Invariant Manifolds for Stochastic Partial Differential Equations

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Randomness or uncertainty is ubiquitous in scientific and engineering systems. Stochastic effects are not just introduced to compensate for defects in deterministic models, but are often rather intrinsic phenomena. Taking stochastic effects into account is of central importance for the development of mathematical models of many phenomena in physics, mechanics, biology, economics and other disciplines. Macroscopic models in the form of partial differential equations for these systems contain such randomness as stochastic forcing, uncertain parameters, random sources or inputs, and random initial and boundary conditions. Stochastic partial differential equations are appropriate models for randomly influenced systems.

Although many useful techniques exist to investigate deterministic partial differential equations as nonlinear dynamical systems, fundamental issues about studying stochastic partial differential equations as random dynamical systems remain unsolved.

In order to investigate stochastic partial differential equations from a dynamical systems point of view, we need to establish a theory for invariant manifolds for stochastic partial differential equations. As in deterministic systems, we expect that invariant manifolds, especially stable and unstable manifolds, to be essential for describing and understanding dynamical behavior of nonlinear random systems.

Recently, Duan, Lu and Schmalfuss [4, 5] have proved the existence of stable and unstable invariant manifolds at deterministic stationary points for a special class of stochastic partial differential equations with a multiplicative or additive white noise. The approaches are based on a random graph transform with a random fixed point theorem and Lyapunov Perron's method. On the other hand, Caraballo, Langa and Robinson [3] proved the existence of a local unstable manifold at the origin for a stochastic Chafee-Infante reaction-diffusion equation by truncating the equation in a suitable way, and proving the existence of inertial manifolds for the truncated equation. To establish the general theory of invariant manifolds for general stochastic partial differential equations, new ideas are needed to be developed.

As a research team at Banff, we have investigated the existence of invariant manifolds at a stationary process for general stochastic partial differential equations under the framework of exponential dichotomy which has a quantitative interpretation as a gap condition. The graph of this manifold is a fixed point of a certain transformation. Under the assumption that the nonlinearity is differentiable, we obtain manifolds with a sufficiently smooth graph. We have made much progress and expect to complete a paper by the end of this year. In the meantime, we have initiated two more new projects related to invariant manifolds.

References

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