Dynamics, Probability, and Conformal Invariance

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The study of dynamics in the plane has recently seen a surge in interest due to three recent breakthroughs: the Sullivan-McMullen-Lyubich proof of the Feigenbaum Universality, the introduction by O. Schramm of SLE processes, and the work of S. Smirnov on percolation. The fields of Holomorphic Dynamics, SLE, and Conformal Field Theory (CFT) are now seen to be closely linked, the glue being provided by renormalization arguments, conformal mappings, Brownian Motion, and other methods related to Conformal Invariance. Indeed, there is an emerging field where these different dynamical processes, as well as more classical areas in conformal mappings, are unified into a more general theory. Though it is still early in the game, much progress has been made. The workshop has brought together leading experts from the areas of SLE, Holomorphic Dynamics, Probability Theory, and Conformal Mappings to present the latest developments in these areas and search for further unification of the fields.

1 Holomorphic Dynamics of Rational Maps and Kleinian Groups.

Hyperbolic geometry in 3 dimensions has experienced some very exciting progress recently. The main recent achievement is the completion of the program of Minsky of proving Thurston's Ending Lamination Conjecture (ELC) by J. Brock, D. Canary, and Y. Minsky [15, 3] (both in the incompressible-boundary case). The Conjecture had the same place in the field as the MLC (Mandelbrot set is Locally Connected) Conjecture occupies in Holomorphic Dynamics. It is a rigidity statement which postulates that combinatorial invariants (ending laminations) uniquely describe the geometry of the 3 manifolds. Another exciting recent progress is the proof of the Tameness Conjecture by Agol [1] and Calegari & Gabai

[4]. It implies, in particular, the Ahlfors' Conjecture: If the limit set of a finitely generated Kleinian group has no interior, then its area is zero.

These achievements are of particular interest to holomorphic dynamicists as both have analogues in the dynamics of rational maps (see below) which remain open. This, of course, is largely due to the fact that the geometric objects (hyperbolic 3-manifolds) provide an additional set of tools to the study of the dynamics of Kleinian groups, which are at best still being developed in the dynamics of rational maps. However, the intuition coming from Kleinian groups has historically played a very important role in Holomorphic Dynamics. **Yair Minsky's** mini-course was a major event of the workshop. In his lectures Minsky has outlined the proof of ELC, and tried to present the material in the form understandable to complex analysts and dynamicists.

In dynamics of rational maps the counterpart of the Ahlfors' Conjecture would state that the Julia set of a rational map is either equal to the sphere (that is has non-empty interior), or has area zero. Given parallels between the two fields, it is both exciting and unexpected that the feeling in rational dynamics is now that this statement may be false.

A decade ago A. Douady has initiated a program to the end of constructing a quadratic polynomial whose Julia set has positive measure. **A. Chéritat** [5] has recently been able to push through a large part of this program, and gave a lecture on his results. The Douady's program consists in approximating the candidate quadratic polynomial by a sequence of carefully chosen quadratics with parabolic periodic orbits.

Each step of approximation is done through two stages. If we write the rotation number of a parabolic point using the digits of its continued fraction expansion as $[a_0, a_1, \ldots, a_n, \infty]$, the first stage consists of perturbing the parabolic point to a nearby Siegel disk with rotation number

$$[a_0, a_1, \ldots, a_n, \text{ very large } N, 1, 1, 1, 1, \ldots];$$

and the second stage with going back to a parabolic

$$[a_0, a_1, \ldots, a_n, \text{ very large } N, 1, 1, 1, 1, 1, \ldots, 1, \infty].$$

Geometrically, each successive approximation should correspond to removing some thin cusps from the filled Julia set – the hope is to bound the area of what is left from below. The limit filled Julia set would then have a Cremer point. Such a filled Julia set would coincide with its Julia set and have a positive measure.

