schauder estimate:

$$||\nabla u||_{\dot{C}^a} \lesssim \left(\sup_{k\in\mathbb{N}} \frac{1}{|a-k|}\right) ||F||_{\dot{C}^a} \quad \forall\, a \in \mathbb{R}_+ \backslash \mathbb{N}. \tag{2}$$

As a consequence, we can obtain the Schauder estimate when A is a non-constant matrix,

$$||\nabla u||_{\dot{C}^a} \lesssim \left(\sup_{k\in\mathbb{N}} \frac{1}{|a-k|}\right) (1 + ||A||_{\dot{C}^a})(||F||_{\dot{C}^a} + ||u||_{L^\infty}) \quad \forall\, a \in (0,\infty)\backslash \mathbb{N}. \tag{3}$$

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References:

- Agmon, Douglis, Nirenberg (1959/1964),
- Ladyzhenskaya-Uraltseva's Book (1964),
- Gilbarg-Trudinger's Book (1983).

How about the parabolic equation?

We consider the parabolic equation

$$\partial_t u - \operatorname{div}(\nabla u) = \operatorname{div}(F) \ \text{ in } \ \mathbb{R}^d \times (0,\infty), \quad u(0,x) = u_0(x) \ \text{ in } \ \mathbb{R}^d \times (0,\infty).$$

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Lemma (Chen, Hu and N. 2024)

For any $a \in (0,1)$, *there holds*

$$\sup_{t>0} ||u(t)||_{L^\infty} + \sup_{t>0} t^{\frac{1+a}{2}} ||\nabla u(t)||_{\dot{C}^a} \lesssim \frac{1}{a(1-a)} \left(\sup_{t>0} t^{\frac{1+a}{2}} ||F(t)||_{\dot{C}^a} + ||u_0||_{L^\infty} \right).$$

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As a consequence, we can obtain the Schauder estimate for solution to

$$\partial_t u - \mathrm{div}(A(x)\nabla u) = \mathrm{div}(F),$$

that for $T = T(||A||_{C^a}) \ll 1$.

$$\sup_{t\in(0,T)} \left(||u(t)||_{L^\infty} + t^{\frac{1+a}{2}} ||\nabla u(t)||_{\dot{C}^a} \right) \lesssim \frac{1}{a(1-a)} \left(\sup_{t\in(0,T)} t^{\frac{1+a}{2}} ||F(t)||_{\dot{C}^a} + ||u_0||_{L^\infty} \right).$$

Consider a general nonlocal parabolic equation

$$\partial_t u(t,x) + \mathcal{L}_s u(t,x) = \mathcal{P}_\gamma f(t,x) \quad \text{in } \mathbb{R}^d \times (0,\infty).$$

Here we assume the operator $\mathcal{L}_s$, $\mathcal{P}_\gamma$ satisfy

$$\mathcal{F}(\mathcal{L}_a u)(t,\xi) = A(\xi)\hat{u}(t,\xi), \quad \text{with } A(\xi) \gtrsim |\xi|^s, \quad |\nabla^\beta A(\xi)| \lesssim |\xi|^{s-\beta}, \forall \beta \geq 0, \xi \neq 0,$$
$$\mathcal{F}(\mathcal{P}_\gamma f)(t,\xi) = B(\xi)\hat{f}(t,\xi), \quad \text{with } |\nabla^\beta B(\xi)| \lesssim |\xi|^{\gamma-\beta}, \forall \beta \geq 0, \xi \neq 0,$$

for $0 < \gamma < s$.

Lemma (Chen, Hu and N. 2024)

For any $\kappa \in (s-\gamma, s)$

$$\sup_{t>0} t^{\frac{\kappa}{s}} ||u(t)||_{\dot{C}^\kappa} \lesssim \frac{1}{(s-\kappa)(\kappa-s+\gamma)} \left(||u_0||_{L^\infty} + \sup_{t>0} t^{\frac{\kappa}{s}} ||f(t)||_{\dot{C}^{\kappa-s+\gamma}} \right).$$

We can obtain the Schauder estimate for solution to

$$\partial_t u(t,x) + \lambda(x)\mathcal{L}_s u(t,x) = \mathcal{P}_\gamma f(t,x), \quad \inf_x \lambda \geq c_0 > 0$$

that for $T = T(||\lambda||_{C^s}, c_0) \ll 1$

$$\sup_{t\in[0,T]} \left(||u(t)||_{L^\infty} + t^{\frac{\kappa}{s}} ||u(t)||_{\dot{C}^\kappa} \right) \lesssim ||u_0||_{L^\infty} + \sup_{t\in[0,T]} t^{\frac{\kappa}{s}} ||f(t)||_{\dot{C}^{\kappa-s+\gamma}}.$$

Some basic applications

Recall that for any $s \in \mathbb{R}$ and $\gamma > 0$

$$||f||_{B^s_{\infty,\infty}} \sim \sup_{t>0} t^{\frac{a}{\gamma}} |||D|^s e^{-t|D|^\gamma} f||_{\dot{C}^a}$$

for any $0 < a < \gamma$ (Triebel 1983). Here $|D|^\gamma = (-\Delta)^{\frac{\gamma}{2}}$.

