Structure and representations of exceptional groups

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1 Background

From Cartan and Killing's original classification of simple Lie groups in the 1890s, these groups have been understood to be of two rather different types: the infinite families of classical groups (related to classical linear algebra and geometry); and a finite number of exceptional groups, ranging from the 14-dimensional groups of type G_2 to the 248-dimensional groups of type E_8. Often it is possible to study all simple Lie groups at once, without reference to the classification; but for many fundamental problems, it is still necessary to treat each simple group separately.

For the classical groups, such case-by-case analysis often leads to arguments by induction on the dimension (as for instance in Gauss's method for solving systems of linear equations). This kind of structure and representation theory for classical groups brings tools from combinatorics (like the Robinson-Schensted algorithm), and leads to many beautiful and powerful results.

For the exceptional groups, such arguments are not available. The groups are not directly connected to classical combinatorics. A typical example of odd phenomena associated to the exceptional groups is the non-integrable almost complex structure on the six-dimensional sphere S^6, derived from the group G_2. What makes mathematics possible in this world is that there are only finitely many exceptional groups: some questions can be answered one group at a time, by hand or computer calculation.

The same peculiarity makes the possibility of connecting the exceptional groups to physics an extraordinarily appealing one. The geometry of special relativity is governed by the ten-dimensional Lorentz group of the quadratic form of signature $(3, 1)$. Mathematically this group is part of a family of Lorentz groups attached to signatures (p, q), for any non-negative integers p and q; there is no obvious mathematical reason to prefer the signature $(3, 1)$. A physical theory attached to an exceptional group - best of all, to the largest exceptional groups of type E_8 - would have no such mathematical cousins. There is only one E_8.

2 Recent Developments and Objectives

Two years ago Garrett Lisi proposed an extension of the Standard Model in physics, based on the structure of the 248-dimensional exceptional Lie algebra E_8. Lisi's paper raises a number of mathematically interesting questions about the structure of E_8, for instance this one: the work of Borel and de Siebenthal published in 1949, and Dynkin's work from around 1950, gave a great deal of information on the complex subgroups of complex simple Lie groups. For example, they independently showed that complex E_8 contains (up to conjugacy) just one subgroup locally isomorphic to $SL(5,\mathbb{C}) \times SL(5,\mathbb{C})$. For Lisi's work, one needs to know about *real* subgroups of *real* simple groups: which real forms of $SL(5,\mathbb{C}) \times SL(5,\mathbb{C})$ can appear in a particular real form of E_8? These are subtle questions, not yet completely understood. A mathematical study of these questions is interesting for its own sake, and may provide some constraints on the structure of the physical theories that can be built using E_8.

The goal of this workshop was to introduce mathematicians to these physical ideas, and to describe much of the recent mathematical work on the exceptional Lie groups.

3 Presentation Highlights

There were quite a few outstanding presentations, both formal and informal, concerning semisimple Lie groups (especially E_8) and their possible use in physical models. Among them, in alphabetical order of presenter's name, are

JEFF ADAMS (University of Maryland), "Elliptic elements of the Weyl group of E_8"

An element of a Weyl group W is *elliptic* if it has no fixed points in the reflection representation. An example is the Coxeter element, studied extensively by Kostant. Elliptic elements were classified by Carter in 1972, who discovered a relation with nilpotent conjugacy classes in the corresponding semisimple group G. Lusztig has recently studied this from a new point of view. Each elliptic conjugacy class in W is naturally a semisimple conjugacy class in G. Prof. Adams considered the elementary question: what is the map from elliptic conjugacy classes in W to W-orbits in T? He focused on the example of E_8 and presented a number of computer calculations. These examples suggested a particularly interesting class of elliptic elements, sharing some of the properties established by Kostant for the Coxeter element. He defined an elliptic conjugacy class in W to be ***uniform*** if each element acts freely on the set of roots, and if there is a representative of the class having length equal to the number of orbits on the roots. He showed that there are exactly 12 uniform conjugacy classes in the Weyl group of E_8.

DAN BARBASCH (Cornell University), "The spherical unitary dual for the quasisplit group of type E_6"

The presenter has in recent years described completely the spherical unitary representations of split groups over real and p-adic fields. A central feature of his work is a reduction to calculations in affine Hecke algebras, and ultimately to calculations related to finite-dimesnsional representations of Weyl groups. Attached to any diagram automorphism τ of finite order m for a simple Dynkin diagram, and to a cyclic Galois extension K of degree m of the base field k, there is a quasisplit group G over k. (In the case of exceptional groups, this means that there is a quasisplit group of type E_6 attached to each quadratic extension of k.) This talk offered a description of the spherical unitary dual of such a quasisplit group, in terms of the spherical unitary duals of smaller split groups (for E_6, split groups of type F_4).

