# Symmetries of Graphs and Networks

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## 1 Introduction

Symmetries in graphs and networks are closely related to the fields of group theory (more specifically, permutation group theory) and graph theory. We measure symmetry using group actions, and focus our attention on vertex-transitive graphs: graphs whose automorphism group acts transitively on the set of vertices.

Although networks with high levels of symmetry do not appear often among randomly-chosen networks, they are often used in practical applications. Networks modelled on vertex-transitive graphs have been shown to be very "good" in their balance of cost (measured by the degree of each vertex in the network) against performance (how easy they are to disconnect, and the efficiency of algorithms run on them). Their symmetry often makes them relatively easy to study and understand, and has the huge advantage that "local" algorithms work globally, since the vertex-transitivity implies that all vertices hold equivalent roles within the global network. Vertex-transitive graphs also provide a beautiful context in which to study many of the general problems of graph theory; beautiful not only because of the symmetric pictures, but because of the interactions with group theory, and permutation group theory in particular. Many problems that are difficult to solve in the general context of graphs may prove tractable in this context, given our understanding of symmetry.

As the field of graph theory has emerged over the past 70 years or so (since the appearance of König's seminal book in 1936), symmetric graphs have thus become a very important area of study, and are often looked at closely by computer scientists and other network designers. Despite this, the number of mathematicians who study these graphs is quite small, and major gatherings have rarely (if ever) included symmetric graphs as a central focus. The field is growing rapidly; there were a relative trickle of papers on vertex-transitive graphs prior to 1980, but over 1500 have been published since that year, of which over 1200 have come out since 1990, and over 500 since 2000, yet there are still few "experts."

The goal of our workshop was to bring together these experts, both to share recent developments and techniques among themselves, and to provide a forum where younger, up-and-coming researchers can meet and learn from these established authorities. With just 20 invited participants, we had to be very selective in our choices of experts, younger researchers, and topics on which to focus, but the workshop was a great success. We invited 9 "overview" talks by experts, on topics within their expertise. These served to introduce the other participants to the latest results on major problems, and methods that have been or might be successful. They also included open problems that were suitable for working on during the free time included in the workshop schedule. We also included 7 talks that focussed on the particulars of recent work on more specific problems.

Although there are many regular graph theory conferences, it is rare for them to have a more specific focus. The last major events that focussed exclusively on graph symmetry were the NATO-sponsored Summer

Workshop and fall conference on Symmetry in Graphs, held in Montreal in 1996. Vertex-transitive graphs have also been important components of the "Graph Theory of Brian Alspach" conference held in Vancouver in 2003, and of the Slovenian International Conferences on Graph Theory held in Bled in 2003 and in 2007, but in each case it was one of a number of important components, meaning that major researchers in the field did not attend one or both, and the schedules were too full of talks to allow for much informal interaction. The SIAM Discrete Math conferences, most recently the one in Victoria in 2006, also often include a significant selection of talks on graph symmetry, but again these are only components of very busy conferences. Many important developments have been made in the field since 1996, so there was much to be discussed at this workshop.

# 2 The Main Topics

Although there are many interesting open problems about vertex-transitive graphs, the workshop focussed on 3 broad topics:

1. automorphisms
2. Hamilton paths and cycles
3. approaches and methods

Here we provide a moderately detailed overview of the current state of each of these, as they were discussed at the workshop. Several talks during the workshop were devoted to providing more detailed background on each of these topics, to the participants.

It should be noted that due to both the limited size and the desire to focus the scope of this workshop, we made a deliberate decision to avoid some topics that have generated great interest. These include graph colourings (and the "fixing number" of graphs), graph embeddings and "maps," and infinite graph theory.

Within the class of vertex-transitive graphs, there is an important subclass: the Cayley graphs. These come up again and again in the study of vertex-transitive graphs, so we will provide a definition here. While the usual definition of a Cayley graph is constructive, beginning with a group, given the context we provide a different (equivalent) definition here. A Cayley graph, then, is a graph whose automorphism group contains a subgroup that acts not just transitively, but regularly, on the vertices of the graph. (That is, given any two vertices $u$ and $v$ of the graph, there is a *unique* automorphism in the subgroup that takes $u$ to $v$.) If $H$ is the regular subgroup, then we refer to the graph as a Cayley graph on the group $H$.

