Analysis of nonlinear wave equations and applications in engineering

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1 Overview of the Field

Nonlinear dispersive wave equations arise naturally in scientific and engineering fields such as fluid dynamics, electromagnetic theory, quantum mechanics, optical communication, nonlinear optics etc. Many important questions (both in theory and applications) are related to the interaction of two effects: energy spreading (dispersion, diffraction) and energy concentrating (nonlinear self-trapping, defect modes,. . .) mechanisms. For example, in Korteweg-deVries equation (KdV)

$$u_t = uu_x + u_{xxx},$$

which describes propagation of long waves in shallow water, the term uu_x steepens the wave and causes it to break, while the term u_{xxx} tends to broaden the wave and smoothes the wave profile. For the so-called soliton solution, which is a localized wave that propagates without distortion, these two effects balance each other. There are several other fundamental equations (nonlinear Schrödinger / Gross-Pitaevskii (NLS / GP), nonlinear Klein-Gordon equations (NLKG)) and their modifications which naturally arise in applications and in which similar balance of nonlinearity and dispersion gives rise to coherent structures (solitary waves, vortices, very long-lived *meta-stable* states). The following are central issues in the field:

- Well-posedness of nonlinear dispersive equations.
- Stability of coherent structures such as solitary waves.
- Interaction among coherent structures.

In the past 2-3 decades, new techniques based on harmonic analysis, variational methods, and dynamical systems have advanced our knowledge in all three directions. However, despite relations between the above questions, there has not been sufficiently strong interaction between them. One of the goals of this workshop was to bring together researchers who would benefit from sharing the ideas. One way in which we achieved such interactions was through two tutorials, which bridged scientific communities:

(I) *Derivations of the nonlinear Schrödinger - Gross Pitaevskii (NLS / GP) equations from the quantum $N-$ body problem (speaker: Benjamin Schlein):* This bridged the mathematical physics community, versed in the quantum N- body problem with the nonlinear PDE and nonlinear waves community, more familiar with NLS / GP as a mathematical description of optical and hydrodynamic phenomena.

(II) *Pore Formation in Polymer Electrolytes (speaker: Keith Promislow):* This provided an introduction to an important class of multi-scale problems of huge interest and receiving broad attention, due to applications to energy problems.

2 Recent Developments and Open Problems

2.1 Nonlinear Optics and Stability Problems

In the last decade the study of variable coefficient generalizations of the basic equations became an active area. This was motivated by advances in fabrication technology, enabling the design of precision *photonic structures*, with applications ranging from optical transmission media, optical storage, pulse compression, or more generally, “light processing”, to quantum information science.

For example, in long distance optical fiber communication systems, *dispersion management*, a prescribed variation of the dispersion properties of the optical fiber transmission media, gives rise to a class of NLS equations with time-dependent (periodic, random) coefficients. Also, propagation of light of sufficient intensity in nonlinear and spatially inhomogeneous media as well as the evolution of macroscopic quantum systems (Bose-Einstein condensates) give rise to PDEs of NLS/GP type with spatially dependent variations (*e.g.* compactly supported or periodic) in the linear or nonlinear potentials. Variations in these potentials can be engineered to influence the dynamics of coherent structures.

Stability properties of coherent structures and variable coefficient extensions of standard PDE theory have been under separate study for quite some time. Engineering advances, such as dispersion management in optical fibers and photonic microstructures, motivate studying the interplay between variable coefficients and stability properties of coherent structures.

A mathematical theory of the stability of soliton-like objects in spatially varying media contributes towards a control theory of these objects required for ”light processing”. Currently, there is a need for links between pure mathematicians advancing the rigorous understanding of basic model equations and engineers and applied mathematicians developing new models and applications. One of the goals of this workshop was to encourage this interaction.

2.2 Many-body quantum mechanics

In 2001, the Nobel prize was given for the first experimental evidence of existence of Bose-Einstein condensate (BEC). This discovery generated considerable activity in the study of the evolution of BEC. Mathematical physicists derived rigorously macroscopic evolution equations for interacting many body systems. For example, the nonlinear Schrödinger equation and nonlinear Hartree-Fock equations have been rigorously derived as mean field limits of interacting Bose gases.

2.3 Fundamental PDE problems

Wave maps and Schrödinger maps are fundamental objects of study in modern PDE analysis. These are natural generalizations of the classical wave equations and Schrödinger equations to non-Euclidean spaces. Wave maps also arise as approximations of Einstein’s equations of general relativity.

