BIRS Workshop on Coordinate Methods in Nonselfadjoint Operator Algebras

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An operator algebra is an algebra of operators on a Hilbert space, which is closed in a suitable topology. The most studied operator algebras are the selfadjoint ones: the norm-closed C*-algebras, which can be thought of as the noncommutative generalization of topological spaces, and the weakly-closed von Neumann algebras, the noncommutative generalization of measure spaces. Dropping the requirement that the operator algebra is closed under the adjoint operation gives a wide range of algebras with connections to various branches of mathematics. Examples include the algebras generated by a single operator or by a finite family of operators, and algebras constructed from dynamical systems. Even the algebras generated by finitely many (non-commuting) isometries have an interesting theory quite different from that of the corresponding C*-algebras.

Historically, some of the most challenging problems in the study of operators on Hilbert space showed the necessity of nonselfadjoint techniques. The invariant subspace problem, that every linear operator has a non-trivial invariant subspace, is a classic hard problem in operator theory. Positive results for specific families of operators, such as Lomonosov's result for operators which commute with a compact, or Brown, Chevreau and Pearcy's result for contractions whose spectrum contains the unit circle, rely on algebraic ideas. See, for example, the presentation of Lomonosov's result in Radjavi and Rosenthal's *Invariant Subspaces*. Reed and Enflo showed (independently) that there are operators on Banach spaces without invariant subspaces; the invariant subspace problem for an operator on a (separable) Hilbert space remains open. Also in the 1950's, Sz. Nagy's work on dilations of contractions demonstrated the power of nonselfadjoint techniques, and dilation results continue to play an important role in both operator theory and operator algebras.

A more formal study of nonselfadjoint operator algebras began with Kadison and Singer's 1960 paper on triangular operator algebras, the work of Ringrose on nest algebras in the early sixties, and Gohberg and Krein's book *An introduction of the theory of linear nonselfadjoint operators* from 1965. Also in this decade, connections between these algebras and dynamic systems were made, such as in Arveson and Josephson's work which demonstrated that the nonselfadjoint version of the operator algebra crossed product contained more information about the dynamics than the full selfadjoint crossed product. Understanding the structure of these algebras, in particular describing the ideals (closed or otherwise), became a focus of several important works, including those of Erdos and Power, McAsey, Muhly and Saito, and others. Davidson's text on nest algebras provides a comprehensive reference on the subject, although there have been a number of important development's since then, such as Orr's Interpolation Theorem for continuous nests and Annousis and Katsoulis's Russo-Dye Theorem for nest algebras.

In the last fifteen years, a new front has been opened, studying subalgebras of almost finite (AF) C*-algebras. These algebras, obtained as inductive limits of subalgebras of matrices, provide a tractable class of algebras which nevertheless contain many of the surprising properties of the wider family of all nonselfadjoint algebras. In an analytic sense, the triangular AF algebras are small, and somewhat discrete, which means many of the functional analytic tools required to study them are not overly complicated. Remarkably, the K-theory for these algebras is typically rather simple, contracting to the K-theory of the self-adjoint diagonal. Describing the larger algebra in term of the smaller diagonal is a useful strategy.

Coordinate methods allow for such a description, and form an important general tool in the study of nonselfadjoint operator algebras. They allow these abstract algebras to be realized concretely as function algebras using a suitable convolution multiplication. This conceptually simple description masks serious technical issues, such as building a general framework for coordinates and describing the operator algebra norm in terms of the functions on the coordinates. Nonetheless, coordinates have been fruitfully applied to a wide variety of nonselfadjoint operator algebras, leading to progress on classification, structural results, representation theory, and ideal structure, as well as yielding connections with dynamical systems and crossed product constructions.

The goal of this workshop is to bring together researchers in nonselfadjoint operator algebras and related areas, to unify and broaden the technical machinery of coordinate methods to a wider class of nonselfadjoint operator algebras. Key problems include classifications of these algebras and precise descriptions of the ideal structure, the family of homomorphisms between algebras, and related properties. Applications to the study of single operators, commuting and noncommuting family of operators, dilations, semigroups, free actions, and dynamical systems, are also of interest.

