Geometric flows in mathematics and physics

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1 Overview of the Field

Few, if any, fields of mathematics have experienced the success that geometric analysis since it emerged as a field of its own about 35 years ago, having formed at the nexus of PDEs, Riemannian geometry, and related fields such as Kähler geometry, general relativity, and applied mathematics. Soon after, the study of geometric flow equations commenced in earnest and has since become one of the most fruitful areas of geometric analysis. Examples of problems that have been solved through the study of geometric flows include the Poincaré conjecture, the Thurston geometrization conjecture, the Penrose conjecture, the differentiable sphere theorem, and the n-dimensional Rauch-Hamilton spherical space forms conjecture.

Geometric analysis has always owed much to physics. For example, the positive energy conjecture of general relativity stimulated Schoen and Yau to produce their geometric anaylsis proof. Conversely, physics has gained much from geometric analysis, an example of this being provided by Yau's proof of the Calabi conjecture, making possible the use of Calabi-Yau manifolds in string theory phenomenology. It is therefore natural for physicists to search for ways to exploit the mathematical progress made in geometric flow equations, and natural as well for mathematicians to use physics as a source for new problems to solve using geomtric flows. There have been a series of meetings in recent years, typically consisting mostly of mathematicians together with a smaller number of physicists, who gather together to discuss progress in the field and possble new applications. One such meeting was a workshop held at BIRS in 2008. The meeting was highly successful, so this motivated us to hold another BIRS workshop on the topic in 2011.

In the last three years, new themes have emerged in the subject. Traditionally, Ricci flow has been applied to Riemannian metrics on closed manifolds (i.e., compact manifolds that have no boundary). This endeavour has continued to have new successes, a recent one being the Brendle-Schoen proof of the differentiable sphere theorem. However, Ricci flow on noncompact manifolds is becoming more important. A large group in France, gathered about Gérard Besson at Grenoble, has been working on this, hoping to apply the results to the classification problem for open 3-manifolds. Furthermore, if one imposes that the metric should obey asymptotic fall-off conditions of certain types (prime examples being asymptotically flat metrics and conformally compactifiable metrics), then the Ricci flow of these metrics has applications to physics. The asymptotically flat problem was studied a few years ago by Oliynyk-Woolgar and Dai-Ma, but the conformally compactifiable (or asymptotically hyperbolic) problem has been studied only much more recently by Bahuaud, by Shi and collaborators, and numerically by

Figueras-Pau-Wiseman. The Ricci flow on manifolds-with-boundary is proving to be an even more difficult problem. We have a satisfactory theory only when the boundary is totally geodesic, in which case the manifold can be "doubled", reducing the problem to the compact manifold case.

Numerics are emerging as an area of interest for geometric flows. Mean curvature flow has many practical applications which have driven numerical work using this flow. However, for Ricci flow, there has not yet been much numerical work, btu that can be expected to change. On the last day of the workshop, a discussion session was held on the topic of numerical Ricci flows. Two questions come to mind: What are the Ricci flow problems that will motivate numerical investigations? And how do we overcome the computational challenges that these problems would pose?

2 Presentation Highlights

Peter Topping: Instantaneously complete Ricci flows

Peter Topping spoke on joint work with G Giesen on Ricci flows which are complete for all $t \in (0, T)$ and $x \in \mathbb{R}^2$, but are such that either curvature becomes unbounded as $t \searrow 0$ or the metric becomes incomplete when extended back to $t = 0$. There is a general existence theorem for this flow for "initial data" (in the above sense) given on a Riemann surface, with the flowing metric being complete at all positive times. Topping was also able to give a formula for the maximal existence time of the flow in terms of the volume of the initial data surface. If the initial metric is conformally equivalent to a complete hyperbolic metric H, then the flow actually C^0-converges to H at least as fast as $1/t$.

Gérard Besson: Natural maps, differential rigidity, and Ricci and scalar curvature.

In this talk, Besson posed the question of when a degree 1 map from one closed smooth n-manifold Y to another X is homotopic to a diffeomorphism. He disucssed a result of Bessières, Besson, Courtois, and Gallot that when g is a metric on Y with $\mathrm{Ric}(g) \geq -(n-1)g$ and g_0 is a hyperbolic metric on X, then there is an ϵ depending only on the dimension and diameter of X such that f is homotopic to a diffeomorphsim whenever $\mathrm{vol}(X, g_0) \leq \mathrm{vol}(Y, g) \leq (1+\epsilon)\mathrm{vol}(X, g_0)$. The proof works by constructing "natural maps" $Y_k \to X$ that converge to an isometry, so that X is the Gromov-Hausdorff limit of a sequence of manifolds obeying a lower bound on Ricci curvature. Thus X and Y_k are diffeomorphic (for large k) by a theorem of Cheeger and Colding.

