Affine Schubert Calculus Workshop: Design and Implementation of Research Tools in MuPAD-Combinat

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1 Background

This meeting was part of the NSF Focus Research Group DMS-0652641:

"Affine Schubert Calculus: Combinatorial, geometric, physical, and computational aspects"

This project concerns the development of a vast extension of Schubert calculus to affine Grassmannians and affine flag varieties, called "affine Schubert calculus". Classical Schubert calculus, a branch of enumerative algebraic geometry concerned with counting subspaces satisfying certain intersection conditions, is the outcome of the solution to Hilbert's Fifteenth problem. In the modern formulation, Schubert calculus is usually interpreted in cohomology theories of homogeneous spaces, most notably flag varieties. The full development of affine Schubert calculus will solve long-standing open problems in Macdonald theory and have an impact on physical questions, such as generalizations of Wess-Zumino-Witten conformal field theory models and extensions of Calogero-Sutherland quantum mechanical models whose eigenfunctions are k-Schur functions. The new approach to affine Schubert calculus is made possible by the recent discovery of certain explicitly defined symmetric functions called k-Schur functions. The k-Schur functions, which arose in the study of the seemingly unrelated Macdonald theory, were recently shown to be connected to the geometry and topology of the affine Grassmanian. The novel combinatorics of k-Schur functions can be exploited to deduce formulae for various multiplicities, including intersection multiplicities in the affine Grassmannian and the affine flag manifold. Some of these multiplicities are known to occur in Macdonald theory and as Verlinde fusion coefficients for the WZW model.

This many-faceted project involves and ties together various problems from combinatorics, geometry, representation theory, physics, and computation. The main questions to be addressed can be viewed from several points of view: a geometric perspective (questions such as "how many lines are there satisfying a number of generic intersection conditions?"), a combinatorial perspective ("how many elements are in given sets and what properties do these sets have?"), a physics perspective ("how do fields correlate?"), and computational aspects ("are there efficient algorithms for calculating these numbers or objects?").

The project is an international cooperative research venture, with core group members located in Canada, the United States, Chile, and France, and interdisciplinary, involving mathematicians, physicists, and computer scientists.

The investigation is largely fueled by extensive computational experimentation. The robust implementation of algorithms derived from the project, is leading to the development of major extensions to the algebraic combinatorics open source project *-Combinat. It has a major impact on this project's life and growth.

Reciprocally, the dissemination of this new software through an open-source development model, not only advances the proposed research program but also has an outreach impact on the mathematics, physics, and computer science communities.

2 Purpose and outcome of the workshop

This meeting served as a start-up meeting for the participants of the FRG ”Affine Schubert Calculus”, focusing on computational aspects of the project. All participants attended the BIRS workshop 07w5048 ”Applications of Macdonald Polynomials” on September 10-14 2007 prior to the meeting.

The MuPAD-Combinat developers Florent Hivert, Nicolas Thiéry and Francois Descouens presented the main features of the existing package MuPAD-Combinat and introduced the participants to the technical details of the programming language and the distributed development tools (svn, wiki, etc.). All participants wrote small combinatorial algorithms appearing in their research as an exercise and committed them to the svn repository. For example, affine Stanley symmetric functions were implemented for types A and C. Further work was done on root systems, the k-shape poset, and the integration of the graph drawing tool graphviz.

In addition, the meeting served for discussions and code brainstorm about the design, development schedule, implementation, and integration of the required research tools:

- Kac-Moody algebras
- Representation theory (how to interact with GAP for example)
- k-Schur theory
- Integration of external C/C++ programs ...

The future of the software and the MuPAD platform was also largely discussed. This was an important step in the later decision (June 2008) of migrating from MuPAD-Combinat to Sage-Combinat. Sage is an open-source mathematics software package developed by a worldwide community of over one hundred people. It was started in 2005 by William Stein (now at the University of Washington) and it consists of over two million lines of code. It incorporates several of the best free, open-source mathematics software packages available (GAP, Singular, Macaulay, GMP, MPFR just to name a few), as well as a huge original library, including several new algorithms not yet found elsewhere (for more information on sage, visit www.sagemath.org). The merging of all code from MuPAD-Combinat to Sage-Combinat is in progress.