The key topics are discussed more fully below.

Single operators and their algebras

The earliest results in nonselfadjoint operator theory were theorems that describe basic properties of a single linear operator in terms of the algebra it generates: for instance, questions of invariant subspaces, dilations, nilpotency, and other properties are best revealed by studying a full algebra and not just the single operator. Marcoux, in joint work with Farenick and Forrest, described the connection of amenability of a single operator in terms of the amenability of the Banach space of operators it generates. Such results on amenability are key theorems in harmonic analysis and C*-theory (e.g. equivalent to nuclearity), and these results are fundamental steps for nonselfadjoint algebras.

Families of isometries and graph algebras

From single operators, one next considers algebras generated by families of isometries; these include the free semigroup algebras as well as the graph algebras, wherein the structure of the isometries is encoded in a specified, directed graph. These include a large family of key examples of nonselfadjoint algebras, and one goal of this program is to develop techniques that can analyze this class of operators, as well as extend the results in C*-theory for these type of structures. Davidson, Hopenwasser, Katsoulis, Kribs, Larocque, Peters all spoke on various aspects of these graph algebras.

Dynamical systems and crossed products

The evolution of a physical system over time is the basic model for a dynamical system. More generally, one considers the action of a group on a space, or on another algebra; the crossed product construction encodes both information about the space and the action in one algebra of operators. It is well-known that the nonselfadjoint variant of the crossed product contains more information than the large selfadjoint algebra; techniques to study these algebras effectively are still being developed. One particular challenge is that many of the existing methods involve some assumption of discreteness - an action of a discrete group, an r-discrete groupoid, and so on. An important research direction is to develop methods for the continuous case. Haataja, Lamoureux, Peters spoke on these topics.

Invariants

Invariants are a key tool in developing a classification theory for mathematical algebras. In nonselfadjoint algebras, many of the standard invariants (K-theory, homology, etc) of noncommutative geometry do not apply, or contract to a simple description of the algebra's diagonal, and thus more innovative invariants are needed. Pitts spoke on some joint work with Donsig on isomorphism invariants for triangular subalgebras, including the notion of the twist.

Spectral theorems

A spectral theorem describes an algebra or module as a space of functions on some fundamental object called the spectrum; for nonselfadjoint algebras, typically the spectrum is given by some maximal abelian selfadjoint algebras (masa), and operators are analyzed, and synthesized through a reference to this masa. Bimodules and ideals are often effectively characterized by these techniques. Katavolos spoke on masa bimodules and Todorov on ternary masa-bimodules

Subalgebras of W*-algebras

There is an important distinction between algebras that are closed in the norm topology, in the weak operator topology, and in the W* topology; the analytical techniques used to work with these various closures are quite different, as one sees in the difference between norm-closed C*-algebras and W*-closed von Neumann algebras. Erlijman and Solel both gave talks on such variants, one of subfactors and invariants for von Neumann algebras, the other on W*-correspondences. Davidson presented a Kaplansky-like density theory that it relevant to all three topologies for free semigroup algebras.

