08w5055 Classical Problems on Planar Polynomial Vector Fields

1. Brief overview of the field

At the end of the 19th century Poincaré and Hilbert stated three problems which are still open today: the problem of the center and the problem of Poincaré, stated by Poincaré in 1885 and in 1891, and Hilbert's 16th problem, stated in Hilbert's address at the International Congress of Mathematicians in Paris in 1900. The first two and the second part of Hilbert's 16th problem are on planar polynomial vector fields and were the focus of our Workshop at BIRS. These problems are very easy to state but very difficult to solve. All three problems are essentially of a global nature and this common feature is at the core of the difficulty: we are interested in whole classes of polynomial vector fields defined on the whole plane. These problems stir our interest because understanding this area necessarily involves using ideas and methods from a variety of fields: algebra, geometry, analysis, real and complex, numeric and symbolic computations. In the last two decades also logicians have become interested in this area and new connections were established. During the past twenty years things have been moving slowly but steadily ahead and a fresh new view is beginning to emerge highlighting a subtle unity among these classical problems. The algebraic nature implicit in the first two problems was known but work in the past twenty years brought deeper connections to light. This area is very much alive and growing. The purpose of this workshop was to share new knowledge, take a measure of the new developments and bring new people into the group who could help in confronting the new tasks ahead.

2. The problem of the center

This problem can be formulated as follows: Determine within the class of real planar polynomial vector fields, all the systems possessing a center, i.e. a singular point surrounded by close phase curves. The first result was obtained by Poincaré in 1885. Poincaré asked to determine the class of systems possessing non-degenerate singularities (i.e. with non-zero eigenvalues) which are centers, and for this it suffices to look for the systems with singularities with pure imaginary eigenvalues. Such a singularity could only be a focus or a center. Poincaré's theorem says that a necessary and sufficient condition to have a center at a singular point s_0 with pure imaginary eigenvalues is that there exists a local analytic non-zero first integral in the neighborhood of s_0. This result was generalized by Lyapunov for analytic vector fields. The search for an analytic first integral led Poincaré to an algorithm for computing the so-called Poincaré-Lyapunov quantities V_i, i=1,2,... of the singularity, which are polynomials over $\mathbb{Q}$ in the coefficients of the system. A necessary and sufficient condition to have a center is then the annihilation of all these quantities. In view of Hilbert's basis theorem for this to occur it suffices to have for a finite number of i, $i \leq j$ and j sufficiently large, $V_i = 0$. Bautin showed in 1939 that for a quadratic system, i.e. systems defined by polynomials of degree $n = 2$, to annihilate all V_i's it suffices to have $V_i = 0$ for $i \leq 3$. The three equations yield an algebraic set in the space of coefficients, the so called center variety, which splits into four irreducible algebraic varieties. To points in each one of these correspond integrable systems with analytic first integral in the neighborhood of the singularity. So the problem of the center is solved for $n = 2$. For cubic differential systems ($n = 3$) the problem is open and it is very hard. Although we have an algorithm for computing the Poincaré-Lyapunov constants for singularities with pure imaginary eigenvalues, we have no algorithm to determine how many of them need to be zero to imply that all of them are zero for cubic or higher degree polynomial differential systems. The annihilation of these Poincaré-Lyapunov quantities yields an algebraic set called the center variety, in the parameter space which is the space of coefficients of the vector fields. These polynomials V_i grow as i increases and even for very special classes of cubic systems, such as the Kukles systems (see [8]), very powerful computers needed to be used to obtain the center variety. For a specific family of polynomial vector fields, one usually starts by computing a certain number of the quantities V_i, presumably "enough" of them to assure success. The next step involves the breaking of the algebraic set $V_i = 0$ for $i \leq j$ into its irreducible components. In the third step one proves that for systems corresponding to parameter values in each such component, the singularity is a center. This will of course imply the annihilation of all remaining quantities V_i. To show that the singular point is a center we have two basic mechanisms: we either show that the system is symmetric or using Poincaré's theorem we show that we have a local analytic first integral. In 1908 in his paper [12] on quadratic systems with a center, Dulac used this technique. We point out that the notion of center of Dulac differs from that of Poincaré. Dulac considered complex polynomial differential equations and for such systems a (non-degenerate) center in the sense of Dulac is a singular point s_0 at which the linearization of the system has non-zero eigenvalues whose quotient is negative and rational and there exists a local analytic non-zero first integral of the system in the neighborhood of s_0. As a real quadratic system with a singularity with pure imaginary eigenvalues has the quotient of eigenvalues -1, and as such a system also generates a complex system, the results of Dulac apply to such real systems also. Although in Dulac's normal form the eigenvalues of the singular point are ± 1, the algorithm of Poincaré can still be applied and it leads to the so-called saddle quantities or dual Poincaré-Lyapunov quantities $\tilde{V}_i$. To prove center at the singularity Dulac applies various ad hoc methods of integration and he observes that in one case a method of integration introduced by Darboux can be applied. Darboux gave his geometric method of integration in his seminal work [10] of 1878. In spite of Poincaré's praise of Darboux' work and in spite of Dulac's observation, this method has not played a significant role in the problem of the center until almost the last 20 years of the 20th century when the situation abruptly changed and during the past years the impact of the geometric work of Darboux has been growing steadily in research work on polynomial vector fields.