Philippe G LeFloch: Weakly regular T^2-symmetric spacetimes.

This talk discussed a weak formulation of the Cauchy initial value problems for the spacetime Einstein equations $\mathrm{Ric} = 0$. The problem is to find weak solutions from rough initial data exhibiting T^2-symmetry. LeFloch argues that the initial data constraints and Einstein evolution equations for the Cauch y problem in the weak T^2-symmetric setting can be formulated in the sense of distributions and solved accordingly.

Gerhard Dziuk: Computation of geometric flows

This talk concerned numerical methods for geometric PDEs on surfaces, including moving surfaces, and on applications. One begins by discretizing the surface, for example by approximating the surface by a piecewise polynomial surface which can then be described by n-simplices defined by a grid of a certain spacing h (e.g., for piecewise linear approximation, use polynomials of degree 1). Then one can develop a discrete version of the mean curvature vector. The most simple approach fails to give a good approximation to the true mean curvature vector and to the Willmore energy of a 2-sphere in $\mathbb{R}^3$, but this can be greatly improved through the use of Ritz projection. The speaker gave explicit numerical examples of evolution of certain surfaces under this discrete Willmore flow. The talk ended with some discussion of a discretization of Ricci curvature, based on a suggestion of G Huisken, which may prove suitable for discrete Ricci flow in higher dimensions and without symmetry.

Felix Schulze: On short time existence of the network flow.

This was a report on joint work of the speaker with T. Ilmanen and A. Neves on how to prove the existence of an embedded, regular network moving by curve shortening flow in the plane, starting from a nonregular initial network.

Here a regular network consists of smooth, embedded line segments such that

at each endpoint, if not infinity, there are three arcs meeting under a 120 degree angle. In the nonregular case, an arbitrary number of line segments are allowed to meet at an endpoint, without an angle condition. The proof relies on gluing in appropriately scaled self-similarly expanding solutions and a new monotonicity formula, together with a local regularity result for such evolving networks.

This short time existence result also has applications in extending such a flow of networks through singularities.

Reto Müller: Central blow-ups of Ricci flow singularities.

Scaling invariance properties of the heat equation motivated Hamilton to presume that singularities of the Ricci flow should be modeled by nontrivial gradient shrinking solitons. Perelman's W-entropy monotonicity strengthens this conjecture, showing that for any central blow-up sequence the shrinking soliton equation is approached in a weighted L^2-sense. Moreover, a similar (but more hidden) indication follows from the monotonicity of Perelman's reduced volume functional.

After explaining these motivations, the speaker demonstrated how Hamilton's conjecture can be proved in the case of Type I Ricci flows. This was joint work of the speaker with Joerg Enders and Peter Topping. Then he described possible extensions (in low dimensions) to the general case and presented partial results obtained with collaborators Robert Haslhofer and Carlo Mantegazza.

Eric Bahuaud: Ricci flow of smooth asymptotically hyperbolic metrics.

This was the first of two talks on the interesting topic of Ricci flow of asymptotically hyperbolic (i.e., conformally compactifiable) metrics. The speaker described his proof that, given a smoothly conformally compact metric, there is a short time solution to the Ricci flow evolving from the given metric that remains smoothly conformally compact. Similar results were also obtained in independent work reported by Shi below. The speaker applied recent results of Schnürer, Schulze, and Simon to prove a stability result for conformally compact metrics sufficiently close to the hyperbolic metric.

Yuguang Shi: Normalized Ricci flow on asymptotically hyperbolic manifolds.

The speaker described his recent joint work with Jie Qing and his PhD student Jie Wu. They investigated the behaviour of normalized Ricci flow whose initial metric is epsilon-Einstein on a complete, noncompact Riemannian manifold. They show that a under nondegeneracy condition on the initial metric the solution to the normalized Ricci flow exists for all time and converges exponentially fast to some Einstein metric. In accord with Bahuaud's result described above, they were able to show that if the initial metric can be conformally compactified then the conformal structure of the boundary-at-infinity is preserved by normalized Ricci flow. Furthermore, the limit is a conformally compact Einstein metric.

Robert Haslhofer: A mass decreasing flow in dimension three.