## 2.1 Automorphisms

### 2.1.1 Finding automorphisms

Finding the full automorphism group of a graph is a notoriously difficult problem. Even the problem of testing whether a given graph has any nontrivial automorphisms, belongs to the class NP. While it is known that almost all graphs have no nontrivial automorphisms (a result that is sometimes paraphrased as "symmetry is unusual"), it would be reasonable to suppose that restricting our attention to the class of vertex-transitive graphs might provide a great deal more information, so making this problem more tractable.

A GRR, or *graphical regular representation* of a group $G$ is a Cayley graph $\Gamma$ on $G$, for which the left-regular representation of $G$, $G_L = \mathrm{Aut}(\Gamma)$. That is, such a graph has as few automorphisms as possible, while still being vertex-transitive. Considerable work went into the study of all groups that have a GRR, and the classification was completed by Godsil in 1978 [33], and showed that with two exceptional infinite families and 13 other small exceptions, every group has a GRR. There is a conjecture (by Imrich, Lovász, Babai, and Godsil, in 1982) [8], that for any group $H$ that does admit a GRR, almost all Cayley graphs on $H$ are GRRs. If true, this conjecture suggests that even when there is known to be a great deal of symmetry, extra symmetry is unusual. Several results on significant families support the conjecture. Further, Dobson [21] has produced some results on significant families of groups, showing that for almost every Cayley graph $\Gamma$ on such a group $G$, if $\Gamma$ is not a $GRR$, then $G \lhd \mathrm{Aut}\Gamma$.

While these results do not explicitly determine automorphisms of graphs, they do suggest (roughly speaking) that the probability of a graph having a particular automorphism group decreases as the order of the group increases. We turn now to the problem of directly determining automorphism groups.

A 1901 result by Burnside [10] about permutation groups of degree $p$ where $p$ is prime, showed that if $G$ is any subgroup of $S_p$ that contains the regular representation of $\mathbb{Z}_p$, then either $G \leq \mathrm{AGL}(1, p)$, or $G$ is doubly-transitive. Since the only connected graph on $p$ vertices whose automorphism group is doubly-transitive is $K_p$ (with automorphism group $S_p$), this result makes it fairly easy to determine the full automorphism group of any vertex-transitive graph that has a prime number of vertices. This was explicitly done by Alspach in 1973 [1]. While several generalisations of Burnside's result have been proven for permutation groups whose degree is a prime power [25, 18, 20], these results are harder to state and leave "exceptional" cases that have to be dealt with separately, if one's goal is to determine the full automorphism group of all vertex-transitive graphs on some prime power number of vertices.

Using these and other methods, the full automorphism group of the following vertex-transitive graphs have been determined (where $p$ and $q$ are distinct primes):

- all vertex-transitive graphs of order $p$ [1] (see above);
- all vertex-transitive graphs of order $p^2$ [36, 25];
- all vertex-transitive graphs of order $pq$ [19];
- all Cayley graphs on $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ [22];
- all Cayley graphs on $\mathbb{Z}_p^3$ (the elementary abelian group of order $p^3$) [23];
- all Cayley graphs on $\mathbb{Z}_n$ [38, 39, 26] (the result [47] gives a polynomial-time algorithm for determining the full automorphism group).

Some of these results are very recent, and Ted Dobson presented an overview of this topic at the workshop, mentioning methods and suggesting some open problems that could be within reach.

### 2.1.2 Semiregular Elements

A semiregular element of a permutation group, is an element all of whose cycles have the same length, in its disjoint cycle representation. The existence of a semiregular element in a permutation group is equivalent to the existence of a fixed-point-free element of prime order.

Semiregular automorphisms of vertex-transitive graphs play an important role in recursive reductions that help us understand the structure of large families of vertex-transitive graphs (see Subsubsection 2.3.2 on Normal Quotients, below). They have also been used in finding Hamilton paths and cycles (see Subsubsection 2.2.1, below), and in the enumeration of all vertex-transitive graphs on up to 26 vertices [45]. They are thus of sufficient importance to warrant special attention amongst all automorphisms of vertex-transitive graphs.