1) The fractional Navier-Stokes equation

$$\partial_t u + |D|^\gamma u = -\mathbb{P}\,\mathrm{div}(u \otimes u) \quad \text{in} \quad \mathbb{R}^d \times (0,\infty) \tag{4}$$

where $\gamma \in (1,2)$ and $\mathbb{P} = 1 - \nabla\Delta^{-1}\,\mathrm{div}$ is the Leray projection. The problem (4) is well posed in the Besov space $B^{1-\gamma}_{\infty,\infty}$.

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Remark:

- $\gamma = 1$, well posed in L^∞.
- $\gamma = 2$, $||u(t)||^2_{L^\infty} \lesssim t^{-1}$. The problem is strongly ill-posed in $B^{-1}_{\infty,\infty}$ (Bourgain-Pavlovic, 2010) and well posed in BMO^{-1} (Koch-Tataru 2003).

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2) The surface quasi-geostrophic (SQG) equation

$$\partial_t u + |D|^\gamma u + \mathrm{div}\left(u\nabla^\perp |D|^{-1} u\right) = 0 \quad \text{in} \quad \mathbb{R}^2 \times (0,\infty) \tag{5}$$

where $\gamma \in (1,2)$. The problem (5) is well posed in the Besov space $B^{1-\gamma}_{\infty,\infty}$.

3) The aggregation-diffusion equation

$$\partial_t \rho - \Delta\rho = \nabla \cdot (\rho\nabla|D|^{-s}\rho) \quad \text{in} \quad \mathbb{R}^d \times (0,\infty) \tag{6}$$

where $s \in (0,1)$. The problem (6) is well posed in the Besov space $B^{-s}_{\infty,\infty}$.

Geometric flows

We consider the Mean Curvature Flow (Wang 2004):
$f = (f^1, ..., f^m) : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^m$

$$\partial_t f = \left[Id + \sum_{i=1}^{m} \nabla f^i \otimes \nabla f^i \right]^{-1} \nabla^2 f \quad \text{in} \quad \mathbb{R}^d \times (0, \infty)$$

and $f(0) = f_0$. Note that $\dot{H}^{1+\frac{d}{2}}(\mathbb{R}^d), \dot{W}^{1,\infty}(\mathbb{R}^d)$ are critical spaces associated with this problem.

Theorem (Chen, Hu and N. 2024)

Let $f_0 \in C^1(\mathbb{R}^d, \mathbb{R}^m)$ and and $\beta \in \mathbb{R}^+ \backslash \mathbb{N}$. There exists $T > 0$ such that the problem has a unique solution f in $[0, T]$ satisfying

$$||f||_T := \sup_{t \in [0,T]} \left(||\nabla f(t)||_{L^\infty} + t^{\frac{\beta}{2}} ||\nabla f(t)||_{\dot{C}^\beta} \right) < \infty.$$

Moreove, if $||f_0||_{B^1_{\infty,\infty}} (1 + ||\nabla f_0||_{L^\infty})^4 \ll 1$, we can take $T = \infty$.

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Moreove, if $||f_0||_{B^1_{\infty,\infty}} (1 + ||\nabla f_0||_{L^\infty})^4 \ll 1$, we can take $T = \infty$.

Remark: when m = 1, the problem has a global unique solution f with any Lipschitz initial data by Ecker and Huisken (1989/1991).

Proof

Set

$$||f||_T := \sup_{t\in[0,T]} \left(||\nabla f(t)||_{L^\infty} + t^{\frac{\beta}{2}} ||\nabla f(t)||_{\dot{C}^\beta} \right),$$

$$A_\varphi(x) = \left[Id + \sum_{i=1}^{m} \nabla\varphi^i \otimes \nabla\varphi^i \right]^{-1} \quad \text{for} \quad \varphi \in C_c^3(\mathbb{R}^d, \mathbb{R}^m).$$

Note that $A_\varphi(x) \geq \frac{1}{\langle\nabla\varphi\rangle^2} Id$, $\langle b\rangle = (1+|b|^2)^{\frac{1}{2}}$.
We rewrite the equation as

$$\partial_t f - A_\varphi \nabla^2 f = (A_f - A_\varphi)\nabla^2 f,$$

with $||\nabla\varphi - \nabla f_0||_{L^\infty} \leq \varepsilon \ll 1$.