Chéritat has shown that the second stage of an approximation step may be carried out with an arbitrarily small loss of measure. Moreover, due to the work of X. Buff and A. Chéritat, making the loss arbitrarily small at the first stage boils down to several conjectures about *cylinder renormalization*. The latter was introduced by M. Yampolsky for proving the hyperbolicity of renormalization of critical circle maps. Geometrically, this renormalization boils down to successive blow-ups of the golden-mean Siegel Julia set. A convergence result for this procedure has been established earlier by McMullen; what is required now is a proof of the hyperbolic properties of the limiting fixed point.

The appearance of renormalization-type arguments is common for this class of problems: for example, Shishikura [19] used a parabolic renormalization procedure to demonstrate the existence of quadratic Julia sets of Hausdorff Dimension 2 used in his proof of HDim$(\partial M) = 2$. The one-dimensional renormalization theory (see e.g. [13]) has seen a spectacular progress since the works of Douady, Hubbard, and Sullivan which related it to Holomorphic Dynamics, culminating in a proof of the Feigenbaum Universality by Sullivan, McMullen, and Lyubich [11, 14, 21]. Many important problems of scaling invariance and universality still remain open, however, even in the setting of One-Dimensional Dynamics.

In particular, a renormalization hyperbolicity result for Siegel disks which would imply positive measure is still missing. However, a lot of numerical evidence exists in its favour, and moreover, the recent unpublished work of Shishikura opens an approach for settling this conjecture. In the light of the recent proof of Ahlfors' Conjecture, the existence of positive measure Julia sets would truly be surprising.

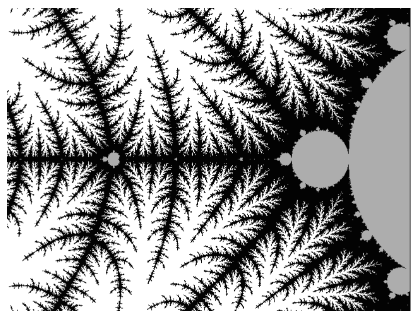

Figure 1: The Mandelbrot set near the Feigenbaum parameter: "hairy", but locally connected?

Of course, renormalization has been the main tool for the attack at the MLC Conjecture. This counterpart of ELC in the dynamics of quadratic polynomials, and its higher degree generalization, the Fatou Conjecture are arguably the main open problems in Holomorphic Dynamics.

In the early 1990's Yoccoz proved that MLC holds at all parameter values c in the boundary of the Mandelbrot set which are at most finitely many times renormalizable. His proof also showed that the corresponding Julia sets are also locally connected, provided all periodic orbits are repelling. Several partial results exist for infinitely renormalizable values of c, however, MLC is still not known in full generality. A particular example of an infinitely-renormalizable quadratic for which MLC is still open is the celebrated Feigenbaum quadratic polynomial. This particular map is infinitely renormalizable with the same combinatorial type (a particular case of *sattelite type*.

M. Lyubich has spoken on a new progress on this front in his recent joint works with J. Kahn [7, 8]. In these works the authors have introduced a new analytic tool for study of Julia sets, which they call a *Quasi-Additivity Law*. This law is a statement about extremal lenghts of families of curves, in the vein of the classical Grötsch Inequality, which is strongly motivated with the analogy with Kleinian groups. They use this law to prove that the Julia set $J(f)$ of at most finitely renormalizable unicritical polynomial $f : z \mapsto z^d + c$ with all periodic points repelling is locally connected, thus providing a higher-degree analogue of the results of Yoccoz. The theorem of Yoccoz was a major step towards MLC, and with the new tool, further progress can be expected. In particular, it is likely that the Quasi-Addittivity Law will lead to MLC for certain infinitely renormalizable values of c of satellite type, thereby bringing the conjecture closer to completion.

As we have seen above, Cremer Julia sets are conjecturally capable of having extreme measure-theoretical properties. Shishikura [19] has established earlier that these Julia sets can have Hdim $= 2$; and it is well-known that they are bad topologically, in particular, never locally connected. All the more surprising that the recent work of I. Binder, M. Braverman, and M. Yampolsky shows that these sets are *algorithmically computable*. That is, an algorithm may be produced to, given the value of the parameter c, draw such Julia sets on the computer screen with an arbitrary magnification. This nothwithstanding the fact that informative pictures of Cremer Julia sets have never been produced. **M. Braverman** reported on this work, as well as on his earlier result with M. Yampolsky demonstrating the existence of non-computable Julia sets in the quadratic family. These results led to a lively discussion, as a number of natural questions follow. Computability results for limit sets of Kleinian groups are not yet known. And in the quadratic case, the size of the set of values c for which the Julia set is uncomputable is interesting – and in particular, whether some such values are actually computable reals themselves.

