Loss of compactness in nonlinear PDE: Recent trends

Pierpaolo Esposito (Universitá degli Studi Roma Tre),
Frank Pacard (Université Paris XII-Val de Marne),
Gabriella Tarantello (Universitá degli Studi di Roma “Tor Vergata”)

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1 Overview of the Field

In the study of nonlinear elliptic PDEs, variational and topological methods are the essential tools to attack the existence question. They essentially rely on some compactness properties of the set of solutions or quasi-solutions. As we explain below, such properties fail in general for a large class of interesting problems which are relevant in physical, biological and theoretical models. This fact has opened in the sixties a very rich line of research which is far from being exhausted yet. In this context, there are several typical issues which we can roughly resume as follows:

- the basic description of the way a specific elliptic PDE exhibits a loss of compactness;
- the identification of the “good” situations where compactness is recovered and existence results can be established directly;
- a deeper description of the asymptotic behavior in the “bad” cases where usually both existence and non-existence can occurr;
- the construction of explicit solutions with a non-compact behavior through a combination of perturbative techniques and variational/topological devices in some specific “bad” situation.

The most famous and paradigmatic problem in this context is represented by the Yamabe equation in conformal geometry. On a compact $n-$dimensional manifold M, $n \geq 3$, with a background metric g_0, the problem of finding a conformal metric g to g_0 is equivalent to solve

$$-\frac{4(n-1)}{n-2}\Delta_{g_0} u + S_{g_0} u = c u^{\frac{n+2}{n-2}} \qquad \text{in } M, \tag{1}$$

for some $c = 0, \pm 1$ depending on the background metric g_0. Here, Δ_{g_0} is the Laplace-Beltrami operator with respect to g_0 and S_{g_0} is the scalar curvature of g_0, defined as a suitable trace of the Riemann tensor Riem_{g_0} of g_0.

Yamabe [22] proposed the following method to attack the existence. First, we replace the nonlinear term $u^{\frac{n+2}{n-2}}$ with $u^{\frac{n+2}{n-2}-\epsilon}$, $\epsilon > 0$. Compactness is recovered for every $\epsilon > 0$ yielding to a “ground-state” solution u_ϵ by standard variational methods. The solution of (1) is then obtained as the limit of u_ϵ as $\epsilon \to 0$. Yamabe in [22] claimed to be able to carry out this limiting procedure on u_ϵ.

Later, Trudinger [20] found a gap in Yamabe's proof, closely related to non-compactness phenomena for (1). In fact, not every solutions sequence u_ϵ has goood compactness properties, and the key point is to identify the "energy" levels where compactness might fail (the "energy" functional is here the associated Rayleigh quotient). In particular, the first "energy" threshold can be computed explicitly as $S^{\frac{n}{2}}$, where S is the Sobolev constant S of the embedding $D^{1,2}(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. Indeed, whenever $\|u_\epsilon\|_\infty \to +\infty$ as $\epsilon \to 0$, around the maximum points x_ϵ of u_ϵ the sequence u_ϵ has an asymptotic profile given by a solution U of

$$\begin{cases} -\Delta U = cU^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n \\ 0 < U \le U(0) = 1 & \text{in } \mathbb{R}^n. \end{cases} \tag{2}$$

For $c = 0, -1$ problem (2) does not possess solutions. Trudinger [20] solved (1) for compact manifolds with non-positive Yamabe invariant ($c = 0, -1$) where the sign of c in (2) prevents non-compactenss phenomena.

In the difficult case of positive Yamabe invariant $c = 1$, the problem remained open for several years and became popular as the Yamabe conjecture. When $c = 1$ all the solutions of (2) are known, coincide with the extremals of the Sobolev inequality and have the same "energy" $S^{\frac{n}{2}}$. The limiting procedure on u_ϵ is still effective for $c = 1$ and produces a positive solution to (1) whenever the "energy" level of u_ϵ does not approach the first bad level $S^{\frac{n}{2}}$. Since the flat space $\mathbb{R}^n$ and the round sphere (S^n, h) are in correspondence through the stereographic projection, the key point is how to "measure" the difference between the geometry of a manifold (M, g) and that of (S^n, h). In this respect, the complete resolution has been given in two steps by Aubin [2] in $'76$ and then by Schoen [17] in $'84$.