Marušič asked in 1981 [43] whether or not there is a vertex-transitive digraph that has no semiregular automorphism. To date, none has been found. Group theoretic arguments can prove that any transitive permutation group has a fixed-point-free element of prime power order, but there are transitive permutation groups that have no fixed-point-free elements of prime order (and thus, no semiregular automorphisms). Such groups are called "elusive." No known example of an elusive permutation group is "2-closed;" that is, can occur as the automorphism group of an edge-coloured (di)graph, where automorphisms must preserve the colours. In 1997, Klin [35] generalised Marušič's question, asking if every transitive finite 2-closed permutation group has a semiregular element.

Early results on this problem proved Klin's conjecture when the number of vertices is a prime power, or has the form $mp$ where $p$ is prime and $m < p$ [43]. Marušič's question was also answered (in the negative) in the cases of cubic vertex-transitive graphs, and vertex-transitive digraphs of order $2p^2$ [44].

By determining limitations on the classes of groups that can be elusive, recent results by Giudici [28], in some cases collaborating with Xu [31], have the following general consequences:

- every vertex-primitive graph has a semiregular automorphism;

- every vertex-quasiprimitive graph (so the automorphism group acts transitively on the vertices, and all nontrivial normal subgroups are transitive) has a semiregular automorphism;
- every vertex-transitive bipartite graph where the only system of imprimitivity is the bipartition, has a semiregular automorphism;
- all minimal normal subgroups of a counterexample to Klin's conjecture must have at least three orbits;
- every 2-arc-transitive graph has a semiregular automorphism; and
- every arc-transitive graph of prime valency has a semiregular automorphism.

Michael Giudici spoke about this problem at the workshop, explaining the methods and suggesting directions for future work. This is a very active research area, and his talk also mentioned a number of results on this problem by other researchers that have appeared in the past year, including:

- Every quartic vertex-transitive graph has a semiregular automorphism [24].
- Every vertex-transitive graph of valency $p+1$ that admits a transitive group whose order is divisible only by 2 and $p$ (where $p$ is an odd prime) has a semiregular automorphism [24].
- There are no elusive 2-closed groups of square-free degree [24].
- Every arc-transitive graph with valency $pq$ (where $p$ and $q$ are prime) whose automorphism group has a minimal normal subgroup with at least 3 vertex orbits, has a semiregular automorphism [56].
- If $G$ is a transitive permutation group all of whose Sylow subgroups are cyclic, then $G$ contains a semiregular element [37].

Recent research has also considered the possible order of a semiregular automorphism. It was shown first that all cubic vertex-transitive graphs have a semiregular automorphism of order greater than 2 [11], and more recently, that there is a function $f$ that grows unboundedly with $n$, such that the automorphism group of a connected vertex-transitive cubic graph on $n$ vertices has a semiregular subgroup of order at least $f(n)$ [40].

#### 2.1.3 Isomorphic Factorisations

In the past 12 years, results have determined that there are self-complementary vertex-transitive graphs of order $n$ if and only if each $p$-part of $n$ is congruent to $1 \pmod 4$ [46], and that a self-complementary Cayley graph on the cyclic group of order $n$ exists if and only if each prime divisor of $n$ is congruent to $1$ (mod $4$) [27, 4]. More recently, in 2001 Li and Praeger [41] found a purely permutation group-theoretic criterion that determines whether or not a given group is a transitive subgroup of the automorphism group of some self-complementary vertex-transitive graph. This enabled them to find self-complementary vertex-transitive graphs that are not Cayley graphs, though the smallest known example has $45^2$ vertices.

Li and Praeger then generalised the concept of self-complementary vertex-transitive graphs, to homogeneous factorisations of the complete graph. These are factorisations of the complete graph into isomorphic subgraphs (on the same vertex set) that admit a common vertex-transitive action. Using the theorem that every transitive permutation group has a fixed-point-free element of prime-power order, enables us to reduce any homogeneous factorisation to one that has prime index. This leads us to the study of homogeneous factorisations of prime index. Li and Praeger's permutation group-theoretic criterion that determines whether or not a given group is a transitive subgroup of the automorphism group of some vertex-transitive graph, has a direct generalisation to the prime-index case. Working with others, Li and Praeger were able to determine which types of primitive group actions on the vertices of $K_n$ can induce a homogeneous factorisation of prime index [34].