Proof

Set

$$||f||_T := \sup_{t\in[0,T]} \left(||\nabla f(t)||_{L^\infty} + t^{\frac{\beta}{2}} ||\nabla f(t)||_{\dot{C}^\beta} \right),$$

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Note that $A_\varphi(x) \geq \frac{1}{\langle\nabla\varphi\rangle^2} Id$, $\langle b \rangle = (1+|b|^2)^{\frac{1}{2}}$.
We rewrite the equation as

$$\partial_t f - A_\varphi \nabla^2 f = (A_f - A_\varphi)\nabla^2 f,$$

with $||\nabla\varphi - \nabla f_0||_{L^\infty} \leq \varepsilon \ll 1$. Set $\tilde{f}(t,x) = f(t,x) - \varphi(x)$. So,

$$\partial_t \partial_i \tilde{f} - \mathrm{div}(A_\varphi \nabla \partial_i \tilde{f}) = \partial_i \left((A_f - A_\varphi)\nabla^2 f \right) + l.o.t.$$

Then,

$$\begin{aligned} ||\tilde{f}||_T &\lesssim ||\nabla \tilde{f}(0)||_{L^\infty} + ||\tilde{f}||_T ||f||_T + C(\varphi) T^{c_0} \left(||f||_T + 1 \right)^2 \mathbf{1}_{\varphi \not\equiv 0} \\ &\lesssim \varepsilon + ||\tilde{f}||_T^2 + C(\varphi) T^{c_0} \left(||\tilde{f}||_T + 1 \right)^2 \mathbf{1}_{\varphi \not\equiv 0}. \end{aligned}$$

for some $c_0 > 0$.

Geometric flows

- The Willmore flow:

$$u_t + \langle \nabla u \rangle \operatorname{div}\left(\frac{1}{\langle \nabla u \rangle} \left(\left(Id - \frac{\nabla u \otimes \nabla u}{\langle \nabla u \rangle^2} \right) \nabla \left(\langle \nabla u \rangle H \right) - \frac{1}{2} H^2 \nabla u \right) \right) = 0 \tag{7}$$

  in $\mathbb{R}^2 \times (0, \infty)$ is well posed in $Lip(\mathbb{R}^2)$ (and $C^1(\mathbb{R}^2)$). Here $H = \operatorname{div}\left(\frac{\nabla u}{\langle \nabla u \rangle} \right)$, $\langle b \rangle = (1 + |b|^2)^{\frac{1}{2}}$.

- The Surface diffusion flow:

$$\partial_t u + \operatorname{div}\left((\langle \nabla u \rangle Id - \frac{\nabla u \otimes \nabla u}{\langle \nabla u \rangle}) \nabla H \right) = 0 \quad \text{in} \quad \mathbb{R}^d \times (0, \infty) \tag{8}$$

  is well posed in $Lip(\mathbb{R}^d)$ (and $C^1(\mathbb{R}^d)$).

- The Thin-film equation:

$$\partial_t u = \Delta(e^{-\Delta u}) \quad \text{in} \quad \mathbb{R}^d \times (0, \infty) \tag{9}$$

  is well posed in $\{f : \Delta f \in L^\infty(\mathbb{R}^d) (\text{and } C(\mathbb{R}^d))\}$.

Our method can be applied for these equations with homogeneous Dirichlet boundary condition.

Nonlocal quasilinear evolution equations

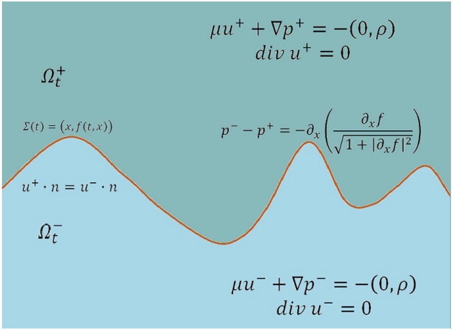

Figure: $\partial_t f = \langle \nabla f \rangle u^- . \vec{n}|_{\partial \Omega_t}$

Assume $\mu = 1$. We consider the 2D Muskat equation with surface tension

$$\partial_t f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1 + \partial_x f(x) \Delta_\alpha f(x)}{1 + (\Delta_\alpha f(x))^2} \partial_x \left(\frac{\partial_x^2 f(x-\alpha)}{\langle \partial_x f(x-\alpha) \rangle^3} \right) \frac{d\alpha}{\alpha} \quad \text{in} \quad \mathbb{R} \times (0, \infty).$$

Here $f : \mathbb{R} \times (0,\infty) \to \mathbb{R}$, $\langle a \rangle = (1+a^2)^{\frac{1}{2}}$, $\Delta_\alpha f(x) = \frac{\delta_\alpha f(x)}{\alpha} = \frac{f(x) - f(x-\alpha)}{\alpha}$.
Note that $\dot{H}^{\frac{3}{2}}(\mathbb{R}), \dot{W}^{1,\infty}(\mathbb{R})$ are critical spaces associated with this problem.