Advances in the problem of the center, conjectures and work on this problem presented in the workshop. The geometric method of Darboux uses algebraic invariant curves of a system to prove integrability and Darboux's theorem affirms that if we have a sufficient number of such invariant algebraic curves then the system had a first integral which is analytic on the complement of the union of these curves. This connection between the problem of the center and algebraic geometry has come to the forefront in recent years and was present in several of the talks at the meeting. There were numerous publications on the problem of the center during the last part of the 20th century and the beginning of the 21st century and we only mention here a few of these results. The first proof that the method of Darboux can be uniformly applied to prove centers in all cases in the quadratic context was given by Schlomiuk in [27]. The same method can be used to establish conditions for center and integrability in finite terms for families of cubic polynomial vector fields (see [8], [26]). For some families of vector fields defined by polynomials which could be of arbitrary degrees, Cherkas obtained necessary and sufficient conditions in [6, 7]. In Cherkas' work the relation to Darboux integrability is not however apparent. In 1992, in a provocative manuscript entitled "The solution to the center-focus problem" [32], **Zoladek formulated the following conjecture**: *The set of planar polynomial vector fields with center can be divided into three groups: with Darboux integral, rationally reversible and with first integral of the Darboux-Schwarz-Christoffel type.* This paper was never published as the proof did not withstand the refereeing process. But ideas launched in this paper survived the passage of time. It was in this paper that Zoladek talked about *mechanisms for producing center* mentioning in the first place the method of Darboux and in the second place *rational reversibility* whose simplest form appears in symmetry of the phase curves of the vector field with respect to a line. Mechanisms for producing centers are today at the core of progress in this problem. In [9] Christopher and Schlomiuk expressed necessary and sufficient conditions for Cherkas' class of systems in terms of two basic algebraic mechanism: Darboux integrability and pull-back along an algebraic map of a first integral leading to a form of algebraic symmetry. A systematic investigation of algebraic symmetries was done by Zoladek in [33]. There were four talks on systems with center at the workshop. **The close relationships existing between reversibility, the center problem and integrability** was explored in J. Giné's lecture. For degenerate singularities the theorem of Poincaré no longer holds. So in such cases looking for a local analytic first integral is of no use as such a first integral may not exist. It is in such cases that the alternative method of showing symmetry or algebraic reversibility applies. A known fact is the following: An analytic system having either a non–degenerate or nilpotent center at the origin is analytically reversible or orbitally analytically reversible, respectively. In this talk Giné showed the existence of a smooth map that transforms an analytic system having a degenerate center at the origin into a reversible linear system (after rescaling the time). Existence of a local analytic first integral was shown in some cases of degenerate centers in reversible systems. **Interesting computer algebra calculations leading to experimental results for the Poincaré–Center–Problem** were presented in H. C. Graf v. Bothmer's talk, a report about recent work with Martin Cremer, Jakob Kröker and Ulrich Rhein. He explained how to obtain information about the component structure of the center variety for cubic polynomial systems by using computer experiments over finite fields of small characteristic. The experimental evidence indicates that *at least up to degree $n = 7$, Zoladek's conjecture is true.* **An interesting result obtained is a proof that the vanishing of 11 focal values (Poincaré-Lyapunov values) is not sufficient to prove that a plane cubic system has a center (this improves the previously known bound by one).** In this work the Poincaré-Lyapunov quantities are computed modulo a prime p. An algorithm is found which computes modulo p the Poincaré-Lyapunov quantities up to $V_{p-3/2}$. **An algebraic approach to the center-focus problem** was presented in the talk of Alexander Brudnyi. By passing to polar coordinates the problem of the center for planar differential systems could be reduced to a problem, also called the center problem but for ODE's of the form $dv/dx = \sum a_i(x)v^{i+1}$, $x \in [0, 2\pi]$. More generally, Brudnyi considers the previous equation with coefficients a_i from the Banach space of bounded measurable complex-valued functions on $[0, T]$ and discusses algebraic aspects of the center problem for these equations. The algebraic structure of the Poincaré-Lyapunov constants and applications were discussed in D. Boularas' lecture.