After reviewing the relationship between Perelman's energy-functional, the stability of Ricci-flat spaces, and the ADM mass from general relativity, the speaker introduced a mass-decreasing flow for asymptotically flat three-manifolds with nonnegative scalar curvature. This flow is defined by iterating a suitable Ricci flow with surgery and conformal rescalings and has a number of nice properties. In particular, wormholes pinch off and nontrivial spherical space forms bubble off in finite time. Moreover, a noncompact variant of Perelman's energy-functional is monotone along the flow. The long time behaviour seems to be quite delicate to analyze, but the speaker conjectured that the flow always squeezes out all the initial mass. He can prove this under an *a priori* assumption relating the ADM-mass and the Perelman energy.

Artem Pulemotov: A mass decreasing flow in dimension three.

A very important issue for Ricci flow is the choice of appropriate boundary conditions for the flow on compact manifolds-with-boundary. To be useful, such boundary conditions would have to produce some degree of control over the flow of curvature through a boundary. However, (short time) existence and well-posedness ought to require that the boundary conditions should be not worse than first differential order in the metric, so any control of curvature through the boundary would have to follow rather indirectly from conditions on, say, the induced metric and extrinsic curvature. One can attempt the DeTurck approach, in which case a natural boundary condition would be to set the DeTurck vector field to be either zero or perhaps just tangent to the boundary. However, it is not entirely clear that this condition will yield existence (and, though the speaker did not raise it in the talk, it is not at all clear that it would then produce a useful degree of curvature control).

The speaker instead attempts to work directly with the Ricci flow without DeTurck vector field. Then the number and type of boundary conditions that one can impose is unclear, since the principal symbol of the operator that defines this flow has nontrivial kernel. The speaker is able to show existence of solutions under the boundary condition that the mean curvature be held constant. The question of what additional conditions to impose to obtain uniqueness and well-posedness remains open.

The Ricci flow with boundary has many applications. Among them, the speaker noted the regularization of Riemannina metrics and the numerical work on phase transitions in quantum gravity of Headrick and Wiseman.

Joakim Arnlind: Poisson algebraic geometry of Kähler submanifolds.

A Poisson algebra is an algebra with skew bracket that obeys the Jacobi identity and the identity $\{ab, c\} = a\{b, c\} + \{a, c\}b$. This talk described joint work of the speaker with J Hoppe and G Huisken. They want to know which geometric quantities can be written as algebraic expressions in the Poisson algebra of functions on an embedded surface. The answer, apparently, is "almost everything". They studied that the differential geometry of almost Kähler submanifolds and formulated this in terms of the Poisson structure induced by the inverse of the Kähler form. More precisely, the submanifold relations, such as Gauss' and Weingarten's equations (as well as many other objects), can be expressed as Poisson brackets of the embedding coordinates. It is then natural to ask the following question: Are there abstract Poisson algebras for which such equations hold? They answer this question by introducing Kähler-Poisson algebras and show that an affine connection can be defined, fulfilling all the desired symmetries and relations; e.g., the Bianchi identities. Furthermore, as an illustration of the new concepts they are able to derive algebraic versions of some well known theorems in differential geometry. In particular, they prove that Schur's lemma holds and that a lower bound on the Ricci curvature induces a bound on the eigenvalues of the Laplace operator. It will be very interesting to see how far this approach can be pushed.

Jingyi Chen: Lagrangian mean curvature flow for entire graphs.

Chen reported on his joint work with Albert Chau and Yu Yuan. They prove longtime existence and estimates for solutions to a fully nonlinear parabolic equation $\frac{\partial u}{\partial t} = \sum_i \arctan \lambda_i$, where the λ_i are the eigenvalues of the Hessian $\nabla\nabla u$, for $C^{1,1}$ initial data u_0 subject to one of two conditions. These conditions are that either u_0 is convex or the Hessian of u_0 obeys $-(1+\delta)I \leq \nabla\nabla u_0 \leq (1+\delta)I$ for some δ that depends on the dimension. They also show that a supercritical condition on the Lagrangian phase is preserved under evolution by this equation.

Maria Buzano: Homogeneous Ricci flow.

Let (M, g) be a Riemannian manifold of dimension n and let $g(t)$, $t \in [0, T)$ be a Ricci flow on M with initial metric g. An important property of the Ricci flow is that it preserves symmetries of the initial metric g. It is then natural to investigate the Ricci flow on Riemannian manifolds which admit a transitive action by a closed group G of isometries. These are called homogeneous Riemannian manifolds. They are diffeomorphic to quotients G/K, where K is the isotropy group of a point in M with respect to the G-action. On this kind of space, we can restrict attention to those Riemannian metrics which are invariant under the action of G. This invariance is then preserved under the Ricci flow.