In the Euclidean space $\mathbb{R}^n$, the Yamabe problem takes the simpler form (2) and the solution set is well understood. On a bounded domain $\Omega \subset \mathbb{R}^n$, the Yamabe problem can be supplemented by a Dirichlet boundary condition:

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The "ground state" energy of (3) is independent of Ω and coincides with $S^{\frac{n}{2}}$–the one on $\mathbb{R}^n$. No hope to produce solutions with the Yamabe procedure. The idea of Brezis and Nirenberg [8] was to introduce a perturbing term λu^q, $1 < q < \frac{n+2}{n-2}$, to see a similar effect as in the Yamabe problem. This line of research has been continued later by several authors along the last twenty years, but the results have not had a deep scientific impact due to the limited relevance of such class of PDEs.

We won't discuss here other very interesting equations in nonlinear PDEs, where similar non-compactness phenomena arise, such as the prescribed $H-$curvature problem for surfaces with given boundary [6] and the problem of harmonic maps [16].

More generally, the identification of a limiting problem and the knowledge of its solution set give insights on non-compact sequences of solutions. To the above list of typical issues we then add:

- identification of the limiting problem and the corresponding solutions set.

The Yamabe problem is the higher-dimensional version of the constant Gauss curvature problem for compact surfaces– referred to as the Uniformization Theorem– which can be written as the two-dimensional equation

$$-\Delta_{g_0} u + K_{g_0} = e^u \qquad \text{in } S,$$

where K_{g_0} denotes the Gauss curvature of g_0. The statistical mechanics of point vortices in the mean field limit leads to variants of it:

on a compact surface S

$$-\Delta_{g_0} u = \lambda \left(\frac{Ve^u}{\int_S Ve^u} - \frac{1}{\text{vol } S} \right) \qquad \text{in } S, \tag{4}$$

and on a bounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} -\Delta u = \lambda \frac{Ve^u}{\int_\Omega Ve^u} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\lambda > 0$. When $\inf_\Omega V > 0$ a first aymptotic description of non-compact sequences of solutions to (5) has been given by Brezis and Merle [7] without assuming any boundary conditions. Additional information (Dirichlet boundary conditions or compact surfaces without boundary) allow a complete picture of the good/bad situations in terms of λ (and not in terms of "energy" levels): solutions with λ away from the set $8\pi\mathbb{N}$ form a compact set in a strong norm, while for $\lambda \to 8\pi k$ non-compact sequences can generally be found. The limiting problem, which is responsible for such a quantization of critical situations, reads in this case as

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2 \\ U \le U(0) = 0 & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^u dx < +\infty. \end{cases} \tag{6}$$

Chen and Lin [11, 12] have derived very precise asymptotic estimates on non-compact sequences and the Leray-Schauder degree d_λ, $\lambda > 0$, of the associated nonlinear map has been explicitly computed. Fine existence results readily follow whenever $d_\lambda \neq 0$. A somehow related problem (for example, the limiting equation is the same) is the Euler-Lagrange equations associated to the Moser-Trudinger functional on $H_0^1(\Omega)$:

$$\begin{cases} -\Delta u = \lambda u e^{u^2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $\lambda > 0$. See [1, 14].

We conclude this overview with the mention of the huge class of singularly perturbed problems. The scalar case is of interest in the study of standing waves of Schrödinger operators in the semi-classical limit [15], in the study of phase transitions [13] (the so-called Allen-Cahn equation) and in the Gierer-Meinhardt model for the dynamics of biological populations. The complex case has been largely studied in connection with the classical Ginzburg-Landau theory in super-conductivity for type II super-conductors [4].