3. The problem of Poincaré

In [22, 23] Poincaré formulated the problem of determining within the class of planar polynomial systems those systems which are *algebraically integrable*, by this meaning those which admit a rational first integral. Poincaré stated this problem inspired by the work of Darboux which he called"admirable" and "oeuvre magistrale". To attract attention to this problem the French Academy of Sciences proposed it as a topic for the Grand Prize of Mathematical Sciences, prize which was won by Painlevé. Inspired by the work of Darboux, Poincaré wrote two papers on the subject. He observed that to decide if a system has a rational first integral it suffices to find an upper bound for the degrees of invariant algebraic curves of the systems. Having such an upper bound *it suffices afterwards to perform algebraic calculations* says Poincaré, meaning of course the calculations of invariant algebraic curves of the systems. In the current literature authors mean by the *problem of Poincaré*, the problem of bounding the degrees of the irreducible invariant algebraic curves of a planar polynomial differential system in terms of the degree of the system.

3.1 Summary of the most significant advances in the problem of Poincaré and connections with the problem of Poincaré in work presented in the workshop. The work of Jouanolou [18] published in 1979 revived interest in this problem. Jouanolou gave a sufficient condition for algebraic integrability saying that if a system of degree n has a number of invariant algebraic curves which is greater than or equal to $n(n+1)/2+2$ then the system has a rational first integral. During the past 20 years a number of articles on this problem appeared in the literature, among them [3]. A solution to the problem has not yet been published but an announcement in this direction was made on January 12, 2009 in ArXiv (See arXiv:0812.2434v1). Carnicer's main result is that if we assume that an algebraic curve C is invariant and that there is no dicritical singularity on C of the complex foliations with singularities generated by the system then the degree of C is bounded by the degree of this foliation plus two. The problem of Poincaré was featured in two lectures of the workshop. **Results obtained on rational and polynomial integrability for some families of planar polynomial differential systems** were discussed in the talk of N. Vulpe presenting joint work with J.C. Artes and J. Llibre. This work involves the **new use of algebraic invariants** for this problem. The theory of algebraic invariants for planar polynomial differential systems was developed by the school of Sibirschi [29]. An interesting feature appearing in this work is the need to use subgroups of the affine group for constructing invariants. Another interesting feature of the work is giving invariant conditions for existence of rational first integrals of degree two or three for quadratic systems. A connection between the polynomial (and rational) integrability of some families of systems and the properties of the roots of the constructed polynomials, whose coefficients are absolute affine invariants of these systems was also presented.

3.2 Connections among the three classical problems and new questions raised were discussed in the lecture of D. Schlomiuk. Some of these connections appeared clearly in her joint work with Artés and Llibre [2]. An intriguing connection is between the problem of Poincaré which is of an entirely algebro-geometric nature and Hilbert's 16th problem whose character is essentially transcendental. In [2] the authors show that all phase portraits with the maximum number of two limit cycles occurring in the class of quadratic differential systems with a second order weak focus, can be obtained by perturbing within this class a purely algebraic object which is a quadratic system having a center and a rational first integral, quotient of second degree polynomials, and whose points at infinity are all singularities. The above mentioned result prompts the following questions: *Is it true that perturbations of the most degenerate polynomial differential systems with a center and a rational first integral produce the largest number of limit cycles in the whole class of quadratic differential systems? Is this true for higher degree systems?* The above mentioned work brought to the forefront the need to study perturbations of this very degenerate quadratic system which possesses a center and a rational first integral. As this system also has a line of singularity at infinity, this involves singular perturbations and three talks in the workshop were devoted to the study of singular perturbations. (See Singular perturbations below).