The speaker considers the Ricci flow on compact homogeneous spaces of the following type. Let G/K be a compact homogeneous space, with G compact and connected and K a closed connected subgroup of G such that G/K is effective. Suppose that there exists an intermediate Lie group H, with $G > H > K$, such that H/K is isotropy irreducible and every G-invariant Riemannian metric on G/K is obtained from a fixed Riemannian submersion $H/K \to G/K \to G/H$, by

rescaling the metric on the fibre and the base. In particular, these include the

case of compact homogeneous spaces G/K with two isotropy irreducible inequivalent summands and such that K is not maximal in G. Such homogeneous spaces are interesting because they include many examples of compact homogeneous

spaces which do not admit any invariant Einstein metric. The lowest dimensional

non-existence example is the 12-dimensional manifold $\mathrm{SU}(4) = \mathrm{SU}(2)$ found by Wang and Ziller. Subsequently, Böhm and Kerr have shown that

this is the least dimensional example of a compact homogeneous space which

does not carry any G-invariant Einstein metric. Note that this example is of the type described above.

More non-existence examples can be obtained from the following fact. On a compact manifold, Einstein metrics are characterised variationally as the critical points of the Hilbert action. Hence, G/K does not carry any G-invariant Einstein metric if the traceless Ricci tensor restricted to one of the isotypical summands of the isotropy representation is positive definite for all G-invariant Riemannian metrics. By a theorem of Böhm, this implies that G/K is of the type described above, yielding many new examples of compact homogeneous spaces which do not carry any invariant Einstein metric. On the homogeneous spaces considered here, the speaker is able to show that the Ricci flow always develops a type I singularity in finite time. In particular, two possible behaviours can occur. Either the whole space shrinks to a point in finite time or the fibre $H = K$ in the above Riemannian submersion shrinks to a point in

finite time and the total space G/K converges in the Hausdorff-Gromov topology

to the base space $G = H$. Moreover, the first behaviour can happen only if

$G = K$ admits G-invariant Einstein metrics. However, there is always an open

subset of initial conditions such that the Ricci flow develops the second kind of

behaviour. It would be interesting to rule out type II singularities in the homogeneous case and investigate the Ricci flow in the case where K is a

maximal subgroup of G, at least in the two isotropy summands case.

Brett Kotschwar: Ricci flow and the holonomy group.

The speaker described his proof that the restricted holonomy group of a complete smooth solution to the Ricci flow of uniformly bounded curvature cannot spontaneously contract in finite time. It follows then from an earlier result of Hamilton that the holonomy group is exactly preserved by the equation. In particular, a solution to Ricci flow may be Kähler or locally reducible (as a product) at some time in the evolution only if the same is true at all previous times. The problem can be reduced to one of backwards uniqueness for a certain system satisfying coupled parabolic and ordinary differential inequalities via the interpretation of the evolution equations in terms of the natural Lie bracket on two-forms. The backwards uniqueness of this system then follows from a earlier general result of the speaker. As the (Carleman-type) estimates responsible for this result measure — and, in principle, limit — the rate at which the curvature operator can asymptotically "acquire" null directions, there is hope that these estimates (or improvements thereof) may have application in future work on analysis up to and including the singular time.

Fuquan Fang: Ricci flow on 4-manifolds and Seiberg-Witten equations.

The speaker described his work with Zhang and Zhang over the last several years on the Ricci flow of both closed and noncompact 4-manifolds. The idea is to try to classify the asymptotic behaviour of immortal (fixed-volume normalized) Ricci flows on such manifolds, as has been done for 3-manifolds. Working with the normalized flow on a closed 4-manifold, the speaker and his collaborators have shown that the flow either collapses in the Cheeger-Gromov sense, subconverges to a Ricci soliton, or subconverges in the pointed Gromov-Hausdorff sense to a finite collection of complete (noncompact) negative Einstein metrics. They further prove that closed 4-manifolds admitting an immortal normalized Ricci flow either have a positive rank F-structure, admit a shrinking Ricci soliton, or satisfy the Hitchin-Thorpe inequality $2\chi(M) \geq 3|\tau(M)|$, for $\chi(M)$ the Euler invariant of M and $\tau(M)$ the signature. If the Yamabe invariant is $\sigma(M) \leq 0$ this inequality can be strengthened to $2\chi(M) - 3|\tau(M)| \geq \frac{1}{96\pi^2}\sigma^2(M)$. A corollary is that the Euler characteristic is ≥ 0 and vanishes iff the solution collapses along a subsequence of times.