Dispersive estimates have been an active area of research for a few decades. More recently the emphasis is on Strichartz estimates on non-Euclidean manifolds and with additional terms, e.g. magnetic or potential field.

3 Presentation Highlights

3.1 Application in optics, stability problems

Gadi Fibich described construction of singular solutions of nonlinear evolution equations that become singular on a sphere. The asymptotic profile and blowup rate of these solutions are the same as those of solutions of the corresponding one-dimensional equation that become singular at a point. These results were obtained for the nonlinear Schrödinger equation, the biharmonic nonlinear Schrödinger equation, the nonlinear heat equation, and the nonlinear biharmonic heat equation.

Jared Bronski considered periodic solutions to equations of Korteweg-de Vries type. The stability of periodic wave nonlinear wave-trains is a fundamental problem, whose analytical theory is far less developed that of the solitary wave stability, due to significant mathematical challenges and new phenomena.

Bronski demonstrated a proof of an index theorem giving an exact count of the number of unstable eigenvalues of the linearized operator in terms of the number of zeros of the derivative of the traveling wave profile together with geometric information about a certain map between the constants of integration of the ordinary differential equation and the conserved quantities of the partial differential equation. This index can be regarded as a generalization of both the Sturm oscillation theorem and the classical Lyapunov stability theory for solitary wave solutions for equations of Korteweg-de Vries type (Benjamin, Bona-Souganidis-Strauss, Weinstein, ...)

In the case of a polynomial nonlinearity this index, together with a related one introduced earlier by Bronski and Johnson, could be expressed in terms of derivatives of hyperelliptic integrals on a finite genus Riemann surface. Since these hyperelliptic integrals satisfy a Picard-Fuchs relation these derivatives can be expressed in terms of the integrals themselves, leading to a closed-form expression in terms of a finite number of moments of the solution.

Boaz Ilan described band-edge solitons of Nonlinear Schrödinger equations with periodic potentials (joint work with M.I. Weinstein). Nonlinear Schrödinger (NLS) / Gross-Pitaveskii equations with periodic potentials admit positive bound states (solitons). For focusing nonlinearities these solitons bifurcate from the zero state with frequencies (propagation constant) lying in the semi-infinite spectral gap and near the spectral band edge. A multiple scale expansion leads to a constant coefficient homogenized / effective medium NLS equation that depends on the band-edge Bloch wave through an effective-mass tensor and nonlinear coupling constant. The multiple scales argument is made rigorous via a Lyapunov Schmidt reduction to Bloch-modes sufficiently near the spectral band edge. To leading order the soliton is constructed from the Bloch wave that is slowly modulated by a ground state of the homogenized equation. In the L^2-critical case, for any non-constant periodic potential the power (L^2 norm) of the soliton is *strictly lower* than the power of the Townes mode, which has the critical power for collapse. The implications to collapse dynamics and self-focusing instability were elucidated using computations of bound states and direct computations of critical NLS equations in 1D and 2D.

Milena Stanislavova presented conditional stability theorems for Klein-Gordon type equations. She considered positive, radial and exponentially decaying steady state solutions of the Klein-Gordon equation with various power nonlinearities. The main result was a precise construction of infinite-dimensional invariant manifolds in the vicinity of these solutions. The precise center-stable manifold theorem for the Klein-Gordon equation includes the co-dimension of the manifold, a formula for the asymptotic phase and the decay rates for even perturbations.

Yoshio Tsutsumi gave a presentation on stability of cavity soliton for the Lugiato-Lefever equation with additive noise. He considered the stability of stationary solution for the Lugiato-Lefever (LL) equation with periodic boundary condition under perturbation of additive noise. The LL- equation is a nonlinear

Schrödinger equation with damping and spatially homogeneous forcing terms, which describes a physical model of a unidirectional ring or Fabry-Perot cavity with plane mirrors containing a Kerr medium driven by a coherent plane-wave field. The stationary solution of (LL) is called a "Cavity Sliton". Tsutusmi showed the stability of certain stationary solutions under the perturbation of additive noise from a viewpoint of the Freidlin-Wentzell type large deviation principle.

Roy Goodman described bifurcations of *nonlinear defect modes*. The nonlinear coupled mode equations describe the evolution of light in Bragg grating optical fibers. Defects (localized potentials) can be added to the fiber in order to trap light at a specialized location as a nonlinear defect mode. In numerical simulations these defect modes are seen to lose (linear) stability through several types of bifurcations. Inverse scattering is used to design defects in which the bifurcations can be easily observed and studied via the derivation of finite-dimensional reduced equations. Goodman gave conditions for the existence of Hamiltonian pitchfork and Hamiltonian Hopf bifurcations.