4. Limit cycles in planar polynomial vector fields

It is well known that the trajectories or orbits of any differential system are homeomorphic either to a point (called *singular point* of the system), or to a circle (called *periodic orbit*), or to straight lines (called *regular orbits*). When a periodic orbit of a differential system is isolated in the set of all periodic orbits of the system it is called a *limit cycle*. Planar limit cycles have the property that other trajectories spiral towards them either as time approaches $+\infty$ or as time approaches $-\infty$. Such behavior is exhibited in some nonlinear systems. In the case where all the neighboring trajectories approach the limit cycle as time $t \to +\infty$, the cycle is called a *stable* or *attractive* limit cycle. If instead all neighboring trajectories approach the cycle as time $t \to -\infty$, the cycle is an *unstable* or *repeller* limit cycle. In all other cases it is neither stable nor unstable. Stable limit cycles imply self sustained oscillations. Any small perturbation from a stable limit cycle would cause the system to return to the limit cycle. Limit cycles of planar vector fields were defined in the famous paper *Mémoire sur les courbes définies par une équation differentielle* [24]. At the end of the 1920's van der Pol [31], Liénard [20] and Andronov [1] proved that a periodic orbit of a self–sustained oscillation occurring in a vacuum tube circuit was a limit cycle as considered by Poincaré. After this observation, the existence and non–existence, uniqueness and other properties of limit cycles have been studied extensively by mathematicians and physicists, and more recently also by chemists, biologists and economists. In the qualitative theory of differential equations the research on limit cycles is a very interesting and difficult topic. We mention that the only families of planar differential systems for which the bifurcation of limit cycles is very well understood are the so called *rotated families* of differential systems, see for instance [11, 4, 5].

4.1 New results on singular perturbations and limit cycles. A series of three lectures was presented in the workshop which started with Dumortier's talk on canard cycles based on joint work with Roussarie. In this talk singular perturbations problems occurring in some planar slow-fast systems were studied. The authors analyze the number of limit cycles that can bifurcate from limit periodic sets (see Definition in [25]) formed by a fast orbit and a curve of singularities containing a unique turning point. The notion of slow-fast Hopf point was introduced, by this meaning a turning point. The authors investigate under very general conditions, the number of limit cycles that can appear near a slow–fast Hopf point (i.e. the limit periodic set is reduced to a turning point). Dumortier's talk was continued in the lecture of Roussarie in which an unfolding of a slow-fast Hopf point was considered. **The conjecture that a generic slow-fast Hopf unfolding with n parameters produces at most n limit cycles** was mentioned and to prove this conjecture is a goal of this work. The result could only be obtained modulo a **conjecture about a system of generalized Abelian integrals**. The lecture of Roussarie concentrated on important steps towards this proof. The last lecture on canard cycles was of P. De Maesschalck. This talk dealt with the cyclicity of limit periodic sets that occur in families of vector fields of slow–fast type. The limit periodic sets are formed by a fast orbit and a curve of singularities containing a unique turning point. The presence of periodic orbits in a perturbation is related to the presence of canard orbits passing near this turning point. The lecture discussed what happens when the slow dynamics exhibit singularities. In particular this study includes the cyclicity of the slow–fast 2–saddle cycle, formed by a regular saddle–connection (the fast part) and a part of the curve of singularities (the slow part). It was shown that the relevant information is no longer merely contained in the slow divergence integral.