Open problems

One afternoon was devoted to presenting open problems, many of which had been hinted at in the expositions above. Listing relevant and important new challenges is a valuable contribution to the discipline. Kribs pointed out a number of basic questions concerning the free semigroupoid algebra L_G arising from a directed graph that remain to be resolved; for instance, when can $n \times n$ matrices of functionals be represented by collections of n-vectors (property A_n). Also with graph algebras, Peters suggested there should be some way to introduce a shift on standard Bratelli diagrams, to obtain a groupoid and a corresponding nonselfadjoint algebra; how would this compare to the AF C*-algebra described by the Bratelli diagram? Solel noted here there is a connection with the fixed point algebra, which may be a useful direction to pursue. Is the a classification for co-cycles on Cuntz-Krieger algebras, in analogy with Solel's classification of co-cycles on AF-groupoids by extended asymptotic ranges? Given an isomorphism of a Cuntz-Krieger groupoid, does the k-th level set map onto itself (or it's negative)? It's not even known if the zero-th set maps to itself. An old problem of Larry Brown was brought up by Davidson: is there an isomorphism of the Calkin algebra which reverses Fredholm index? Also for free semigroups algebras, when are there wandering vectors? Can one find a description of all completely contractive Schur idempotents; patterns in the matrix representation are important, but it is not easy to see how to build them all. One could ask the same question in the case of continuous nest algebras. Pitts considers some problems in AF algebras: for instance, given an operator with zero spectrum, can one find a triangular limit subalgebra containing this operator? Marcoux illustrated a more specific problem in UHF algebras: given an operator with zero trace, can it be expressed as a commutator? The answer is yes in finite dimensions, in other examples it can be expressed as a sum of two commutators, and there seems to be no obstruction to doing it in general with just one, but the answer is not known. Grossman considered some nonselfadjoint problems that arise in real problems with seismic imaging, including characterizing minimum phase operators which represent physical attenuation of seismic waves.

Titles and abstracts:

Ken Davidson: A Kaplansky theorem for free semigroup algebras

Abstract: A free semigroup algebra is the unital weak operator topology closed algebra generated by n isometries with pairwise orthogonal ranges. We show that the unit ball of the norm closed algebra is weakly dense in the whole ball if and only if the weak-$*$ closure agrees with the weak operator closure. This fails only when the weak closure is a von Neumann algebra but the weak-$*$ closure is not — and no examples of this phenomenon are known to exist.

Juliana Erlijman: On braid type subfactors and generalisations

Abstract: I will discuss a few aspects related certain construction of families of subfactors from braid group representations and of their extension to subfactors from braided tensor categories, as well as some techniques for computing important invariants for some of the examples.

Jeff Grossman: Minimum-phase preserving filters

Abstract: This talk is intended as part of the "open problems" session for the workshop. I'll begin by introducing causality and minimum phase conditions that come up in wave propagation and seismic imaging problems. In signal processing and imaging, we typically think of linear operators acting in L^2 as filters. The class of stationary (translation-invariant) linear filters corresponds to the convolution operators; and it is known that among these stationary filters, the ones which preserve minimum phase are precisely those described as convolution by a minimum phase function. So we ask the question: which nonstationary filters, if any, preserve minimum phase?

Steve Haataja: Inverse semigroups and crossed products

Alan Hopenwasser: Subalgebras of graph C*-algebras

Abstract: After a review of the groupoid associated with a graph C*-algebra, I will discuss the spectral theorem for bimodules. This says that a bimodule over a natural masa is determined by its spectrum iff it is generated by its Cuntz-Krieger partial isometries iff it is invariant under the gauge automorphisms. This contrasts notably with the situation for principal groupoids. If the edges of the graph are suitably ordered then (for finite graphs), there is a natural nest and a natural nest subalgebra associated with the order. I will describe the Cuntz-Krieger partial isometries which are in the nest algebra and the spectrum of the nest algebra.

Aristides Katavolos: Some results and problems on masa bimodules

Elias Katsoulis: Isomorphisms of algebras associated with directed graphs

Abstract: Given countable directed graphs G and G', we show that the associated quiver algebras A_G, $A_{G'}$ are isomorphic as Banach algebras if and only if the graphs G are G' are isomorphic. For quiver algebras associated with graphs having no sinks or no sources, the graph forms an invariant for algebraic isomorphisms. We prove that the quiver algebra A_G, associated with a graph G with no sources, is isometrically isomorphic to the disc algebra $alg(G)$ of the universal Cuntz-Krieger graph C*-algebra $C^*(G)$. This allows us to extend our classification scheme to subalgebras of graph C*-algebras of Cuntz-Krieger type. We also show that given countable directed graphs G, G', the free semigroupoid algebras L_G and $L_{G'}$ are isomorphic as dual algebras if and only if the graphs G and G' are isomorphic. In particular, similar free semigroupoid algebras are unitarily equivalent. For free semigroupoid algebras associated with locally finite directed graphs with no sinks, the graph forms an invariant for algebraic isomorphisms as well. (Joint work with D. Kribs.)