2 Random shapes and conformal invariance

2.1 SLE

One of the central topics of the workshop was Stochastic (or Schramm) Loewner evolution (SLE) (see [18, 17]). It is a process defined by using one-dimensional Brownian motion as the driving parameter in Loewner's differential equation. There is one free parameter in SLE, which is the speed of the Brownian driving process. Thus the whole family of conformally invariant processes, SLE$_\kappa$, is defined. Introduction to the properties of SLE and the general overview of the subject was given by **Oded Schramm**, the inventor of SLE in the first talk of his three-lecture mini-course.

The SLE paths are conjectured to be the scaling limits of various natural random processes in the plane, such as the interface of critical percolation, the Ising model or the self avoiding walk. Some such statements have been recently proved by several authors: Smirnov [20] for Critical Percolation on the triangular lattice ($\kappa = 6$); Lawler, Schramm, and Werner [10] for the Uniform Spanning Tree ($\kappa = 8$), and Loop Erased Random Walk ($\kappa = 2$). Quite a few other statements of this sort remain unproven. The direction of the research is currently extremely active. Two of the talks of the mini-course series by Oded Schramm were devoted to the problem. First, he discussed the so-called harmonic explorer process, which, as proven by the speaker and Scott Sheffield, converges to SLE$_4$. Using the result, they establish that the level lines of the discrete Gaussian Free Field also converge to SLE$_4$. It was also explained how one can find SLE$_\kappa$ for $\kappa \neq 4$ in the Gaussian Free Field.

Although the conjectures about the value of κ for scaling limits of different lattice models are widely believed, it is not clear in a few cases which particular form of the SLE is obtained in the limit – there can be different parameterizations, boundary conditions, etc. To understand this situation in a few specific cases, Monte Carlo simulations of the two dimensional self-avoiding walk (SAW) were discussed at the talk by **T. Kennedy** entitled *Monte Carlo comparisons of the self-avoiding walk and SLE*. This simulations have given support to the conjecture that the scaling limit of the SAW is SLE with parameter 8/3. These past simulations treated the SAW and SLE as subsets of the plane, i.e., the parameterization of the curves was ignored. In this talk the speaker considered the SAW and SLE as parameterized curves and compared things that depend on the parameterization.

Another very active area of research is understanding of the fine geometric properties of the SLE. It is known that for $\kappa \leq 4$ SLE is almost surely a simple path (Rohde and Schramm [17]), for $\kappa > 4$ SLE is not a simple path almost surely, but is still generated by a curve called *trace*(Rohde and Schramm for $\kappa \neq 8$, Lawler, Schramm, and Werner for $\kappa = 8$). The estimate for the upper bound, $1 + \kappa/8$ on the Hausdorff dimension of SLE trace was established by S. Rohde and O. Schramm. It was shown by V. Beffara that the Hausdorff dimension of the SLE_κ-trace is actually equal to $1+\kappa/8$. On the other hand, the conjecture that the dimension of the boundary of the hull when $\kappa > 4$ is equal to $1 + 2/\kappa$ still remains open.

Normalized Schwarzian derivatives of SLE maps and other geometric properties of the SLE boundary were discussed by **Nam-Gyu Kang** in his talk *Boundary Behavior of SLE*. He showed that the normalized (pre-)Schwarzian derivatives of SLE maps with higher order terms are continuous square integrable martingales with second moment obeying the Duplantier duality. Also he showed that they have correlations that decay exponentially in the hyperbolic distance. The BMO space, or the space of functions of bounded mean oscillation, is the appropriate substitute for L^∞ in many results concerning singular integrals. This notion can be modified in the setting of continuous martingales. The normalized (pre-)Schwarzian derivatives of SLE maps with negligible terms are BMO martingales. As a corollary, they satisfy the John-Nirenberg inequality. This result may lead to an estimate on the lower bound for the Hausdorff dimension of the boundary of SLE hull. The results obtained by Kang allows to make a formal argument for the lower bound.