2 Recent Developments and Open Problems

Despite of the complete resolution of the Yamabe problem, there are still interesting questions to be addressed. Schoen [18] attempts to describe the set of metrics having constant curvature in a given conformal class, with an interest towards a-priori estimates and multiplicity results of the Yamabe equation. In particular, for locally conformally flat compact manifolds M which are not conformally equivalent to the round sphere he shows that the Yamabe solutions in a given conformal class with prescribed volume form a compact set in $C^{2,\alpha}(M)$. He leaves open the case of non locally conformally flat compact manifolds, in literature referred to as the Schoen conjecture. Later, several authors (Druet, Li, Marques, Zhu, Zhang) have established the validity of this conjecture in low dimension $n \le 7$ and in every dimension under an additional assumption on g. Brendle has constructed counter-examples to this conjecture for dimensions $n \ge 52$, and it is still open to know the exact dimensions to have the validity of the Schoen conjecture.

The constant Gauss curvature problem extends to a fourth order equation on $4-$dimensional compact manifold, as well as the Yamabe problem to $n-$dimensional compact manifolds with $n \ge 5$. On $4-$manifolds, Paneitz in the first '80 discovered a fourth order operator P_g having the same transformation law of the Laplace-Beltrami operator Δ_g under conformal changes of the metric. The operator P_g is built with a principal part given by Δ_g^2. A notion of curvature –the so-called $Q-$curvature– was then introduced in terms of S_g and the Ricci tensor Ric_g of g. The problem of prescribing a constant $Q-$curvature on a compact $4-$manifold leads to a fourth-order elliptic PDE with esponential nonlinearity:

$$P_{g_0} u + 8 Q_{g_0} = 8 c e^u \qquad \text{in } M, \tag{8}$$

where $c = 0, \pm 1$ depends on the sign of $\int_M Q_{g_0} dv(g_0)$. The interest in solving this problem is in the same spirit of the Uniformization Theorem and relies on a $4-$dimensional Gauss-Bonnet formula:

$$\int_M (Q_g + \frac{|W_g|^2}{8}) dv(g) = 4\pi^2 \chi(M),$$

where W_g is the Weyl tensor of (M, g) and $\chi(M)$ is the Euler characteristic of M. Existence results are available for compact manifolds with non-negative Yamabe invariant and

$$0 \leq \int_M Q_{g_0} dv(g_0) < 8\pi^2.$$

It is completely open such a question when $\int_M Q_{g_0} dv(g_0) \geq 8\pi^2$.

In the flat case, equation (8) can be considered also on bouded domains $\Omega \subset \mathbb{R}^4$ in the mean field form

$$\Delta^2 u = \lambda \frac{V e^u}{\int_\Omega V e^u} \qquad \text{in } \Omega \tag{9}$$

and possibly supplemented by either Dirichlet boundary conditions

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega$$

or Navier boundary conditions

$$u = \Delta u = 0 \qquad \text{on } \partial\Omega.$$

Similarly as in two dimensions, we can consider the Euler-Lagrange equations associated to the Moser-Trudinger functional:

$$\Delta^2 u = \lambda u e^{u^2} \qquad \text{in } \Omega, \tag{10}$$

with either Dirichlet or Navier boundary conditions. Quantization issues for (9) and (10) are not known, and the "bad" situations have not been identified yet.

Another direction has been recently pursued in conformal geometry. Letting σ_k the $k-$th elementary symmetric polinomial, we have that S_g coincides with (a multiple of) $\sigma_1(\lambda_1(g), \ldots, \lambda_n(g))$, where $\lambda_1(g), \ldots, \lambda_n(g)$ are the ordered eigenvalues of the Schouten tensor of g –defined in terms of the Ricci tensor Ric_g of g and $S_g g$. Viaclowsky [21] has proposed the so-called $k-$Yamabe problem, which corresponds to finding a metric g in a given conformal class with $\sigma_k(\lambda_1(g), \ldots, \lambda_n(g))$ being a constant. The case $k = 2$ has been considered by Chang, Gursky and Yang [10], and in its Euclidean counter-part by Caffarelli, Nirenberg and Spruck [9] for every $k = 1, \ldots, n$. The geometric implications of solving the $k-$Yamabe problem are strong and make their study very interesting. Analytically, the problem is delicate and is still under investigation.

