Approximation Theory and Harmonic Analysis on Spheres and Related Domains

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This is a program of Research in Team of three participants, the two organizers and Heping Wang of Capital Normal University, Beijing.

1 Overview of the Field

There have been continuing researches in approximation theory and harmonic analysis on the unit sphere throughout the last century. For approximation theory, one of the historical highlights is the complete characterization of best approximation by polynomials on the sphere in terms of a modulus of smoothness defined via the spherical means, the accumulation point of decades of works by many authors and finally materialized in [13, 1994]. For harmonic analysis on the sphere, besides the general results that hold for the homogeneous spaces, including the sphere, the essential results are those on multiplier theorems and convergence of projection operators and the Cesàro means in [1, 1972] and [14, 1986].

In recent years analysis on the sphere has been revitalized by applications in applied mathematics, such as earth sciences, computational mathematics and statistics. The topics have fruitful connections with many branches of mathematics, such as numerical integration, computer tomography, coding theory, data fitting, special functions, group representation, spectral theory, random matrices, and differential equations.

2 Recent Developments and Open Problems

In recent years, there have been several significant development in these two fields:

1. The first significant progress is the introduction of orthogonal polynomials on the sphere for a family of wright functions invariant under a finite reflection group, called the h-harmonics, in the work of Dunkl, highlighted by the introduction of the Dunkl operators [11, 1989], a family of commuting first order differential-difference operators that play the role of differential operators in the ordinary harmonic analysis on the sphere. The study of harmonic analysis in terms of h-harmonic expansions on the sphere was started in [15], but picked up pace only in the past a few years. By now, all results in [1, 14] have been extended to the case of weighted space [5, 6, 7].

2. The second one is the realization that h-harmonic expansions in the weighted L^p spaces are closely related orthogonal expansions on the unit ball and on the simplex [16], which lead to significant progress in these two compact domains, of which little quantitative works were known previously. This realization further opens up the possibility of quantitative studies of analysis on the weighted spaces on the sphere for doubling weights [3, 4, 16].

3. The modulus of smoothness defined via spherical means have the drawback of difficult, or impossible, to compute, making the results impractical. Ditzian [10, 1997] introduced a new class of moduli of smoothness that has interesting features but does not solve the problem of computability. Only last year, two of the organizers introduced a new modulus of smoothness that essentially reduced many problems in approximation theory on the sphere to the circle, and provides a most satisfactory solution for the problem [8, 9].

All three participants of the Research in Team have been actively involved in the recent developments of these directions.

3 Scientific Progress Made

The recent development and new results have changed the landscape of approximation theory and (weighted) harmonics analysis. The two participants, Dai and Xu, have been working on a research monograph entitled "Approximation theory and Harmonic Analysis on Unit Sphere", which will be the first research monograph dedicated entirely to this subject. One of the main objectives of the program is for the two authors to discuss the organization and details of the book. Besides the book, the participants in the research team also discussed in depth several research problems, which are listed below.

1. Intertwining operator. This operator, usually denoted by V_κ, intertwines between Dunkl operators $\mathcal{D}_i$ and the differential operator ∂_i in the sense that $\mathcal{D}_i V_\kappa = V_\kappa \partial_i$. It is known to be a positive linear operator that encodes essential information on the h-harmonics. For example, the reproducing kernel $P_n^h(\cdot,\cdot)$ of the space of h-harmonics of degree n is given by

$$P_n^h(x,y) = \frac{n+\lambda_\kappa}{\lambda_\kappa} V_\kappa\left[C_n^{\lambda_\kappa}(\langle \cdot, y\rangle)\right](x), \qquad \lambda_\kappa := \sum \kappa_v + \frac{d-2}{2},$$

   which becomes the well-known formula of zonal harmonics for ordinary harmonics when all parameters $\kappa_v = 0$ and, hence, $V_\kappa = id$. In the case of the weight function $w_\kappa(x) = \prod_{i=1}^d |x_i|^{2\kappa_i}$, $\kappa_i \ge 0$, associated with the reflection group $\mathbb{Z}_2^d$, the intertwining operator is an integral operator given by

$$V_\kappa(x) = c_\kappa \int_{[-1,1]^d} f(t_1 x_1, \ldots, t_d x_d) \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} dt.$$

   Currently many deeper results are established only for the case of $\mathbb{Z}_2$, such as the critical index of the Cesàro means, because of the availability of this explicit formula of V_κ.