The concept of homogeneous factorisations of complete graphs have been further generalised, to homogeneous factorisations of arbitrary vertex-transitive graphs. Interestingly, the Petersen graph never occurs as a factor in a homogeneous factorisation, but the disjoint union of 11 Petersen graphs does.

Cai Heng Li gave an overview talk on the status of all of these problems, and discussed the related problems in permutation group theory. Again, a lot of progress has been made on this topic in the past 5 years, so there was a great deal of new material to be presented.

His talk also touched on symmetrical factorisations of arc-transitive graphs: factorisations for which the restriction of the an arc-transitive subgroup of the automorphism group to each factor, acts arc-transitively on the factor. Much less is known about these.

## 2.2 Hamilton Paths and Cycles

### 2.2.1 Paths and Cycles

Although the problem of finding Hamilton paths and cycles in vertex-transitive graphs has been much-studied since it was first raised as a question by Lovász in 1969 [42], it remains frustratingly obdurate. There are only 4 known connected vertex-transitive graphs (on at least 3 vertices) that do not have a Hamilton cycle, and none of them are Cayley graphs. Thomassen (in 1991) [53] and Babai (in 1979) [7] have made contradicting conjectures as to whether the number of connected vertex-transitive graphs without a Hamilton cycle is finite or infinite.

The families of vertex-transitive graphs for which Hamilton cycles or paths are known to exist are still quite restricted; the most significant result is that all Cayley graphs of $p$-groups have Hamilton cycles; this was proven by Witte in 1986 [54]. Much of the work of finding Hamilton paths and cycles has focussed on cubic vertex-transitive graphs, for two reasons: a paucity of edges intuitively makes it harder to find paths or cycles, and the vertex-transitive graphs that do not have Hamilton cycles are all cubic. Many people have minor results on cycles in families of Cayley graphs, and Marušič has generalised many results to families of vertex-transitive graphs.

Dragan Marušič gave an overview talk about this problem, in which he suggested that given the amount of effort that has been put into solving this problem with relatively small progress, it seems that a new approach is called for. He presented two approaches that have been used recently with some success.

First he presented an approach used by Glover, Kutnar and Marušič (for example in [32]), on cubic Cayley graphs. Under this strategy, they embed the graph onto an orientable surface, and find a tree of faces in the embedding that spans all of the vertices. Then tracing the border of the tree of faces, gives a Hamilton cycle. Naturally, it is not always easy to find a tree of faces that spans all of the vertices, and sometimes a smaller tree produces a smaller path or cycle that later needs to be adjusted to produce a Hamilton path or cycle. In his talk, Marušič discussed some of the deep results that have been drawn upon to deal with such issues.

Another strategy that has been used very successfully to find Hamilton paths and cycles in vertex-transitive graphs, is to reduce the problem to a quotient graph (where an appropriate quotient exists, as for example when there is a semiregular automorphism that is not regular), find a Hamilton cycle in the quotient, and "lift" this back to the original graph, adjusting if necessary to produce a single cycle rather than a union of cycles. Although this approach has been around since at least 1948 [51], it continues to be effective. It was discussed at the workshop in the context of some individual recent results, as well as in the overview talk.

The other approach presented by Marušič has been used in recent work by himself, Du, and Kutnar, that is very close to proving that every connected vertex-transitive graph of order $pq$ (other than the Petersen graph) has a Hamilton cycle. This uses a 1972 result by Chvátal [13] that guarantees a Hamilton cycle if a condition on the degrees of the vertices of a graph is satisfied. The strategy consists of taking a quotient of the graph with respect to a semiregular automorphism of order $p$, using the Chvátal theorem to find a Hamilton cycle in the quotient graph, and lifting this cycle to a Hamilton cycle in the original graph.

### 2.2.2 Hamilton Connectivity and Laceability

A graph is Hamilton-connected if there is a Hamilton path between any pair of terminal vertices. Similarly, a bipartite graph is Hamilton-laceable if there is a Hamilton path between any pair of terminal vertices in distinct parts. We say that a family of graphs is $H^*$-connected if the bipartite graphs in the family are Hamilton-laceable, while the other graphs are Hamilton-connected.