In several theories in super-conductivity (for example, in the Ginzburg-Landau and Chern-Simons models), there is a very special physical regime, referred to as the selfdual regime, where the Euler-Lagrange equations considerably simplify. The function $u = \ln|\phi|$ –where the Higgs complex function ϕ is an order parameter measuring the superconductivity state in the sample Ω– satisfies exactly equations like (4) or (5) with an additional singular source term supported at the zero set of ϕ. In all these models, such a set is always composed by finitely many points –the so-called vortices– with integer multiplicities.

Since the presence of a singular source term can be re-absorbed into a vanishing potential V, in many physical applications there is a definite interest in considering potentials V in (4) or (5) which vanish at finitely many points in Ω. From the physical point of view, the simplest situation of interest is a potential V in the form

$$V = |x - p|^\alpha K\,, \qquad \inf_\Omega K > 0$$

with $\alpha \in \mathbb{N}$. As when $\inf_\Omega V > 0$, there is a quantization of the "bad" levels in the form

$$\lambda \in 8\pi\mathbb{N}$$

as shown by Tarantello [19]. In the easier case $\alpha \notin \mathbb{N}$ a similar quantization does hold according to [19] and sharp asymptotic estimates are available in [3]. When $\alpha \in \mathbb{N}$ these estimates are still missing, no spike solutions are known and a formula for the Leray-Schauder degree is not available yet.

In the context of Ginzburg-Landau theories, physicists have proposed several other models to take into account specific effects which have physical relevance. For example, an anisotropic Ginzburg-Landau model does not adequately describe the magnetic properties of a layered high-temperature superconductor. The interest in layered structures relies on the fact that many of high-temperature superconductors are obtained as highly anisotropic crystalline materials composed of stacks of copper oxide superconducting planes separated by insulating or weakly superconducting material. The Lawrence-Doniach model has been instead proposed to properly capture the effect of the layered structure, where stacks of parallel superconducting planes are coupled via the Josephson interaction. This effect is particularly noticable when an external "in-plane" field is applied to the sample, i.e. the magnetic field is parallel to the superconducting planes. As $\epsilon \to 0$ it is not clear the behavior of the Lawrence-Doniach model.

More generally, for singularly perturbed problems it is often possible to construct for small $\epsilon > 0$ solutions which are strongly localized around finitely many concentration points. They are obtained in a constructive way as small perturbations of a specific family of approximating solutions, built on as a suitable gluing of several local profiles of the associated limiting problem centered at such concentration points. The concentration points can't be chosen freely but are prescribed by the problem under consideration.

In the last years, in collaboration with several authors, Malchiodi has been able to construct solutions localized around higher dimensional concentration sets Γ. The limiting problem, which provides the good local profiles to build the approximating family, has to be considered in $\mathbb{R}^{n-\dim\Gamma}$, and Γ is prescribed by some geometric condition (when dim Γ =1 , Γ is usually a geodesic with respect to some distance function). There are many analytical difficulties which have been overcome and clarified by Malchiodi's work. His ideas could be fruitfully applied in many problems (not only for the singularly perturbed ones) to construct solutions which exhibit a concentration and/or a blow-up behavior on a manifold Γ of positive dimension (in case the situation $\Gamma = \{p_1, \dots, p_k\}$ is already well understood).

3 Scientific Progress Made

The speakers of our workshop have reported about the results they have recently established. They have given partial/complete answers to the open questions in our field as listed in the previous section. We will report in the sequel only about the most striking achievements and we won't attempt to give a complete account of the talks in our workshop.