4.2 New results on limit cycles in cubic and higher degree differential systems and Abelian integrals. Both Chengzhi Li and Jibin Li gave constructions of cubic differential systems with at least 13 limit cycles. So far thirteen is the maximum number for which we can prove that we have cubic systems with at least 13 limit cycles. The talk of Chengzhi Li is based on joint work with Changjian Liu and Jiazhong Yang. The result is essentially based on counting the number of zeros of some Abelian integrals, all the steps of the proof are analytic. The lecture of Jibin Li was based on joint work with Yirong Liu. The authors gave new and relevant results on Z_q–equivariant planar polynomial vector fields. First for the planar Z_2-equivariant cubic systems having two elementary foci, the characterization of a bi–center problem and shortened expressions of the first six Liapunov constants are completely solved. They use these results to show the existence of cubic systems having 12 and 13 limit cycles. They then extend these results to Z_6-equivariant planar polynomial vector field of degree 5, showing that there are quintic systems with at least 24 limit cycles. Extending again their results to Z_5-equivariant planar polynomial vector field of degree 5 they show the existence of quintic systems having at least 25 limit cycles. Maite Grau's talk, based on joint work with Francesc Mañosas and Jordi Villadelprat was on Zeros of Abelian integrals and Chebyshev systems. In planar polynomial differential systems the Abelian integrals appear in a natural way when studying bifurcations of limit cycles from the periodic orbits of a period annulus (i.e. a topological annulus fulfilled of periodic orbits). These integrals reduce the problem of finding the limit cycles which bifurcate from a limit periodic set, to searching for the zeros of a function, which is an Abelian integral, defined on an open interval. The authors provide a new and powerful criterion for studying the zeros of Abelian integral using the Chebyshev systems.

4.3. New results on limit cycles in perturbations of Hamiltonian systems. Maoan Han's talk, based on joint work with Junmin Yang and Dongmei Xiao, was on limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle. In this talk the authors consider general analytic near-Hamiltonian systems with parameters on the plane. They suppose that the unperturbed Hamiltonian system has a double homoclinic loop passing through a nilpotent saddle. The system presents three families of periodic orbits, inside or outside the loop which yield three Melnikov functions. They study the analytical property of the three first order Melnikov functions and obtain asymptotic expansions of them near the loop together with the computation formulas of the first coefficients of the expansions. Using these coefficients they give a sufficient condition for the perturbed system to have 8, 10 or 12 limit cycles in a neighborhood of the loop with seven different distributions. They finally consider some polynomial systems and find a lower bound of the maximal number of limit cycles as an application of their main results. Pei Yu's talk, based on joint work with Han Maoan was about bifurcation of limit cycles occurring in 3rd–order perturbations of 2nd–order Hamiltonian planar vector fields. The authors show that a Z_2–equivariant 2nd–order Hamiltonian planar vector fields with 3rd–order symmetric perturbations can have at least 12 limit cycles. The method combines the general perturbation to the vector field and the perturbation to the Hamiltonian function. The Melnikov function is evaluated near the center of vector field, as well as near homoclinic orbits. It is shown that 10 small limit cycles bifurcate from two symmetric centers, and at least 2 large limit cycles exist, each enclosing 5 small limit cycles.

4.4 Limit cycles in Liénard polynomial differential systems. Polynomial Liénard systems are of the form:

$$\dot{x} = y, \quad \dot{x} = -g_m(x) - f_n(x)y,$$

where m and n are the degrees of the polynomials $g_m(x)$ and $f_n(x)$, respectively. This is a subclass of an important class of systems, the more general Liénard systems of the same form but where $g_m(x)$ and $f_n(x)$ are replaced by more general functions $g(x)$ and $f(x)$. A survey of results on limit cycles in this family was presented in the lecture of Jaume Llibre.