While these properties are interesting in their own right, proving that the cubic graphs in some family are $H^*$-connected and using induction to prove that all graphs in the family are $H^*$-connected, may actually in some cases be the easiest way of establishing the weaker result that all graphs in the family have a Hamilton path. The stronger hypothesis may make the induction step easier.

In 1980, Chen and Quimpo [12] proved that the family of connected Cayley graphs of valency at least 3 on abelian groups is $H^*$-connected. It was some time before further results emerged. In 2001, Alspach and Qin [5] proved that the family of connected Cayley graphs of valency at least 3 on Hamilton groups is $H^*$-connected. Quite recently, Alspach and Dean [3] proved that the family of connected Cayley graphs of valency at least 3 on generalised dihedral groups whose orders are a multiple of 4 is $H^*$-connected.

Alspach gave a talk in which he sketched the proofs of these results, each of which is inductive. Few other results about $H^*$-connectedness are known. In 1995, Wong [55] proved that the family of butterfly graphs is $H^*$-connected.

Some other recent results relate to Cayley graphs on symmetric groups:

- Every connected Cayley graph on $S_n$ whose connection set consists of transpositions, is Hamilton-laceable [6].
- For $2 \leq k \leq n$, $D_k$ denotes all permutations in $S_n$ that move $k$ successive elements and fix the other $n-k$ elements. Every Cayley graph on $S_n$ with connection set $D_k$, $4 \leq k \leq n$, is Hamilton-connected [52].

Clearly, this problem remains wide open.

## 2.3 Approaches and Methods

### 2.3.1 Computer Use

Marston Conder gave a presentation in which he demonstrated the use of computational tools (as in GAP and Magma [9]) in his research. He pointed out the usefulness of computer-generated information in revealing patterns that can lead to new discoveries and insights.

He gave examples of the use of different algorithms to produce families of vertex-transitive graphs, and showed how clever choice of the algorithm can significantly speed up the computation, or allow us to generate much more information within a reasonable time-frame. This approach has enabled him, for example, to find all arc-transitive 3-valent graphs of small order, extending the Foster census up to 2048 vertices [15]. He also discussed how his computer-assisted determination of all orientable regular maps and hypermaps of genus 2 to 101 [14] had provided enough data that he could discern patterns never seen before, and then (in collaboration with a number of co-authors, cf. [16, 17]) use combinatorial group theory and other techniques to prove many new results about the genus spectrum of various classes of maps.

### 2.3.2 Normal Quotients

If a graph $\Gamma$ is edge-transitive and connected, $G$ is a group that acts transitively on the edges, and $\mathcal{P}$ is a partition of the vertices of $\Gamma$ that is preserved under the action of $G$, then we can form a quotient graph $\Gamma_{\mathcal{P}}$, whose vertices are the sets in the partition, with an edge between two sets if and only if there is an edge between some pair of vertices, one of which lies in each of the two sets. Notice that the edge-transitivity and connectedness of $\Gamma$ forces $\Gamma_{\mathcal{P}}$ to be connected, and $G$ to be edge-transitive on $\Gamma_{\mathcal{P}}$. Further, no edges of $\Gamma$ can lie within a block of the partition. If $G$ is actually arc-transitive on $\Gamma$, then it will also be arc-transitive on $\Gamma_{\mathcal{P}}$. If the partition $\mathcal{P}$ is maximal, then $G$ acts primitively on the vertices of $\Gamma_{\mathcal{P}}$.

This reduction enables us to analyse the structure of various families of edge-transitive graphs. If it happens that, for all graphs $\Gamma$ in the family, all such quotients $\Gamma_{\mathcal{P}}$ remain in the family, then there is a strong relationship between typical graphs and vertex-primitive graphs in the family. We then consider the vertex-primitive graphs in the family to be the "basic" graphs in the family. Using the O'Nan-Scott Theorem that characterises the structure of primitive permutation groups, may enable us to come to some understanding of these basic graphs. Then since an arbitrary graph in the family has as a quotient some basic graph, whatever understanding we gain of the basic graphs may enable us to make deductions about the structure of all graphs in the family.