We rewrite the equation as

$$\partial_t f + \frac{1}{\langle \partial_x f \rangle^3} |D|^3 f = R(f) + \frac{1}{\pi} N(f),$$

where

$$N(f) = \int \frac{\Delta_\alpha f (\partial_x f - \Delta_\alpha f)}{\langle \Delta_\alpha f \rangle^2} \partial_x \left(\frac{\partial_x^2 f(. - \alpha)}{\langle \partial_x f(. - \alpha) \rangle^3} \right) \frac{d\alpha}{\alpha}$$
$$+ \int \partial_x^2 f(. - \alpha) \delta_\alpha \left(\frac{1}{\langle \partial_x f(.) \rangle^3} \right) \frac{d\alpha}{\alpha^2}.$$

Theorem (Chen, Hu and N. 2024)

Let $f_0 \in C^1(\mathbb{R}^d)$ *and* $\beta \in \mathbb{R}^+ \backslash \mathbb{N}$. *There exists* $T > 0$ *such that the problem has a unique solution* f *in* $[0, T]$ *satisfying*

$$||f||_T := \sup_{t \in [0,T]} \left(||\nabla f(t)||_{L^\infty} + t^{\frac{\beta}{3}} ||\nabla f(t)||_{\dot{C}^\beta} \right) < \infty.$$

Moreove, if $||\nabla f_0||_{L^\infty} \ll 1$, *we can take* $T = \infty$.

Nonlocal quasilinear evolution equations

The 2D Peskin equation with general force (Lin-Tong, Mori-Rodenberg-Spirn 2019)

$$\partial_t \boldsymbol{X} = \int_{\mathbb{T}} \partial_\alpha G(\delta_\alpha \boldsymbol{X}(s)) \left(\mathbf{T}(|\partial_s \boldsymbol{X}|)\partial_s \boldsymbol{X}\right)(s-\alpha) d\alpha.$$

Here $\boldsymbol{X} : \mathbb{T} \times (0,\infty) \to \mathbb{R}^2$ and $G(\boldsymbol{x}) = \frac{1}{4\pi}\left(-\log(|\boldsymbol{x}|)I + \frac{\boldsymbol{x}\otimes\boldsymbol{x}}{|\boldsymbol{x}|^2}\right)$ is the fundamental solution of the Stokes equation in $\mathbb{R}^2$, $\delta_\alpha \boldsymbol{X}(s) = \boldsymbol{X}(s) - \boldsymbol{X}(s-\alpha)$,
$\mathbf{T}(r) + r\mathbf{T}'(r), \mathbf{T}(r) > 0$, $\forall\ r \geq 0$.

The Peskin problem describes a closed one dimensional elastic structure immersed in a two dimensional field of incompressible viscous fluid and is free to move with the fluid velocity. It is inspired by the numerical immersed boundary method introduced by Peskin (1972, 2002) to study the flow patterns around heart valves.

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- Arc-chord assumption: There exists $\lambda_0 > 0$ such that

$$\inf_{s,\alpha} \frac{|\delta_\alpha \boldsymbol{X}_0(s)|}{|\alpha|} \geq \lambda_0,$$

  where $|\alpha| = \inf_{k\in\mathbb{Z}} |\alpha - k\pi|$.
- As Muskat equation, the 2D Peskin equation also has the following scaling propriety : if $\boldsymbol{X}(s,t)$ is a solution then the familly $\lambda^{-1}\boldsymbol{X}(\lambda s, \lambda t)$ is a solution with initial data $\lambda^{-1}\boldsymbol{X}_0(\lambda s)$ for all $\lambda > 0$. Under this scaling, $\dot{W}^{1,\infty}, BMO^1$ and $B^1_{\infty,\infty}$ are critical spaces.

We can rewrite the equation as

$$\partial_t \boldsymbol{X}(s) + \frac{1}{4} A(\partial_s \boldsymbol{X}(s)) \Lambda \boldsymbol{X}(s) = \tilde{\mathcal{N}}(\boldsymbol{X})(s),$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and $A(Z) = \mathbf{T}(|Z|)id + \frac{\mathbf{T}'(|Z|)}{|Z|} Z \otimes Z > 0$.