In this respect, the first result to quote is the complete resolution of the Schoen conjecture as reported by Marques. In collaboration with Khuri and Schoen, they establish compactness of the set of metrics with volume one and constant scalar curvature in a given conformal class for all the non locally conformally flat compact manifolds M of dimension $n \leq 24$. In this context, the counter-examples of Brendle for $n \geq 52$ are extended by Marques and Brendle to all the remaining dimensions $25 \leq n \leq 51$. The picture is then complete in the compact case: the Schoen conjecture is valid on locally conformally flat manifolds (different from the round sphere) and on non locally conformally flat manifolds of dimension $n \leq 24$, and is false in general in the remaining situations.

In a joint work with Djadli, Malchiodi solves the constant $Q-$curvature problem on compact $4-$manifolds with non-negative Yamabe invariant and $\int_M Q_{g_0} dv(g_0) \geq 0$. For the associated "energy" functional J, by looking at the "energy" sublevels $J^a = \{J \leq a\}$ for a very negative, they see how to define a suitable min-max scheme to produce a critical "energy" level c. They also introduce in the equation a parameter λ so that the original problem corresponds to $\lambda = \int_M Q_{g_0} dv(g_0)$. To recover compactness, taking advantage of the Struwe monotonicity trick, they show the validity of the Palais-Smale condition for λ in a small dense subset aorund $\int_M Q_{g_0} dv(g_0)$ and find an associated solution. By a previous result of Malchiodi, compactness does hold when λ is far from $8\pi^2\mathbb{N}$ and, by a limiting procedure, an existence result is deduced for $\int_M Q_{g_0} dv(g_0) \in [0,+\infty)\setminus 8\pi^2\mathbb{N}$. Malchiodi has also shown that a Morse theoretical approach, similar in spirit to the one just described, works also for (4) when $\inf_S V > 0$. It yields to existence results and simplifies the proof of a degree formula for d_λ obtained by Chen and Lin [12]. Let us also point out the contribution of the talk of Lucia on a general deformation lemma for a class of functionals that do not satisfy Palais-Smal condition, including the ones arising in the stuy of the mean field equation (4).

As far as (9), Robert in his talk has given a beautiful and complete description of the general state-of-art. When $\inf_\Omega V > 0$, in dimension two a non-compact sequence u_n always satisfies a concentration property of the nonlinear term which is equivalent to have

$$u_n - \int_\Omega V e^{u_n} dx \to -\infty \qquad \text{in } C_{\text{loc}}(\Omega \setminus \{p_1, \dots, p_k\}) \text{ as } n \to +\infty,$$

for a finite number of points $p_1, \dots, p_k \in \Omega$. Surprisingly, it generally does not hold in dimension four and the situation can become quite weird. However, adding a L^1-bound on Δu_n or a Dirichlet/Navier boundary condition, the situation becomes similar to that of the two-dimensional case.

Wei has reported about a later work, in collaboration with Lin, Robert and Wang, concerning (9) on a domain $\Omega \subset \mathbb{R}^4$ with Dirichlet or Navier boundary conditions. They have established compactness for λ in compact sets of $[0,+\infty)\setminus 64\pi^2\mathbb{N}$, sharp asymptotic estimates for $\lambda \to 64\pi^2\mathbb{N}$ as well as a degree-counting formula for d_λ, in the same line of [11, 12]. Similarly, Struwe has reported on a quantization property for (10) with Navier boundary conditions: for positive solutions $u_k \rightharpoonup 0$ weakly in $H^2(\Omega)$ the concentration energy

$$\Lambda = \lim_{k\to+\infty} \int_\Omega |\Delta u_k|^2 dx$$

is quantized in integer multiples of $\Lambda_1 = 16\pi^2$. A discussion grows out on the possibility of obtaining for (10) sharp asymptotic estimates and a degree formula as for (9).