A small modification of this strategy works in the case of distance-transitive graphs [50]. If a group $G$ acts vertex-primitively and distance-transitively on a graph $\Gamma$, then either $\Gamma$ comes from a known list, or $G$ is of affine type or almost simple type (two of the varieties of primitive permutation groups as characterised in the O'Nan-Scott Theorem). Further (great) effort by many researchers has come close to pinning down a complete classification of the primitive distance-transitive graphs. It is also known that an arbitrary graph in the family will be either a bipartite double or an antipodal cover of a basic graph.

In the case of $s$-arc-transitive graphs, additional structure is required, because a quotient graph of an $s$-arc-transitive graph need not itself be $s$-arc-transitive. However, if the partition $\mathcal{P}$ comes from the orbits of a normal subgroup of an $s$-arc-transitive group having at least three orbits, then Praeger showed in 1985 [48] that the quotient graph will be $s$-arc-transitive, and the original graph will be a cover of the quotient. Such a quotient graph is called a normal quotient.

When we use normal quotients to reduce, the reduction comes to an end when there are no further normal subgroups of $G$ whose orbits form a nontrivial partition of $\Gamma$ - that is, when every nontrivial normal subgroup of $G$ has at most two orbits. Then by definition, $G$ is quasiprimitive if each normal subgroup is transitive, or $G$ is biquasiprimitive if $G$ has a normal subgroup with two orbits and all nontrivial normal subgroups have at most two orbits. In every other case, there is a normal subgroup of $G$ with at least 3 orbits. Thus, the "basic" graphs in the family are those on which the group action on the vertices is quasiprimitive or biquasiprimitive, and arbitrary graphs are covers of the basic graphs. In 1993, Praeger [49] produced a classification of quasiprimitive permutation groups along similar lines to the O'Nan-Scott classification of primitive permutation groups. This can be used to help us further understand the "basic" graphs in a family after normal quotient reduction.

Further work by a variety of researchers has moved towards a classification of the "basic" $s$-arc-transitive graphs, using this strategy.

Cheryl Praeger presented an overview of the quotient and normal quotient strategies, and discussed the kinds of families of graphs on which these strategies are likely to work well (locally $Q$ graphs for various group-theoretic symmetry properties $Q$, for example). She presented the example of locally $s$-arc-transitive graphs, which have been fairly well classified using this approach (cf. [29, 30]). At the end of her talk, she suggested that this approach might be productive in determining the structure of half-arc-transitive graphs, and asked for guidance from people who had worked with these graphs, on how the reduction could best be adapted to avoid reducing to graphs that are fully arc-transitive.

#### 2.3.3 Schur Rings

Although Schur Rings have been used extensively to prove very deep results about vertex-transitive graphs, very few of the researchers who work in vertex-transitive graph theory understand them well. István Kovács gave an introductory talk about Schur rings, and how they have been and can be used to prove results about vertex-transitive graphs.

Determining the Schur rings over a particular group $G$ provides extensive information about the possible automorphism groups of Cayley graphs of $G$. Schur rings have been used to determine the full automorphism groups of certain classes of graphs (see Subsection 2.1.1), and extremely effectively in determining what groups have the CI (Cayley Isomorphism)-property. The CI-property is the property that an arbitrary pair of isomorphic Cayley graphs on a fixed group $G$, must be isomorphic via an automorphism of $G$. The property can be restated as the property that any 2-closed group that contains a regular representation of $G$, has just one conjugacy class of regular subgroups isomorphic to $G$. Several of the most significant results in proving that families of groups have the CI-property, have involved classifying Schur rings over that family of groups.

#### 2.3.4 Association Schemes

Association schemes are a generalisation of Schur Rings. Again, few of the researchers in vertex-transitive graph theory know much about them, although they are starting to appear more and more often in deep results. Chris Godsil gave an introductory talk about association schemes. He considered the natural special case of association schemes as sets of matrices, and focused on the massive quantity of information that is encoded in the spectrum of an association scheme. As an example to illustrate the techniques of association schemes and the information that can be obtained from them, he considered the Johnson scheme and ended with the result that no automorphism of the Petersen graph maps each vertex to a neighbor.

## 3 Open Problems

The following lists some of the open problems that were presented and discussed at the workshop.