4.5 New results on the study of cyclicity of some limit periodic sets. Limit cycles bifurcate from singular points, periodic orbits or graphics. There are two kinds of graphics the ones having finitely many singularities, called non–degenerate graphics, and the degenerate ones. A *non–degenerate graphic* of a planar differential system consists of finitely many singular points $s_1, \ldots, s_r$ not necessarily distinct, and of finitely many regular orbits $\gamma_1, \ldots, \gamma_r$ such that the orbit γ_i tends to s_i when t tends to $-\infty$ and to s_{i+1} when t tends to $+\infty$ for $i = 1, \ldots, r$ and where we put $s_{r+1} = s_1$. A *degenerate graphic* of a system is defined in a similar way but instead of connecting *all* points s_i with s_{i+1} by regular orbits we only do this for some of them and for the remaining ones we ask that they be connected by arcs of curves having all their points as singularities of the system. In her lecture Christiane Rousseau talked about finite cyclicity of center graphics. She defined the notion of center graphic, for graphics with no first return map and discussed a method for proving the finite cyclicity of center graphics. The method is a mixture of a good normal form and the use of the "Bautin trick". Some examples were discussed. Isaac A. Garcia's talk, based on joint work with Hector Giacomini and Maite Grau, was on the study of cyclicity using the inverse integrating factors. This work is concerned with planar real analytic differential systems with an analytic inverse integrating factor defined in a neighborhood of a regular orbit. A function $V : U \to \mathbb{R}$ is an *inverse integrating factor* of a differential system

$$\dot{x} = \frac{dx}{dt} = P(x, y), \qquad \dot{y} = \frac{dy}{dt} = Q(x, y), \tag{0.1}$$

if V verifies the partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V$$

in U. The following result became very useful for studying the limit cycles: *Assume that the C^1 differentiable system* (0.1) *is defined on the open subset U of $\mathbb{R}^2$, and that $V : U \to \mathbb{R}$ be an inverse integrating factor of that system. If γ is a limit cycle of system* (0.1)*, then γ is contained in* $\{(x, y) \in U : V(x, y) = 0\}$. Garcia, Giacomini and Grau show that the inverse integrating factor defines an ordinary differential equation for the transition map along the orbit. When the regular orbit is a limit cycle, they can determine its associated Poincaré return map in terms of the inverse integrating factor. In particular, they show that the cyclicity of a limit cycle coincides with the vanishing multiplicity of an inverse integrating factor over it. They also apply this result to study some kind of homoclinic loop bifurcation. This interesting work is a new example that the inverse integrating factor is relevant for the study of the dynamics of planar differential systems, and in particular for their limit cycles.

4.6 Complex limit cycles and moduli spaces of complex dynamical systems. Nikolay Dimitrov's talk was on rapid evolution of complex limit cycles in $\mathbb{C}^2$. A complex cycle is defined as a nontrivial element of the fundamental group of a leaf from the foliation. A complex cycle is called a limit cycle provided that it corresponds to an isolated fixed point of a Poincaré map iteration. New results and open questions were presented. This lecture aroused interest in this line of research. Another lecture in which complex dynamical systems was considered was the lecture of W. Arriagada Silva on the modulus for the analytic classification of unfoldings of order one weak foci in $\mathbb{C}^2$. A characterizeation the modulus of analytic classification, under orbital equivalence and under conjugacy, for generic analytic 1 parameter dependent unfoldings of a generic weak focus of order 1 at the origin of coordinates in $\mathbb{C}^2$ was given.