On the $k-$Yamabe problem, we can point out the talk of Ge on an analytic foundation for the fully non-linear equation

$$\frac{\sigma_2(\lambda_1(g), \dots, \lambda_n(g))}{\sigma_1(\lambda_1(g), \dots, \lambda_n(g))} = f$$

on compact manifolds M with positive Yamabe invariant. As an application, in a joint work with Wang and Lin, they prove that, if a compact $3-$dimensional manifold M admits a metric g with positive scalar curvature $S_g > 0$ and $\int_M \sigma_2(\lambda_1(g), \lambda_2(g), \lambda_3(g)) dv(g) > 0$, then it is topologically a quotient of the sphere. In the fully nonlinear context, Y.Y. Li has explained his contribution in terms of a Liouville-type result for entire solutions of general conformally invariant fully nonlinear elliptic equations of second order, motivated by the study of the limiting problem along non-compact sequences. The interest is strictly related to a-priori estimates for this class of problems on compact manifolds M.

Thanks to the quantization property in [19] and to the sharp asymptotic estimates in [3], Lin has been able to compute a degree formula for (4) with a potential V in the form

$$V = |x-p|^\alpha K\,, \qquad \inf_S K > 0$$

with $\alpha \notin \mathbb{N}$. In his talk, he gives the explicit expression for the degree formula when $\alpha \notin \mathbb{N}$ and describes how to get, with a limiting procedure, a similar degree formula for $\alpha \in \mathbb{N}$. In this way, for $\alpha \in \mathbb{N}$ it is possible to overcome the difficult identification of all the possible non-compact sequences (and their contribution to the changes of the degree when λ crosses the values in $8\pi\mathbb{N}$). A discussion grows out on how try to establish sharp asymptotic estimates for $\alpha \in \mathbb{N}$ and how to construct explicit non-compact sequences. In this respect, del Pino and Musso carry to the audience attention a recent partial result in collaboration with Esposito on non-compact sequences for simply connected domains. A general result should be in order via a suitable gluing argument.

Sandier in his talk presents the rigourous derivation of an anisotropic Ginzurg-Landau theory as the limit, in a certain regime, of the Lawrence-Doniach model. In particular, if the interlayer distance goes to zero no faster than ϵ, an extension of the order parameter between the layers converges weakly to the canonical harmonic map away from the vortices, and the magnetic potential converges to a vector field whose magnetic field satisfies the anisotropic London equation. In an ongoing joint work with Alama and Bronsard, they also find that anisotropic $3D$ models have interesting Gamma limits as $\epsilon \to 0$.

In collaboration with Bronsard and Millot, Alama is interested to describe the asymptotic behavior as $\epsilon \to 0$ of the energy minimizers of a two-dimensional superconductor under the effect of an external applied magnetic field. They are interested in determining the number and the distribution of the vortices, which are defects of the superconductive state and appear in the model as quantized singularities. For an external applied magnetic field near the "lower critical field", as $\epsilon \to 0$ these vortices concentrate along a curve determined by a classcial problem from potential theory. Here, the "lower critical field" represents the critical value of an external applied magnetic field for which vortices first appear in the superconductor.

Solutions of the Allen-Cahn equation with finite energy as $\epsilon \to 0$ admit a limiting profile given as an entire solution of the Allen-Cahn equation (with $\epsilon = 1$) having at most a growth R^{n-1} of the energy on $B_R(0)$, as $R \to +\infty$. Del Pino presents the results of some joint works with Kowalczyk, Pacard and Wei, where a solution of this type is constructed having a finite number of nearly parallel transition layers. The solution is constructed as a gluing of one-dimensional profiles with a single transition located very far appart one to each other. Similarly, for the stationary nonlinear Schrödinger equation mulitple bump lines are found, while Toda system is shown to rule out the asymptotic shape of these transition lines.