1. Find the full automorphism group of every vertex-transitive graph whose order is a product of three (not necessarily distinct) primes.
2. Prove that for any group $G$ that admits a GRR, almost every Cayley graph on $G$ is a GRR.
3. Prove that almost every Cayley graph on $G$ has $G \triangleleft \mathrm{Aut}(\Gamma)$.
4. Prove that almost every Cayley graph on $G$ whose automorphism group is not as small as possible, has $G \triangleleft \mathrm{Aut}(\Gamma)$.
5. Determine whether or not there is any vertex-transitive graph with no semiregular automorphism.
6. Determine whether or not there is any 2-closed permutation group with no semiregular element.
7. Determine whether or not there is any arc-transitive graph with no semiregular automorphism.
8. Determine whether or not there is any distance-transitive graph with no semiregular automorphism.
9. Find new constructions of "elusive" groups.
10. For which degrees do elusive groups exist? (Answer is known up to degree 40.)
11. Does the set of all degrees of elusive groups have density 0?
12. Is every self-complementary vertex-transitive graphs on fewer than $45^2$ vertices, a Cayley graph?
13. Determine fixed-point-free automorphisms of $\mathbb{Z}_p^d$ that have 2-power order.
14. Study transitive permutation groups that have an orbital-fixed-free automorphism of prime-power order.
15. Characterise vertex-transitive graphs that have a homogeneous factorisation.
16. Characterise vertex-transitive graphs that occur as a factor of a homogeneous factorisation.
17. Classify arc-transitive graphs that have an arc-symmetrical factorisation whose factors are connected and cubic.
18. Characterise arc-transitive graphs that arise as arc-symmetrical factors of a given arc-transitive graph.
19. Does every connected vertex-transitive graph have a Hamilton path?
20. Does every connected Cayley graph have a Hamilton cycle?
21. Determine whether the number of connected vertex-transitive graphs without a Hamilton cycle is finite, or infinite.
22. When is a vertex-transitive cover of a hamiltonian graph, also hamiltonian?
23. Use normal quotients to determine the structure of families of arc-transitive graphs.
24. Work out how to use normal quotients to determine the structure of families of (bipartite) vertex-intransitive, edge-transitive graphs.
25. Work out how to use normal quotients to determine the structure of families of half-arc-transitive graphs.
26. Is every direct product of two CI-groups of coprime orders, itself a CI-group?
27. Classify all $p$-Schur Rings over $\mathbb{Z}_p^4$.

## 4 Outcomes of the Meeting

All of the participants were vocal in their appreciation of the workshop. The organisers are aware of several new collaborations that began at the workshop and seem likely to produce exciting new results, particularly on using normal quotients to classify families of graphs. Progress on a number of other problems has also been reported to us, though it is too early for any proofs to have been polished. One young post-doc referred to this as his "best-ever" conference, and a student reported that she had made progress on a problem as a result of discussions at the workshop, so will be able to include additional results in her thesis.

The workshop brought together major teams of researchers from Australia and Slovenia, with more isolated vertex-transitive graph theorists who work in North America, China, etc. Even though a number of researchers from Israel and China were forced to decline our invitations due to work or (in one case) health considerations, we were very fortunate in the people who were able to attend the meeting. We are also pleased to be able to say that despite the theme of the workshop being a field that remains fairly male-dominated, 5 of our 20 participants were women, and another 2 women were invited but had to decline.

The organisers have created a web page of resources from the workshop, that includes the slides from all of the talks, and Magma code from Marston Conder's talk. Preprints may also be posted there as they are produced. The url of this web page is www.cs.uleth.ca/˜morris/banff-symmetries/.

A special issue of *Ars Mathematica Contemporanea* will publish papers that were presented at the workshop, or that are closely related to the themes of the workshop.

Dragan Marušič has agreed to host a conference in Slovenia in 2010, to build on the success of this workshop and to maintain the relationships that have been created.

It seems fair to say, then, that the workshop accomplished all of its major goals:

- to bring together experts in the field, who would be able to share their methods, approaches, recent results, and favourite problems and help each other make progress in their research;
- to include dynamic young researchers, who could both help the more established experts, and learn from them; and
- to provide the connections and impetus to begin a regular series of conferences focussed on vertex-transitive graph theory.

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