Stas Smirnov in his talk entitled *Conformally invariant fractals* discussed some recent progress and techniques in the study of the fine geometric properties of the SLE. In particular, he explained the multifractal analysis of harmonic measure on SLE. In a joint work with D. Beliaev the speaker derived a formula for one of the multifractal spectra, the so-called integral means spectrum, of the SLE. The spectrum reflects the behavior of the Riemann map for SLE near the boundary. Using these calculations one can see that the fine behavior of harmonic measure of the boundary, predicted by B. Duplantier, is very plausible.

Many geometric properties of the SLE were predicted by theoretical physicists (for example Cardy, Duplantier).

Overview of the physics point of view on SLE (see [6]) was given in the talk by **Bertrand Duplantier** entitled *Conformal fractal geometry and Quantum Gravity*. More specifically, he discussed the fractal geometry of conformally-invariant (CI) scaling curves. He focused on deriving critical exponents associated with interacting random paths, by exploiting an underlying quantum gravity (QG) structure, which uses KPZ maps relating exponents in the plane to those on a random lattice, i.e., in a fluctuating metric. This was accomplished within the framework of conformal field theory (CFT), with applications to well-recognized critical models, like $O(N)$ and Potts models, and to the Stochastic Löwner Evolution (SLE). Two fundamental ingredients of the QG construction are relating bulk and Dirichlet boundary exponents, and establishing additivity rules for QG boundary conformal dimensions associated with mutually-avoiding random sets. These rules are established from the general structure of correlation functions of arbitrary interacting random sets on a random lattice, as derived from random matrix theory. The physics derivation of the multifractal spectra was also discussed.

An essential role in the derivation is played by the Quantum Gravity, i.e. the theory of random two-dimensional Riemann surfaces, and especially by Knizhnik-Polyakov-Zamolodchikov (KPZ) equation [9]. It would be extremely important both for SLE theory and for String Theory to obtain the rigorous mathematical justification of the Quantum Gravity and KPZ.

The talk of **Angel** was devoted to the construction of the rigorous theory of discrete random Riemann surfaces.

One can also consider the talk of **J. Dubedat** entitled *Commutation of SLEs* related to this program. In the talk he discusses questions pertaining to the definition of several SLEs in a domain (i.e. several random curves). In particular, the speaker derived infinitesimal commutation conditions, discussed some solutions, and show how to lift these infinitesimal relations to global relations in simple cases. All these relations agree to what is predicted by the means of Quantum Gravity, and they give some insights on how Quantum Gravity can be defined using SLE.

The workshop finished with an informal talk by **Peter Jones**. In the talk he presented his new result related to the welding problem for the SLE. Peter Jones proposed a family of (random) homeomorphisms of the circle which are conjectured to be the welding homeomorphisms of SLE. The family of the homeomorphisms is related to the Gaussian Free Field on the unit circle. He also discussed the connection of this new family with some previous conjectures.

2.2 Other random shapes

Other random shapes were discussed during the workshop.

One of the most important questions in the Geometric function theory is understanding of the extremal behavior of the multifractal spectra. The answer to the question would incorporate the Makarov's and Jones-Wolff's dimension theorem, affirm the famous Brennan's conjecture, and answer many classical questions related to the coefficient growth problem for the univalent functions. It is known that the extremal behavior of the spectra is the same for general and for the simply-connected domains, and that this behavior "almost" occur on Julia sets. Nice upper estimates on the spectra were obtained by H. Hedenmalm and S. Shimorin using the technique of Bergman spaces.