Björn Sandstede gave a presentation on pointwise estimates and nonlinear stability of waves. Over the past decade, pointwise Green's function estimates have proved very useful in establishing nonlinear stability of viscous shock profiles under the assumption of spectral stability. He reported here on recent work with Beck and Zumbrun on extending this approach to the nonlinear stability of time-periodic viscous shocks. Key to the derivation of the required *pointwise* bounds in the time-periodic setting are meromorphic extensions of exponential dichotomies of appropriate time-periodic eigenvalue problems. He also showed how spectral stability of weakly time-periodic shocks can be established near Hopf bifurcation using a spatial-dynamics approach. The motivation for this work comes from sources in reaction-diffusion systems. He also outlined the challenges and hopes for nonlinear stability proofs in this context.

Justin Holmer considered dynamics of KdV solitons in the presence of a slowly varying potential. He studied the dynamics of solitons as solutions to the perturbed KdV (pKdV) equation $\partial_t u = -\partial_x(\partial_x^2 u + 3u^2 - bu)$, where $b(x,t) = b_0(hx, ht)$, $h \ll 1$ is a slowly varying potential. This result refined earlier work of Dejak-Sigal and an estimate on the trajectory of the soliton parameters of scale and position was obtained. In addition to the Lyapunov analysis commonly applied to these problems, a local virial estimate due to Martel-Merle was used. The proof did not rely on the inverse scattering machinery and could be expected to carry through for the L^2 subcritical gKdV-p equation, $1 < p < 5$. The case of $p = 3$, the modified Korteweg-de Vries (mKdV) equation, is structurally simpler and more precise results can be obtained by the method of Holmer-Zworski. This was joint work with Galina Perelman.

Eduard Kirr considered asymptotic stability of nonlinear bound states and resonances for nonlinear Schrödinger equations with *subcritical* nonlinearities. What makes the extension to the subcritical case possible is his recent method for obtaining dispersive estimates for perturbations of linear Schrödinger operators with time-dependent and spatially localized coefficients. The method currently works in dimensions two and higher. Kirr discussed obstacles in extending his method to one space dimension. Also some applications to nonlinear equations, in particular to asymptotic stability and radiative damping of ground states in NLS were presented.

Gideon Simpson presented a poster on numerical simulations of the energy-supercritical NLS equation. These computations were motivated by recent works of Kenig-Merle and Kilip-Visan who considered some energy supercritical wave equations and proved that if the solution is *a priori* bounded in the critical Sobolev space (i.e. the space whose homogeneous norm is invariant under the scaling leaving the equation invariant), then it exists for all time and scatters. Simpson numerically investigated the boundedness of the H^2-critical Sobolev norm for solutions of the NLS equation in dimension five with quintic nonlinearity. It was found that for a class of initial conditions, this norm remains bounded, the solution exists for long time, and it scatters (disperses to zero).

Young-Ran Lee presented a proof of exponential decay of dispersion managed solitons with vanishing average dispersion. It was shown that any L^2 solution of the Gabitov-Turitsyn equation describing dispersion managed solitons with zero average dispersion decays exponentially in space and frequency domains. This confirmed in the affirmative Lushnikov's conjecture of exponential decay of dispersion managed solitons. This work was done jointly with M. Burak Erdogan and Dirk Hundertmark.

3.2 Many-body quantum mechanics

Benjamin Schlein gave a two-hour tutorial on derivation of equations of nonlinear Schrödinger / Gross-Pitaevskii type as the mean field limit of N-body quantum problems, as $N \to \infty$. In particular, he showed that the nonlinear Hartree equation can be used to describe the macroscopic properties of the evolution of a many body system in the so called mean field limit. He also explained how Gross-Pitaevskii equation, a cubic nonlinear Schrödinger equation, can be used to describe the dynamics of Bose-Einstein condensates.

Mathieu Lewin considered variational models for infinite quantum systems with an example of the crystal with defects. Describing quantum particles in a quantum medium often leads to strongly indefinite (sometimes unbounded from below) theories, for which it is usually quite hard to establish the existence and the stability of bound states. Two well-known, important examples are relativistic electrons described by the Dirac operator and electrons close to a defect in a quantum crystal. Lewin presented a new method for constructing and studying a variational model for such systems. He concentrated on the Hartree model for the crystal with a defect.

The main idea is to describe at the same time the electrons bound by the defect and the (nonlinear) behavior of the infinite crystal. This leads to a (rather peculiar) bounded-below nonlinear functional whose variable is however an operator of infinite-rank.