The collaborators can find similar results by assuming that the Seiberg-Witten equations can be solved on the 4-manifold, giving a nontrivial Seiberg-Witten invariant (which counts generic irreducible solutions of these equations). If this invariant is nonzero and if $|\mathrm{Ric}[g(t)]| < const$ along the flow, then they obtain $\chi(M) \geq 3\tau(M)$.

The speaker also considered the case of finite-volume noncompact manifolds. These can also be studied with the fixed-volume normalized Ricci flow. Immortal solutions then either collapse along a subsequence or subconverge to a collection of complete negative Einstein manifolds; the shrinking solitons never occur.

Miles Simon: Expanding solitons with non-negative curvature operator coming out of cones.

The speaker described joint work that he did with Felix Schulze. They show that a Ricci flow of any complete Riemannian manifold without boundary with bounded non-negative curvature operator and asymptotic volume ratio $\lim_{r\to\infty} \frac{\mathrm{vol}(M^n)}{r^n} \neq 0$ exists for all time and has constant asymptotic volume ratio.

They show that there is a limit solution, obtained by scaling down this solution at a fixed point in space, which is an expanding soliton coming out of the asymptotic cone at spatial infinity.

Tobias Lamm: Geometric flows with rough initial data.

Given a flow problem $\frac{\partial u}{\partial t} = F(x, t, u, Du, D^2u)$, the speaker asks the question, "What are the right assumptions on the initial data $u_0(x) = u(x, 0)$ to get well-posedness?" He is able to show the existence of a global unique and analytic solution for the mean curvature flow and the Willmore flow of entire graphs for Lipschitz initial data with small Lipschitz norm. He also shows the existence of a global unique and analytic solution to the Ricci-DeTurck flow on Euclidean space for bounded initial metrics which are close to the Euclidean metric in L^∞ and to the harmonic map flow for initial maps whose image is contained in a small geodesic ball.

Sylvain Maillot: Ricci flow with surgery and applications to open 3-manifolds.

The speaker discussed his joint work with L Bessières and G Besson, some of which is still in progress. Applications of Ricci flow with surgery include Perelman's proof of Thurston's geometrization conjecture and Coda Marques's result on connectedness of spaces of metrics of positive scalar curvature. The speaker discussed extensions of this work to noncompact manifolds. One goal is a Ricci flow proof of the geometrization conjecture in the noncompact, finite volume case, a result originally proved by Thurston using other methods. The finite volume restriction allows the use of the fixed-volume normalized Ricci flow. This would provided a unified approach to geometrization of closed manifolds and of compact manifolds with non-empty boundary comprised of a union of torii.

The speaker posed a conjecture that would generalize a result of Codá Marques for closed manifolds: that for a noncompact manifold M with sifficiently simple topology, the space of complete metrics on M with scalar curvature having a positive minimum should be path-connected.

3 Numerical Ricci Flow Discussion Group

The Friday morning session was given over to a group discussion of numerical geometric flows. The main idea was to find mathematical problems that could be addressed by numerical Ricci flow, to anticipate the technical issues that might arise, and to begin to define an approach.

The session was surprisingly well attended, considering it was informal and was on the last morning of the workshop. At least two thirds of the participants were in attendance.

One of the problems suggested to the group is the long-standing problem of constructing an explicit nontrivial metric on a Calabi-Yau manifold. This requires real dimension 4, but is most interesting in real dimension 6, so it is something of a challenge numerically (though there has been some work in dimension 4 already). In a more general setting, Ricci flow might not converge to an Einstein metric, for the same reason that applies to any "relaxation method". Namely, the linearized flow is governed in this case by the Lichnerowicz Laplacian of the Einstein metric, which may contain a negative (i.e., repulsive) mode. However, an advantage is that Kähler Ricci flow is known to converge exponentially to a Kähler-Einstein metric.

Maria Buzano proposed a different problem, which is the study of possible Type II singularities in the flow of manifold of low cohomogeneity that are not rotationally symmetric. Then the computer no longer has to deal with a high-dimensional spatial grid. Gerhard Dziuk pointed out that because the problem involves evolution close to a singularity, during the flow the time step will have to be reduced near the region of singularity formation as the singularity forms Multi-grid methods will have to be employed. The question arose as to whether there might be a computationally efficient way to do this. Miles Simon pointed out the Perelman's pseudolocality theorem gives a technique to predict the change in curvature over a spatial region during for an interval of time. This may be just the tool necessary to allow for the development of an efficient multi-grid code to study singularities in Ricci flow.