David Kribs: Directed graph operator algebras

Abstract: Every directed graph generates a family of operator algebras. They go by such names as Cuntz-Krieger or C-K-Toeplitz algebras, free semigroupoid algebras, quiver algebras, etc. Initial motivations came from dynamical systems, but now the study of these algebras has taken on a life of its own. Work on the nonselfadjoint subclass has been fruitful because it has been possible to link deep properties of the algebras with simple properties of the underlying directed graph in ways not possible for the C*-algebra case, and at the same time many new interesting examples have been discovered. I shall begin with a general discussion then touch on some specific results from joint works with Jury, Katsoulis and Power.

Michael Lamoureux: Continuous coordinate methods in nsa algebras

Abstract: Many of the tractable examples of nonselfadjoint operator algebras involve some assumption of discreteness: r-discrete groupoids, discrete group actions, graph algebras, and atomic nests, to name a few. To deal with more general nsa algebras that arise from dynamical systems and continuous group actions, we need more powerful tools to analyze the structure of the these algebras. We examine the continuous analogues for useful discrete coordinate methods.

Philippe Larocque: A spatial model for m λ-commuting isometries

Abstract: In this talk, we will describe a model for m isometries satisfying $V_iV_j = \lambda_{i,j}V_jV_i$ (in a Hilbert space). Basically, to (almost) every such m-tuple, a subset of Z^m can be chosen in such a way

that m isometries can be defined on it and these isometries are approximately unitarily equivalent to the original m isometries.

Laurent Marcoux: On amenable operators

Abstract: A Banach algebra A is said to be amenable if all (continuous) derivations of A into dual Banach A-bimodules M are inner. In this talk, we shall discuss the amenability of norm closed, singly generated algebras of operators on a Hilbert space. (Joint work with D.R. Farenick [Regina] and B.E. Forrest [Waterloo].)

Justin Peters: Cocycles on Cuntz-Krieger groupoids

Abstract: We examine $Z^1(G; R)$ where G is a Cuntz-Krieger groupoid. We begin with a representation theorem for cocycles. This theorem yields a connection between the dynamics of the shift map on path space, and properties of cocycles. In AF groupoids, the bounded cocycles and the integer-valued Cocycles play important roles. We look at these classes in the Cuntz-Krieger context.

David Pitts: Isomorphism invariants for subdiagonal triangular subalgebras of regular C*-inclusions

Abstract: A pair of unital C*-algebras (C, D) is a regular C*-inclusion if D is a MASA in C whose normalizers span C and is such that every pure state on D has a unique extension to a state on C. When this occurs, there exists a faithful conditional expectation $E : C \to D$. Following Arveson, a norm-closed subalgebra A with $D \subset A \subset C$ is triangular and subdiagonal if $A \cap (A)^* = D$ and $E|_A$ is a homomorphism.

For C*-diagonals, Kumjian introduced an isometric isomorphism invariant, which he called the twist. I will describe a class $E(A)$ of linear functionals on A which plays an role in the context of triangular subdiagonal algebras similar to that of the twist and which gives an invariant under bounded isomorphism. I will also discuss several questions about when this invariant is a complete invariant. (Joint work with Allan Donsig.)

Baruch Solel: Hardy algebras associated with W*-correspondences

Abstract: I shall discuss the construction of the Hardy algebras (which are the weak closures of the tensor algebras), their representations, canonical models for the representations and Schur-class operator functions.

Ivan Todorov: Normalisers, ternary rings of operators and reflexivity

Abstract: A ternary ring of operators is a subspace of $B(H, K)$ closed under the triple product $(T, S, R) \to TS^*R$. A ternary masa-bimodule is a ternary ring of operators which is also a bimodule for two maximal abelian selfadjoint algebras. In this talk a relation between ternary masa-bimodules and normalisers of some classes of operator algebras will be exhibited. The role ternary masa-bimodules play in operator synthesis will be described.

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