   The participants discussed in depth how to deduce information of V_κ, for reflection groups other than $\mathbb{Z}_2$, without knowing explicitly an explicit formula. After applying appropriate cut-off functions, some of the problems reduces to a sharp upper bound of $V_\kappa \chi_{c(x,r)}$, where $c(x,a)$ denotes a spherical cap centered at x with radius $r > 0$ and χ_E denotes the characteristic function of E. They also discussed the case of V_κ for the group of D_3, the simplest case beyond $\mathbb{Z}_2^d$.

2. The participants also discussed the differential operators $D_{i,j} := x_i \partial_j - x_j \partial_i$, which can also be written as partial derivative in terms of the $\theta_{i,j}$ angle of the polar coordinates in (x_i, x_j)-plane. These operators are also the infinitesimal operator of the group representation of $SO(d)$. The Laplace-Beltrami operator Δ_0 on the sphere satisfy a decomposition

$$\Delta_0 = \sum_{1 \le i < j \le d} D_{i,j}^2,$$

   which indicates that $D_{i,j}$ are more primitive than Δ_0 and, as a result, should encode more information. Their essential presence in the recent work of characterization best approximation on the sphere [8] suggests that they deserve a more through look. As results of the discussions, several new identities are derived and the boundedness of the Riesz transforms defined via $D_{i,j}$ is established.

3. Another problem being discussed is the proof of the equal area partitions of the sphere, which states that, for each given integer N, the sphere S^{d-1} can be partitioned into N connected pieces with all pieces having the same area. This is a useful concept that has potential applications in cubature rules and play a role in the recent announced proof on optimal spherical design [2], which states that for each positive integer n there is a cubature rule of equal weight on the sphere S^{d-1} that uses $O(1)n^{d-1}$ many points on the sphere. This announced proof is scrutinized but the participants were not able to verify the entire proof.

4 Outcome of the Meeting

1. For the book project, the discussion settled a number of issues, including the treatment of homogeneous spaces, details on multiplier theorems, on operators defined by cut-off functions, on approximation on the sphere, as well as on technical details of h-harmonics. As a result of the discussion, the first four chapters of the book are more or less in shape and substantial progresses are made on Chapters 5 and 6.

2. The Riesz transforms on the sphere can be defined by $R_{i,j}f := D_{i,j}(-\Delta_0)^{-\frac{1}{2}}f$. Making use of an operator defined by a smooth cut-off function, this operator is shown to be weak type $(1,1)$ and, hence, bounded in $L^p(S^{d-1})$ for $1 < p < \infty$.

3. Based on the discussions on the intertwining operators, Wang has established a theorem on Cesàro means on the sphere for the weight function $w_\kappa(x)$ associated with the group $\mathcal{G} := G \times \{e\}$, where G is any finite reflection group and w_κ is invariant under $\mathcal{G}$. Let $S_n^\delta(w_\kappa; f)$ denote the Cesàro means of the h-harmonic expansions associated with w_κ. The new result states that

   (1) For all $1 \le p \le \infty$,

   $$\sup_{n\in\mathbb{N}} \|S_n^\delta(w_\kappa; f)\|_{\kappa,p} \le c\|f\|_{\kappa,p} \text{ for all } f \in L^p(w_\kappa; S^{d-1})$$

   if and only if $\delta > \lambda_\kappa := \frac{d-1}{2} + \sum_v \kappa_v$.
   (2) Let $1 < p < \infty$ satisfy $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{2\lambda_\kappa+2}$. If

   $$\sup_{n\in\mathbb{N}} \|S_n^\delta(w_\kappa; f)\|_{\kappa,p} \le c\|f\|_{\kappa,p} \text{ for all } f \in L^p(w_\kappa; S^{d-1}),$$

   then
   $$\delta > \lambda_\kappa := \max\left\{(2\lambda_\kappa + 1)\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right\}.$$

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