5. Hilbert's 16th problem

This is a problem on the Topology of Algebraic Curves and Surfaces [15] but in its second part Hilbert says: *In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limites) for a differential equation of the first order and degree of the form (E): $dy/dx = Y/X$ where X and Y are rational integral functions of the nth degree in x and y.* Hilbert asked for the "maximum number and position of Poincaré's boundary cycles (cycles limites)..." of an equation with X, Y of the n-th degree in x and y." Presumably Hilbert felt that there is a finite smallest upper bound number of the number of limit cycles occurring for equations with X, Y of the nth degree, so what is this smallest upper bound $H(n)$ which is called the Hilbert number? Poincaré said that each individual polynomial differential equation has a finite number of limit cycles. In a long paper published in 1923 Dulac, a former student of Poincaré, claimed to have proved this statement. In the late 1970's Dumortier called attention to this proof as he noticed a gap and later Ilyashenko found an error (see [16]). Chicone and Shafer proved that a quadratic system has a finite number of limit cycles in any compact subset of the plane. Later Bamon proved in 1986 that for a quadratic equation, also in the vicinity of infinity, there are a finite number of limit cycles. In the late 1980's the general theorem for equations of any degree n was proved independently by Ilyashenko and Ecalle (see [16] and [13]). They actually proved a more general theorem: *Every analytic vector field on a closed surface has a finite number of limit cycles.* This theorem, called the **individual finiteness theorem** (IFT), is the deepest result obtained so far in connection with the second part of Hilbert's 16th problem but the proof of this theorem is not understood beyond the generic case. Smale points this out by saying in [30]: "this theorem is still to be digested by the mathematical community". As the second part of Hilbert's 16th problem is an extremely hard problem, a number of other problems, thought to be less hard to solve but which are still difficult were stated, among them problems for Abel's equation or for Liénard equations, the infinitesimal Hilbert's 16th problem, and the Hilbert-Arnold problem. A survey of these problems is given in [17]. Another problem is **the finiteness part of Hilbert's 16th problem** asking to prove that there exists a finite number $H(n)$, the Hilbert number, which is the maximum number of limit cycles occurring in planar polynomial systems of degree n. In 1989 Roussarie suggested a program to solve this problem (see [25]) and during the past 20 years many results were obtained in this direction for quadratic systems but the problem for this case is still open. The study of the cyclicity of the more degenerate limit periodic sets, among them those involving lines of singularities turned out to be very difficult. These are limit periodic sets where singular perturbation methods need to be used and we reported on these in section 4.1 above. **A breakthrough will come when finite cyclicity will be proved for these very degenerate limit periodic sets involving singular perturbations.** Assuming that the finiteness part were solved we would still have no value for the Hilbert number $H(2)$. In spite of hundreds of paper on quadratic differential systems, **misconceptions about this class** appear even in papers written by well-known mathematicians and published in major journals. Why? This is partly due to the view that *general* results should be our goal. The mathematicians who share this view consider the quadratic class to be *just an example*. This is an *"example"* which has affine moduli and which by the way is expected to produce more than 2000 distinct phase portraits! Quadratic systems form an example just as much as elliptic curves form an example. Let us remind ourselves that Andrew Wiles' theorem which solved Fermat's last theorem is a theorem about elliptic curves saying that *every semi-stable elliptic curve is modular.* In the recommendations we received for writing this report we were asked to include in it **outstanding conjectures and problems, the points of controversy, new directions**. So far we have mentioned conjectures, problems and even new directions but no controversy. The moment has come to talk about the tension between two opposed poles in research orientation: 1) moving away from the hard Hilbert's 16th problem towards less arid, perhaps more fertile ground, by formulating and working on (general) *weaker versions* of this problem (its offspring) or on some finiteness problems involving limit cycles and 2) staying close to Hilbert's 16th problem in its original form rather than to its offspring, by trying to make progress by getting an insight into what happens in specific families of low degree polynomial vector fields. A survey of results in the first direction can be found in [17]. But an attentive look at the last paragraph of [17] on page 324 about the quadratic case points out errors. Loosing sight of Hilbert's 16th problem in the quadratic case leads to pitfalls and it is not just errors that we need to avoid. There is considerably more at stake here. Quadratic systems intervene in many areas of applied mathematics and it is important to gain some real understanding of this class for its own sake and to achieve this we need to look at the second direction of research. Some new insights into Hilbert's 16th problem for quadratic systems by this second approach are: **the deeper connection between the problem of the center and Hilbert's 16th problem as well as a connection between this last problem and the problem of Poincaré**. In the class of quadratic systems with a weak focus of second order all systems with limit cycles have phase portraits topologically equivalent to perturbations of quadratic systems with a center. This was established in joint work of Artes, Llibre and Schlomiuk [2] and was mentioned by Schlomiuk in her lecture. This work was made possible by a global approach to Hilbert's 16th problem for quadratic systems to be discussed below. Finally we observe that in the second part of Hilbert's 16th problem Hilbert also asked for a description of the possible configurations of limit cycles which polynomial systems can have. In [21] it is proved that *any finite topological configuration of limit cycles is always realizable for a convenient polynomial differential system*, and according with the configuration an estimation of the degree of the polynomial differential system realizing the given configuration is also given. **Global aspects of the theory of families of planar polynomial vector fields**. The global theory of planar quadratic vector fields was initiated in an article of Vulpe and Nicolaiev dealing with the problem of classification of quadratic vector fields in the neighborhood of infinity. They solved the problem by applying the theory of invariants of polynomial vector fields as developed by Sibirschi and his school in the second part of the 20th century. Parallel to this development, Schlomiuk and Pal introduced some algebraic-geometric concepts to deal with the problem of classifying quadratic vector fields. While the first paper involved invariant methods but was lacking in geometric content, the second paper had geometric content but was lacking in invariant content. A fusion of the two approaches was produced in [28] where an invariant and at the same time geometric theorem was given classifying quadratic vector fields in the neighborhood of infinity. After this article was published several other works appeared, in particular an analogous result on finite singularities by Artes, Llibre and Vulpe. Another lecture describing globally a class of quadratic differential systems was Artes' lecture, based on joint work with J. Llibre and D. Schlomiuk in which he talked about the bifurcation diagram of the class of all quadratic differential systems with a first order weak focus and an invariant straight line. This bifurcation diagram was obtained by using global methods, in particular algebraic and geometric invariants. A challenging problem is the study of the number of limit cycles that can appear in a neighborhood of this class inside the full class of quadratic systems. Dana Schlomiuk's lecture reported on joint work with N. Vulpe and with Naidenova on the classification of Lotka-Volterra systems. In this work global algebraic-geometric concepts are used in order to determine and organize the maze of phase portraits into a meaningful whole. Algebraic invariants are used to give an invariant classification. **Controlling the maximum number of limit cycles of some specific families of planar polynomial systems** The lecture of Armengol Gasull, based on joint work with Hector Giacomini, gave upper bounds for the number of limit cycles of some planar polynomial differential systems. The problem considered provides some steps in understanding difficulties related with the 16th Hilbert problem. The authors give an effective method for controlling the maximum number of limit cycles of some planar polynomial systems. It is based on a suitable choice of a Dulac function and the application of the well–known Bendixson–Dulac Criterion for multiple connected regions. The key point is a new approach to control the sign of the functions involved in the criterion. The method is applied to several examples.