BIRNE BINEGAR (Oklahoma State University), "W-graphs, nilpotent orbits, and primitive ideals"

Work of Howe and others in the 1970s attached to any irreducible representation of a semisimple Lie group some geometric invariants: for example, some nilpotent orbits in the dual of the Lie algebra. The presenter described his work using the `atlas` software to compute some of these invariants, using the Kazhdan-Lusztig notion of "W-graphs."

DAN CIUBOTARU (University of Utah), "The Dirac operator for graded affine Hecke algebras" (joint work with D. Barbasch and P. Trapa)

Prof. Ciubotaru defined an analogue of the Dirac operator for graded affine Hecke algebras of p-adic groups, and establish a version of Parthasarathys Dirac operator inequality. He then proved a version of Vogan's conjecture for Dirac cohomology. The formulation of the conjecture depends on a uniform parametrization of spin representations of Weyl groups. Prof. Ciubotaru applied these results to prove new results about unitary representations of graded affine Hecke algebras, and therefore of p-adic reductive groups.

MICHAEL EASTWOOD (Australian National University (Canberra)), "Representations from contact geometry"

Apart from $SL(2)$, each simple Lie group is the symmetry group of a contact manifold equipped with some extra geometric structure. This includes the exceptional groups. This fact can be used to give a geometric construction of the finite-dimensional representations of the simple groups, including the exceptional groups. Prof. Eastwood gave a useful introduction to contact geometry and indicated just how this gives a useful elegant construction of finite dimensional representations..

SKIP GARIBALDI (Emory University), "There is no (interesting) Theory of Everything inside E_8"

In joint work with Jacques Distler, Prof. Garibaldi proved that the real forms of E_8 (and the complex group E_8 regarded as a real group) cannot have subgroups with certain properties. Some widely accepted (this is meant to be a more neutral term than "well established") principles for mathematical models of physics suggest that a physical interpretation of this result is that the "Exceptionally Simple Theory of Everything" conflicts with widely accepted representation-theoretic properties of the Standard Model. He indicated that this interpretation is robust, in that the result also shows that a whole family of related "Theories of Everything" also conflict with these same properties of the Standard Model.

There was quite a bit of lively discussion here, as the mathematicians tried to pin down the precise meaning of various terms and conventions, and as Garrett Lisi questioned aspects of the presentation that were in contrast to his E_8 theory. Each of their viewpoints predicts some ("a handful of") unobserved particles and part of this discussion centered on how many unobserved particles were acceptable.

ALAN HUCKLEBERRY (Ruhr-Universität Bochum), "The role of Kobayashi hyperbolicity in the study of flag manifolds"

Open orbits D of simple real forms G_0 in flag manifolds $Z = G/Q$ of their complexifications G are considered. For any choice K_0 of a maximal compact subgroup of G_0, the minimal K_0-orbit in the flag domain D is a compact complex manfold referred to as the base cycle C_0 with respect to the choice of K_0. It can be regarded as a point in the Chow (or Barlet space) $\mathcal{C}_q(Z)$ of all cycles of the same dimension q. It is known that $\mathcal{C}_q(Z)$ is smooth at C_0 and therefore it makes sense to consider the irreducible compoment of $\mathcal{C}_q(Z)$ which contains C_0 and the open subset $\mathcal{C}_q(Z)$ of those cycles which are contained in D. The complex geometry of $\mathcal{C}_q(Z)$ was the theme of the talk. For example, using analytic properties of the intersection of the cycles with certain special Schubert varieties, the Kobayashi hyperbolicity of $\mathcal{C}_q(Z)$ is proved. This sheds light on the complex geometry of D, e.g., leading to a precise desciption of its group of holomorphic automorphisms. It should be emphasized that for fixed G_0 the cycle space $\mathcal{C}_q(Z)$ varies tremendously as D and Z vary, making it a rich source of interesting complex varieties with the potential of realizing nontrivial G_0-representations in a holomorphic context. A preprint (arXiv:1003:5974) is available.

TOSHIYUKI KOBAYASHI (Kyoto University), "Stable Spectrum on Locally Homogeneous Spaces"

Video: http://www.birs.ca/events/2010/5-day-workshops/10w5039/videos/watch/201007081615-Kobayashi.mp4

Questions of spectra and discontinuity are more delicate for homogeneous spaces G/H with H noncompact, than for those with compact H. Prof. T. Kobayashi discussed several aspects of this situation:

the existence question for $\Gamma \subset G$ with $\Gamma\backslash G/H$ compact
spectral analysis on compact quotient manifolds $\Gamma\backslash G/H$
deformation of Γ, e.g. to a subgroup $L \subset G$ for which $L \cap H$ is compact and Γ is uniform in L, and
stable spectrum of $\Gamma\backslash G/H$

Here G is a noncompact simple Lie group, H is a closed reductive subgroup, and Γ is a discrete subgroup of G. Or G/H may be a pseudo–riemannian nilmanifold, e.g. Minkowski space. In any case, the first step is to find the condition for Γ to act freely and properly discontinuously on G/H, so that $\Gamma\backslash G/H$ is a pseudo–riemannian quotient manifold of G/H. Building on this, the presenter described the current state of these matters and contrasted the general pseudo–riemannian cases with the more classical riemannian cases.

