Hausdorff geometry of complex polynomials, positive charge distributions and normal operators

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June 29, 2008–July 6, 2006

1 Overview

The intensive week spent at the Banff Research Center was focused on discussions of a series of precise quantitative conjectures and pointed questions referring to the Euclidean distance geometry of the critical points of complex polynomials versus the locations of their zeros. The subject goes back to some early studies in electrostatics by Gauss and Maxwell and has penetrated into modern mathematics via approximation theory, specifically the Ilieff-Sendov conjecture.

2 Open Problems

We have begun our attempt at solving the following three conjectures [1,2], the first of which is a strengthening of Sendov's conjecture.

Conjecture 1 [Variance conjecture for complex polynomials]: If $p \in \mathbb{C}[z]$ with $deg(p) \geq 2$ then $H(p) \leq \sigma_2(p)$, where $H(p)$ denotes the symmetrized Hausdorff distance between the zero sets of p and p' while $\sigma_2(p)$ is the standard deviation of the zero set of p.

During the focused research group, we formulated some sufficient conditions for the truth of this conjecture. Before we give an example, we need the following terminology.

Let $p(z) = \prod_{k=1}^{m}(z - z_k)^{m_k}$ where the z_k are distinct. Let $\{w_k\}_{k=1}^{m}$ be the roots of p' which are not also roots of p. We define the Gauss-Lucas matrix of p to be the m by n matrix G with entries $g_{ij} = \frac{|w_i - z_j|^{-2}}{\sum_{k=1}^{n} |w_i - z_k|^{-2}}$. This matrix is a stochastic matrix; a fact which was used by Cesaro in his proof of the Gauss-Lucas theorem [5, pg. 93]. It can be shown that the variance conjecture holds for p if every column of the Gauss-Lucas matrix has an entry greater than or equal to $1/n$. This observation gives us a nice proof of the conjecture for polynomials with at most three distinct roots. The conjecture also holds for polynomials with all roots real. This approach cannot be used to prove the conjecture in general since there are high degree polynomials whose Gauss-Lucas Matrices do not satisfy the above column property. (The 19th degree polynomial from [4] is one such example).

A similar conjecture can be stated for the zeros of the Cauchy transform of compactly supported probability measures. We first need the following notation. Given $K \Subset \mathbb{C}$, let $\mathcal{P}(K)$ be the set of all probability measures μ on $\mathbb{C}$ supported on K. For $\mu \in \mathcal{P}(K)$ denote by $E(\mu)$ and $\sigma_2(\mu)$ the barycenter and the standard deviation of μ, respectively. Let $\mathcal{W}(\mu)$ be the set of finite zeros of the Cauchy transform of μ (i.e., the equilibrium points of the logarithmic potential associated with μ) and let $\mathcal{W}_e(\mu) = \mathcal{W}(\mu) \cup E(\mu) \cup \partial K$. One can show that the minimal radius of the circle containing K (the circumradius) $\rho(K)$ equals the maximal variance of K, that is, $\rho(K) = \sup_{\mu \in \mathcal{P}(K)} \sigma_2(\mu)$.

Conjecture 2 [Variance conjecture for probability measures]: For any $K \Subset \mathbb{C}$ and $\mu \in \mathcal{P}(K)$ one has $H(K, \mathcal{W}_e(\mu)) \leq \rho(K)$, where H is the symmetrized Hausdorff distance.

We have also formulated an operator-theoretic conjecture which is roughly analogous to conjectures 1 and 2.

The solution to these and other related problems would almost certainly involve novel techniques which could be useful in solving other problems in complex analysis, potential theory and operator theory. Progress on the stated conjectures and related results may have implications in numerical analysis, in the study of the propagation of singularities of solutions to linear PDE's in the complex domain, in astrophysics (such as the recent solution to the long standing open question about the number of images in gravitational lensing by Khavinson and Neumann [3]), in statistical physics (e.g. Lee-Yang type results on the zeros of the partition function), in combinatorics as well as to matrix models in quantum field theory and fluid mechanics.

3 Outcome of the Meeting

During the focused research group, we have identified the following five topics for further study.

We have been able to reformulate some of our questions about the Gauss-Lucas matrix in terms of a related dynamical system. We will continue our analysis of the dynamics (such as fixed points, basins of attraction, singular points) of this system. We will work towards a better understanding of the potential theoretic parallel/contrast between finitely many sources and continuous densities, incorporating known phenomena into the larger theory of value distribution for meromorphic functions. We will further examine the statistical interpretation of the Hausdorff distance between critical points and zeros of a polynomial in terms of invariant dispersion measures. We will attempt to construct and solve a hyperbolic version of the Sendov or variance conjecture by elaborating in terms of the hyperbolic distances the relationship between various configurations of the critical points versus the zeros of finite Blaschke products in the hyperbolic metric of the unit disk. It would also be worth considering the spectral analysis of central truncations of Toeplitz matrices as a two dimensional analog of the classical Padé approximations of Markov functions.

Because of the diverse research backgrounds of the group, we were able to find connections among and acquaint one another with many different areas of mathematics. In one case this extended beyond our focused research group. Julius Borcea gave a talk entitled "Negative correlations, phase transitions and zeros of multivariate polynomials" to the BIRS workshop in Recent progress in two-dimensional statistical mechanics which was taking place at the same time.

4 Participants

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