Lewin introduced the appropriate functional analytic setting, stated the existence of global-in-time solutions to the associated time-dependent Schrödinger equation, and discussed the existence, properties and the stability of bound states.

Walid K. Abou Salem presented microscopic derivation of the magnetic Hartree equation. He discussed the rigorous derivation of the time-dependent Hartree equation in the presence of magnetic potentials. He also remarked on how to extend the analysis to the Gross-Pitaevskii equation.

Natasa Pavlovic discussed the quintic NLS as the mean field limit of a Boson gas with three-body interactions. She described the dynamics of a boson gas with three-body interactions in dimensions d=1,2. She and her collaborator, Thomas Chen, prove that in the limit as the particle number N tends to infinity, the BBGKY hierarchy of k-particle marginals converges to a limiting Gross-Pitaevskii (GP) hierarchy for which they proved existence and uniqueness of solutions. For factorized initial data, the solutions of the GP hierarchy are shown to be determined by solutions of a quintic nonlinear Schrödinger equation. This was joint work with Thomas Chen.

Thomas Chen discussed some recent developments on the well-posedness of the Cauchy problem for focusing and defocusing GP hierarchies. He surveyed some recent results, all from joint works with Natasa Pavlovic, related to the Cauchy problem for the Gross-Pitaevskii (GP) hierarchy. First, he addressed the local well-posedness theory, in various dimensions, for the cubic and quintic case. He then introduced new conserved energy functionals which were used in the following contexts: (1) In a joint work with N. Tzirakis, to prove, on the L2 critical and supercritical level, that solutions of focusing GP hierarchies with a negative average energy per particle blow up in finite time. (2) To prove the global well-posedness of the Cauchy problem for energy subcritical, defocusing GP hierarchies, based on the conservation of higher order energy functionals. (3) To prove global well-posedness of focusing and defocusing GP hierarchies on the L2 subcritical level, based on a generalization of the Gagliardo-Nirenberg inequalities which they establish for density matrices.

Manoussos Grillakis spoke on precise $N-$ dependent error bounds, satisfied by the NLS / GP approximation to mean-field scaled quantum N body problem, for large N.

3.3 Fundamental problems in PDE analysis

Wilhelm Schlag gave a presentation on inverse square potentials and applications. He discussed some recent work on dispersive estimates on a curved background. These problems arise in geometry and physics, and are reduced for fixed angular momentum to a one-dimensional problem with an inverse square potential. For the Schwarzschild case, one obtains local pointwise decay rates which increase with the angular momentum. This was joint work with R. Donninger, A. Soffer, and W. Staubach.

Daniel Tataru presented his recent result on large data wave maps. He proved for large data wave maps from $\mathbb{R}^{2+1}$ into a compact Riemmanian manifold, the following dichotomy: either a solution is global and dispersive, or a soliton like concentration must occur. This was joint work with Jacob Sterbenz.

3.4 Energy conversion

Keith Promislow gave tutorial on Pore Formation in Polymer Electrolytes. The efficient conversion of energy from chemical and photonic forms to useful electric voltage requires the development of nanostructured materials with interpercolating structure. In practical applications this is achieved by functionalizing polymers, attaching acid groups to short side chains which extend from long, hydrophobic polymer backbones. When placed in solvent, these functionalized polymers form nanoscale solvent-filled pores lined with the tips of the acid groups, and ideal environment for the selective conduction of properly charged ions.

He presented a family of models, which we call functionalized Lagrangians, which mimic the energy landscape of the functionalized polymer/solvent mixtures. The functionalized energies are higher order, and strongly nonlinear, but with special structure which renders them amenable to analysis. He outlined the properties of the functionalized Lagrangians, and the multi-stage structure of the associated gradient flows.

3.5 Geometric PDEs

Stephen Gustafson gave a talk on Schrödinger and Landau-Lifshitz maps of low degree. The Schrödinger (and Landau-Lifshitz) map equations are a basic model in ferromagnetism, and a natural geometric (hence nonlinear) version of the Schrödinger (and Schrödinger-heat) equation. While there has been recent progress on the question of singularity formation for the wave and heat analogues (wave map and harmonic map heat-flow), the Schrödinger case seems more elusive. He presented results on global regularity and long-time dynamics for equivariant maps with near-minimal energy. He emphasized lower degree (2 and 3) maps, for which the analysis is trickier, and the dynamics more complex, phenomena related to slower spatial decay of certain eigenfunctions. This was joint work with K. Nakanishi, and T.-P. Tsai.