Theorem (Chen, Hu and N. 2024)

Let $\beta \in \mathbb{R}^+ \backslash \mathbb{N}$. *For any initial data* $\boldsymbol{X}_0 \in L^\infty \cap C^1$ *satisfying* $\inf_{s,\alpha} \frac{|\delta_\alpha \boldsymbol{X}_0(s)|}{|\alpha|} \geq c_0 > 0$, *there exists a time* $T_0 > 0$ *such that the problem has a unique solution* $\boldsymbol{X} \in C([0, T_0]; C^{1-})$ *satisfying*

$$\sup_{t \in [0, T_0]} \left(||\partial_s \boldsymbol{X}(t)||_{L^\infty} + t^\beta ||\partial_s \boldsymbol{X}(t)||_{\dot{C}^\beta} \right) < \infty, \quad \inf_{s,\alpha} \frac{|\delta_\alpha \boldsymbol{X}(s,t)|}{|\alpha|} \geq c_0/2 \ \forall t \in [0, T_0].$$

Remark: when $\mathbf{T}(r) \equiv 1$, the problem is local well posed for arbitrary initial configuration $X_0 \in (C^1)^{\dot{B}^1_{\infty,\infty}}$ with the arc-chord assumption.

The Peskin problem in $3D$ with general nonlinear elastic laws:

$$\frac{\partial \boldsymbol{X}}{\partial t}(\widehat{\boldsymbol{x}}) = \int_{\mathbb{S}^2} G(\boldsymbol{X}(\widehat{\boldsymbol{x}}) - \boldsymbol{X}(\widehat{\boldsymbol{y}}))\nabla_{\mathbb{S}^2} \cdot \left(\mathcal{T}\left(|\nabla_{\mathbb{S}^2} \boldsymbol{X}(\widehat{\boldsymbol{y}})|\right) \frac{\nabla_{\mathbb{S}^2} \boldsymbol{X}(\widehat{\boldsymbol{y}})}{|\nabla_{\mathbb{S}^2} \boldsymbol{X}(\widehat{\boldsymbol{y}})|} \right) d\mu_{\mathbb{S}^2}(\widehat{\boldsymbol{y}}),$$
$$\boldsymbol{X}(\widehat{\boldsymbol{x}})|_{t=0} = \boldsymbol{X}_0(\widehat{\boldsymbol{x}}), \tag{10}$$

where $\nabla_{\mathbb{S}^2}$ denotes the surface gradient operator on the unit sphere, $\mu_{\mathbb{S}^2}$ is the standard measure on the unit sphere, $G(x)$ is the 3D Stokeslet tensor: $G(\boldsymbol{x}) = \frac{1}{8\pi}\left(\frac{1}{|\boldsymbol{x}|} I_3 + \frac{\boldsymbol{x}\otimes\boldsymbol{x}}{|\boldsymbol{x}|^3}\right)$ and $\mathcal{T}' \geq c_0 > 0$. This equation describes a two-dimensional elastic membrane immersed in a three-dimensional steady Stokes flow. Similar as in 2D, we need to impose the arc-chord condition

$$\Theta(\boldsymbol{X}_0) := \sup_{\theta,\eta\in\mathbb{S}^2,\theta\neq\eta} \frac{|\theta-\eta|}{|\boldsymbol{X}_0(\theta) - \boldsymbol{X}_0(\eta)|} < +\infty. \tag{11}$$

Theorem (Chen, Hu and N. 2024)

Let $\beta \in \mathbb{R}^+\backslash\mathbb{N}$. There exists $\varepsilon_0 > 0$ such that, if the initial data $\boldsymbol{X}_0 \in W^{1,\infty} \cap (C^1)^{\dot{B}^1_{\infty,\infty}}$ satisfying $\Theta(\boldsymbol{X}_0) < \infty$. Then there exists a time $T_0 > 0$ such that the problem has a unique solution $\boldsymbol{X} \in C([0,T_0]; C^{1-})$ satisfying

$$\sup_{t\in[0,T]} \|\nabla \boldsymbol{X}(t)\|_{L^\infty} + t^\beta \|\nabla \boldsymbol{X}(t)\|_{\dot{C}^\beta} \leq C\|\nabla \boldsymbol{X}_0\|_{L^\infty}, \quad \Theta(T) \leq 2\Theta_0.$$

Juarez, Kuo, Mori and Strain (93 pages, 2023) proved this result with $C^{1,\alpha}$ initial data.

Thank you for your attention