BERTRAM KOSTANT (MIT) "Experimental evidence for the occurrence of E_8 in nature and the radii of the Gossett circles

Video: http://www.birs.ca/events/2010/5-day-workshops/10w5039/videos/watch/201007061330-Kostant.mp4

A recent experimental discovery involving spin structure of electrons in a cold one-dimensional magnet points to a validation of a (1989) Zamolodchikov model involving the exceptional Lie group E_8. The model predicts 8 particles and predicts the ratio of their masses. The conjectures have now been validated experimentally, at least for the first five masses. The Zamolodchikov model was extended in 1990 to a Kateev-Zamolodchikov model involving E_6 and E_7 as well. In a seemingly unrelated matter, the vertices of the 8-dimensional Gosset polytope identify with the 240 roots of E_8. Under the famous two-dimensional (Peter McMullen) projection of the polytope, the image of the vertices are arranged in 8 concentric circles, hereafter referred to as the Gosset circles. The McMullen projection generalizes to any complex simple Lie algebra (in particular not restricted to A-D-E types) whose rank is greater than 1. The Gosset circles generalize as well. Applying results in Prof. Kostant's AJM 1959 paper, he found some time ago a very easily defined operator A on a Cartan subalgebra, the ratios of whose eigenvalues are exactly the the ratios of squares of the radii r_i of the generalized Gosset circles. The two matters considered above relate to one another in that the ratio of the masses in the E_6, E_7, and E_8 Kateev-Zamolodchikov models are exactly equal to the ratios of the radii of the corresponding generalized Gosset circles.

GARRETT LISI, "Group-theoretic models in gravity, the standard model, and old-and-new ideas about unification"

This series of three informal lectures was the keynote of the conference. Meeting after dinner, each consisted of perhaps 30 minutes of exposition and 60 minutes of questions and answers. Most of the latter were clarifications to mathematicians, but some addressed the differences between Dr. Lisi's E_8 theory and the more conservative physical theory criteria described by Prof. Garibaldi, this in terms of properties that that one expects for a good physical model. The titles of the individual talks were "Unification"; "A physicist's topology—a group effort"; and "Massive speculation—trialities and tribulations".

TODOR MILEV (Jacobs Universität Bremen), "Computing regular subalgebras of simple Lie algebras"

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $\mathfrak{h}$ be a fixed Cartan subalgebra. Let $\mathfrak{l}$ be a subalgebra containing $\mathfrak{h}$ (non-zero nilradicals allowed) and let $\mathfrak{k} \supset \mathfrak{h}$ be the reductive part of $\mathfrak{l}$. A Fernando-Kac subalgebra of $\mathfrak{g}$, associated to an infinite dimensional $\mathfrak{g}$-module M, is defined as the set $\mathfrak{g}[M]$ of locally finitely acting elements of $\mathfrak{g}$. A subalgebra $\mathfrak{l}$ for which there exists an irreducible module M with $\mathfrak{g}[M] = \mathfrak{l}$ is called a Fernando-Kac subalgebra of $\mathfrak{g}$. A Fernando-Kac subalgebra is of finite type if there exists a module as above for which the Jordan-Hölder $\mathfrak{k}$-multiplicities of all simple $\mathfrak{k}$-modules are finite. A root system criterion describing all $\mathfrak{l} \supset \mathfrak{h}$ that are Fernando-Kac of finite type was conjectured by I. Penkov based on his joint work [PNZ] with V. Serganova and G. Zuckerman and a paper of S. Fernando. The presenter's Ph.D. thesis proves this criterion for all finite dimensional simple Lie algebras except E_8 (the case $\mathfrak{s}l(n)$ was already proved in [PSZ]). The proofs for exceptional Lie algebras F_4, E_6,and E_7 involved a computer computation. A regular subalgebra of a simple Lie algebra can be defined as a semisimple subalgebra spanned by root spaces of $\mathfrak{g}$. Regular subalgebras were classified in Dynkins fundamental paper "Semisimple Lie algebras of semisimple Lie algebras" (there are 75 proper isomorphism classes in E_8). Dynkin's classification automatically applies to root reductive subalgebras. In order to enumerate all possible nilradicals up to isomorphism one should first compute the $\mathfrak{k}$-module decomposition of $\mathfrak{g}$.