D. Beliaev in the joint work with S. Smirnov proposed a new class of random fractals, so-called *Random Snowflakes*. It is proven that the almost extremal behavior of the integral means spectrum also occur for the class of objects. Because of the stochastic nature of the random snowflakes, the explicit calculations of the multifractal spectra for them are much easier to control. Using the random snowflakes new rigorous lower estimates on multifractal spectra are obtained. The estimates are now extremely close to the conjectured values.

A generalization of the simple random walk and SLE to two- and higher-dimensional processes is another active area of research. One of such analogies was given by **Rick Kenyon** in the talk entitled *Simple random surfaces*. The talk was devoted to the speaker's joint work in progress with David Brydges and Jessica Young. They consider a natural model of random immersed surfaces in a (finite or infinite) 2-complex. This is in many ways a natural generalization of the simple random walk. Although little is known about this model, certain expectations can be computed using the Green's function on 1-forms.

While SLE provide at least conjectural limit for various two dimensional lattice model, nothing like this exists in higher dimensions. **G. Slade** in the talk entitled *Scaling limits and super-Brownian motion* explained how critical percolation and related models can be described by super-Brownian motion, in high spatial dimensions. The talk provided a survey of several results and gave all the necessary background on super-Brownian motion.

3 Complex Analysis

Holomorphic Dynamics, the analytic theory of Kleinian groups, and SLE have their roots in classical complex analysis and geometric function theory. In this section we report on some developments and talks that are dealing with fundamental questions from complex analysis. They are not necessarily directly related to the topics described above, but in most cases the relevance to the central theme of the workshop is very obvious.

Joan Lind discussed how properties of the driving term in the Loewner equation affect the geometry of solutions to the Loewner equation. Since the Schramm-Loewner evolution is the Loewner equation driven by one-dimensional Brownian motion, this can be viewed as the ”deterministic” counterpart to the path properties of SLE. A natural space of driving terms is the space of Hoelder continuous functions with exponent 1/2, with the Hoelder norm c replacing the speed κ in SLE. It was shown that for $c < 4$ the Loewner equation always generates simple curves whereas for $c \geq 4$ selfintersections and even topologically wild compacts can occur. This phase transition at $c = 4$ is the deterministic counterpart to the phase transition in SLE at $\kappa = 4$ from simple to non-simple curves. In her talk she also illustrated by means of examples that there is no simple other phase transition that would correspond to the transition at $\kappa = 8$ from ”swallowing curves” to ”space filling” curves.

Another topic very closely related to the Loewner equation was discussed by **Don Marshall**. He (and independently Rainer Kühnau) discovered in the early 1980's an elementary algorithm for computing conformal maps (see [12]). The algorithm is fast and accurate, but convergence was not known. Given points $z_0, , z_n$ in the plane, the algorithm computes an explicit conformal map of the unit disk onto a region bounded by a smooth curve γ with $z_0, , z_n \in \gamma$. Marshall reported on joint work with S. Rohde, proving convergence for Jordan regions in the sense of uniformly close boundaries, and gave corresponding uniform estimates on the closed disc for the mapping functions. Improved estimates are obtained if the data points lie on a smooth or a K-quasicircle. The algorithm was discovered as an approximate method for conformal welding, however it can also be viewed as a discretization of the Loewner differential equation.

A central topic of complex analysis is quasiconformal mappings. Quasiconformal mappings appear naturally in the deformation theory of Riemann surfaces and are an indispensible tool in Kleinian groups. Since their introduction to complex dynamics in the proof of Sullivan's no wandering domain theorem, they have become one of the most powerful tools in dynamics. They are the main tool in the work of Marshall-Rohde and of Lind, as well as a cornerstone of the work of Peter Jones described above. **Kari Astala** and **Daniel Meyer** both talked about exciting developments related to the theory of quasiconformal maps. Astala described his deep joint work with Päivärinta [2], solving the Calderon's inverse conductivity problem: In tomography, or inverse problems in general, one aims to determine the structure of an object from indirect observations. Such methods have a variety of immediate applications, ranging e.g. from medical imaging to different industrial processes. A typical example is to determine the (conductivity) structure of a body from (electrical) measurements on the boundary. From the mathematical point of view this question has a clear and precise formulation, asking if the Dirichlet-to-Neumann boundary data determines the coefficients of a differential operator in the interior of a domain. In his talk, Astala discussed recent joint work with L. Päivärinta, solving the problem in two dimensions. Complex analysis, quasiconformal methods and, in particular, the function theoretic view to elliptic PDE's developed by Bers, are unavoidable for the solution in its full generality.