The last talk we would like to quote concerns solutions which concentrate and blow-up along a curve. In collaboration with del Pino and Pacard, Musso considers slightly sub-critical Yamabe equation (3) in a domain $\Omega \subset \mathbb{R}^n$. Since they are interested in boundary concentration and $\partial\Omega$ is a $(n-1)-$dimensional manifold, the right exponent in the Yamabe equation is not $\frac{n+2}{n-2}$ but the critical Sobolev exponent in dimension $n-1$, i.e. $\frac{n+1}{n-3}$. For this problem, they construct a family of solutions whose energy density concentrates as a Dirac line measure on Γ, where Γ is a closed geodesic in $\partial\Omega$ with negative curvature and satisfying some non-degeneracy condition.

4 Outcome of the Meeting

The meeting has given the opportunity of all the participants to exchange ideas and to communicate new results and research directions in this field. As planned in our proposal, we have had the big opportunity to gather junior and senior scientists, and let them the possibility of a fruitful exchange of experiences.

We hope to have stimulated new interactions in our mathematical community, whose revenues will certainly become manifest in next years.

References

[1] Adimurthi and M. Struwe, Global compactness properties of semilinear elliptic equations with critical exponential growth, *J. Funct. Anal.* **175** (2000), 125–167.

[2] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pures Appl. (9)* **55** (1976), 269–296.

[3] D. Bartolucci and C.-C. Chen and C.-S. Lin and G. Tarantello, Profile of blow-up solutions to mean field equations with singular data, *Comm. Partial Differential Equations* **29** (2004), 1241–1265.

[4] F. Bethuel and H. Brezis and F. Hélein, *Ginzburg-Landau vortices*, Progress in Nonlinear Differential Equations and their Applications **13**, Birkhuser Boston, Boston, 1994.

[5] S. Brendle, Blow-up phenomena for the Yamabe equation, *J. Amer. Math. Soc.* **21** (2008), 951–979.

[6] H. Brezis and J.-M. Coron, Multiple solutions of H-systems and Rellich's conjecture, *Comm. Pure Appl. Math.* **37** (1984), 149–187

[7] H. Brezis and F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. Partial Differential Equations* **16** (1991), 1223–1253.

[8] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** (1983), 437–477.

[9] L. Caffarelli and L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian, *Acta Math.* **155** (1985), 261–301.

[10] A. Chang and M. Gursky and P. Yang, An equation of Monge-Ampère type in conformal geometry, and four manifolds of positive Ricci curvature, *Ann. of Math.* **155** (2002), 709–787.

[11] C.-C. Chen and C.-S. Lin, Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces, *Comm. Pure Appl. Math.* **55** (2002), 728–771.

[12] C.-C. Chen and C.-S. Lin, Topological degree for a mean field equation on Riemann surfaces, *Comm. Pure Appl. Math.* **56** (2003), 1667–1727.

[13] E. De Giorgi, Convergence problems for functionals and operators, *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*, pp. 131–188, Pitagora, Bologna, 1979.

[14] O. Druet, Multibumps analysis in dimension 2: quantification of blow-up levels, *Duke Math. J.* **132** (2006), 217–269.

[15] A. Floer and A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.* **69** (1986), 397-408.

[16] J. Sacks and K. Uhlenbeck, The existence of minimal immersions of 2-spheres, *Ann. of Math. (2)* **113** (1981), 1–24.

[17] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, *J. Differential Geom.* **20** (1984), 479–495.

[18] R. Schoen, On the number of constant scalar curvature metrics in a conformal class. In *Differential geometry*, Pitman Monogr. Surveys Pure Appl. Math., **52**, 311–320, Longman Sci. Tech., Harlow, 1991.

[19] G. Tarantello, A quantization property for blow up solutions of singular Liouville-type equations, *J. Funct. Anal.* **219** (2005), 368–399.

[20] N.S. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, *Ann. Scuola Norm. Sup. Pisa (3)* **22** (1968), 265–274.

[21] J. Viaclovsky, Conformal geometry, contact geometry and the calculus of varitions , *Duke Math. J.* **101** (2000), 283–316.

[22] H. Yamabe, On a deformation of Riemannian structures on compact manifolds,
Osaka Math. J. **12** (1960), 21–37.