6. Outcome of the Meeting.

Summing up some of the main new gains made in this area of research, highlighted at the meeting we have the following:

Recent research presented at the meeting indicate subtle connections among the three classical problems and a new, more unified view emerges. This view is closer in spirit to Hilbert's work. Hilbert had breakthrough results on algebraic invariants in the late 19th century. In the 20th century a theory of algebraic invariants for differential equations was developed. This particular theory was instrumental in pointing out the deep connections among the classical problems (see [2]) and, as seen in the talks presented at the meeting, is now being used to advance research on these problems.

Work at the meeting indicated a deeper connections of this area with algebra, with algebraic-geometry and with computer algebra. A new and interesting result presented by Graf v. Bothmer, a newcomer to our group, in which algebraic varieties over fields of finite characteristics highlighted such connections. Other work presented at the workshop was about algebraic-geometric concepts necessary for handling global classification problems on planar polynomial vector fields.

Connections of Hilbert's 16th problem and singular perturbation theory were stressed at the meeting and three of the talks presented new results on singular perturbations.

Complex methods and complex limit cycles were present. These methods include the theory of geometric integration of Darboux, stated in terms of complex algebraic curves. This theory was present in several lectures at the meeting. Furthermore in Xiang Zhang's talk, based on joint work with J. Llibre he talked about the theory of Darboux for polynomial vector fields on $\mathbb{R}^n$ or on $\mathbb{C}^n$, for $n \geq 2$. Combining the method developed recently by Chritopher–Llibre–Pereira for characterizing the multiplicity of invariant algebraic curves, the authors improved the Darboux theory of integrability for higher dimensional systems in $\mathbb{C}^n$ taking into account not only the invariant algebraic hypersurfaces but also their multiplicity. An interesting lecture on new results on multi-fold complex limit cycles was presented at the meeting by N. Dimitrov. Three lectures in the workshop involved complex dynamical systems.

The meeting brought new people in the group. There were also three Ph.D. students. The meeting helped in strengthening relationships and also establishing new collaborations among the participants. The cordial atmosphere emphasized collaboration rather than competition. The splendid location and excellent conditions helped in making this a memorable experience and our thanks go to the members of the Staff who were extremely nice and helpful and we are all grateful.

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