KARL-HERMANN NEEB (Universität Erlangen), "Semibounded representations of automorphism groups of Banach symmetric spaces"

The presenter discussed separable unitary representations of the automorphism group of a Hilbert hermitian symmetric space and its central extensions. He assumed that the representations are semibounded in the sense that, some element of the Lie algebra has a neighborhood on which the operators of the derived representation are uniformly bounded above. The methods to analyze such representations come from three sources: (1) Pickrells regularity results on separable representations of orthogonal and unitary groups, (2) some recent insights in the structure of invariant open convex cones in orthogonal and unitary Lie algebras, and (3) procedures to realize representations in Hilbert spaces of holomorphic sections of complex Hilbert bundles over the symmetric space.

BENT ØRSTED (Aarhus University), "Borel-de Siebenthal discrete series for exceptional groups"

For a semisimple Lie group admitting discrete series representations, it remains an interesting problem to find explicit realizations. In this lecture, based on joint work with Joseph Wolf, the presenter described the Borel-de Siebenthal discrete series, giving details about the geometry of the corresponding coadjoint orbits. In particular for some exceptional groups he described realizations allowing continuation in the discrete series parameter.

ROBERTO PERCACCI (International School for Advanced Studies, Trieste), "Elements of a GraviGUT"

A GraviGUT would be a theory where gravity is unified with the other interactions in a way that directly generalizes what is done in the grand unified theories of particle physics. The presenter described what one would need to do to construct such a theory, and the steps that have been successfully carried through so far. He concentrated on the case, developed in his work with Fabrizio Nesti, where the unifying group is $SO(3, 11)$.

HADI SALMASIAN (University of Ottawa), "Unitary representations of supergroups and the method of orbits"

The main goal of this talk was to show that ideas of the orbit method can be applied to describe unitary representations of Lie supergroups. The presenter defined Lie supergroups and their unitary representations (in a global sense) and proved that for nilpotent Lie supergroups there exists a bijective correspondence between irreducible unitary representations and nonnegative coadjoint orbits. A simple branching rule for irreducible unitary representations to the even part followed.

GORDAN SAVIN (University of Utah), "Classifying discrete series representations of G_2 using minimal representations"

The presenter began with the classical inclusions

$$SL(3, \mathbb{C}) \hookrightarrow G_2(\mathbb{C}) \hookrightarrow Spin(7, \mathbb{C}).$$

Using Langlands functoriality conjectures, he deduced some (conjectural) relationships between discrete series representations for G_2 (over a p-adic field k) and representations of $PGL(3, k)$ and $PSp(6, k)$. Finally, he showed how to prove these conjectural relationships using theta-liftings related to minimal representations of E_6, E_7, and E_8.

DANIEL STERNHEIMER (Keio University and Université de Bourgogne) "Some instances of the unreasonable effectiveness (and limitations) of symmetries and deformations in fundamental physics"

The presenter gave a survey of some applications of group theory and deformation theory (including quantization) to mathematical physics. He discussed rotation and discrete groups in molecular physics ("dynamical" symmetry breaking in crystals, Racah-Flato-Kibler); chains of groups and symmetry breaking. He also discussed classification of Lie groups ("internal symmetries") in particle physics. Finally he addressed the topics of space-time symmetries and relations with internal symmetries. Then there was a discussion of deformation of symmetries, specifically deformation quantization, quantum groups and quantized spaces; of field theories and evolution equations from the point of view of nonlinear Lie group representations; of connections with some cosmology, including especially quantized anti-de Sitter groups and spaces; and of prospects for future developments between mathematics and physics.

4 Outcome of the Meeting

We enthusiastically thank BIRS for the opportunity to bring together a group of representation theorists with a group of physicists in circumstances that facilitated communication and understanding. The facilities and setting at BIRS are outstanding, as is the organization and infrastructure. In particular Brenda Williams and her staff made a big contribution to the success of the program.

For one reason or another, physics participation was less than we had hoped. This affected the balance of participants and the composition of the organizing committee. The BIRS staff was extremely helpful in dealing with that, and the organizers warmly thank them for their flexibility, which led to an exciting and fruitful conference.

The main progress was the increased understanding by mathematicians of the Standard Model and of the E_8 models in particle physics. There was some reciprocity here as the physics participants learned a good bit about the representation theory of semisimple Lie groups, E_8 in particular, and the ATLAS project in semisimple structure and representation theory.

A certain amount of mathematical software (especially ATLAS) was demonstrated and circulated. This will certainly have future impact.

With these two items of progress, the program more than satisfied its goals, and as nearly as we can tell all the participants were delighted with the way it worked out. But more than that, a number of participants took advantage of the presence of the others to advance individual or joint research projects; so the benefits of the meeting will continue to develop for some time.

David Vogan

Joseph Wolf