Self-similar sets in two dimensions often can be quasisymmetrically (quasiconformally) mapped to standard sets: For instance, limit sets of quasifuchsian groups and Julia sets of hyperbolic rational maps are quasiconformal circles (if they are topological circles). The powerful tools to prove such statements, an explicit geometric characterization of quasicircles (the Ahlfors three-point condition) and the $\lambda-$ Lemma about holomorphic motions, are not available in dimensions higher than two. Already in three dimensions, there are self-similar surfaces (such as the product of the van koch snowflake with the real line, known as "Rickman's rug") that cannot be quasisymmetrically parametrized by the plane. Daniel Meyer discussed quasisymmetric parametrizations of fractal surfaces in three dimensions. A Quasisphere is the image of the sphere under a quasiconformal map (of $\mathbf{R^3}$). The largest known class of quasispheres are called snowballs, they are topologically 2-dimensional analogues of the snowflake curve. For those surfaces the qc-embedding can be constructed explicitly. Many questions about the mapping behavior can be answered, at least numerically. For instance, Meyer showed that the "harmonic measure on a snowball", i.e. the image of Lebesgue measure of the sphere under the quasisymmetric parametrization, has dimension strictly smaller than the dimension of the snowball. This can be viewed as an analog of the celebrated Makarov theorem concerning harmonic measure of simply connected planar domains, and its higher dimensional generalization by Bourgain. The question if the dimension is greater than two (reminiscent of Wolff's example) was raised, at this point a positive answer is suggested by numerical results.

Nick Makarov reported on joint work with H. Hedenmalm on the quantum Hele-Shaw flow. The Hele-Shaw flow is closely related to the Loewner differential equation and describes the geometry of a growing "cell". It is used to model the interface between two fluids of different viscosity (such as water and oil). It appears as the formal limit of diffusion limited aggregation DLA.

Michel Zinsmeister talked about joint work with S.Rohde. The physicists Hastings and Levitov proposed a stochastic model for Laplacian growth, based on compositions of ”random” conformal maps, depending on a parameter $0 \leq \alpha \leq 2$. For $\alpha = 2$, the process is a version of DLA. SLE can also be viewed as (the scaling limit of) random compositions of conformal maps, but the difference is that in SLE the growth is restricted to a specified boundary point, whereas in the Hastings-Levitov model HL(α) the growth is uniformly distributed with respect to harmonic measure. Incidentally, the special case HL(0) was considered in the late 1980’s by Richard Rochberg and his son and called ”stochastic Loewner evolution”. As no result was published, the name did not stick. Indeed, the celebrated work of Hastings and Levitov appeared about ten years after Rochberg’s unpublished work. Zinsmeister proved some rigorous results about HL(α). In particular, he proved that the scaling limit for $\alpha = 0$ exists, he described this limit in terms of the Loewner equation, and proved that the Hausdorff dimension of the random set is 1 almost surely. For α near two, he explained how the Carleson-Makarov formalism can be adopted to the current setting to obtain nontrivial lower bounds for the dimension of the cluster. He also discussed the formal limit of the model and its relation to the Hele-Shaw equation.

A. Poltoratski described joint work with N. Makarov. He generalized the definition of Toeplitz operators to larger spaces of analytic functions. After that he studied the problem of injectivity of Toeplitz operators in these spaces. It turns out that many problems of classical analysis, such as distributions of zeros of entire functions (Levinson), completeness of bases of reproducing kernels (Beurling-Malliavin), spectral problems for the Schroedinger and string operators (Krein, Marchenko, ...), naturally become a part of the picture. One can use the Toeplitz approach together with some of the recent advances in complex and harmonic analysis to give shorter proves and further generalizations to the classical results.

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