Analytic versus Combinatorial in Free Probability

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1 Overview of the Field

Free probability theory is an area of research which parallels aspects of classical probability, in a non-commutative context where tensor products are replaced by free products, and independent random variables are replaced by free random variables. It grew out of attempts to solve some longstanding problems about von Neumann algebras of free groups. In the almost thirty years since its creation, free probability has become a subject in its own right, with connections to several other parts of mathematics: operator algebras, the theory of random matrices, classical probability, the theory of large deviations, and algebraic combinatorics. Free probability also has connections with some mathematical models in theoretical physics and quantum information theory, as well as applications in statistics and wireless communications.

There exist two different approaches to free probability theory at a very basic level; one is analytic and the other one is combinatorial. These approaches complement each other, and in many situations it is the interaction between both of them which drives the subject forward.

A main theme of the workshop was the discussion of a number of important extensions of free probability that were studied during the recent years, and continue to provide an intense active topic of research at present. In the basic theory of free probability both the combinatorial and the analytic structure, as well as the interaction between them, are quite well understood. However, in these recent extensions of the setting of free probability, the development of these features is not yet clear; progress on these subjects will surely rely on an interaction between analytic and combinatorial considerations.

In the following we present (not exhaustively) several such developments that were covered by the workshop. Most of the sections below are based on write-ups made by workshop participants who work in those directions, and we thank Octavio Arizmendi, Serban Belinschi, Camille Male, Paul Skoufranis, and Moritz Weber for their contributions to this report.

2 Recent Developments and Open Problems

2.1 Bi-freeness and pairs of faces

Although the simplest way to define free independence is via the "alternating centred moments vanish" condition, the connection between free probability and operator algebras is most easily seen by defining free independence as the ability to represent algebras on reduced free product spaces using the left regular representation. As there is also a right regular representation on reduced free product spaces, in [21] Voiculescu introduced a generalization of free independence, known as bi-free independence, which is defined as the ability to represent pairs of algebras on reduced free product spaces where one algebra from each pair acts via the left regular representation and the other acts via the right regular representation. This ability to simultaneously study the left and right regular representations allows for a wider variety of behaviours to be observed and modelled thereby permitting bi-free probability to investigate problems untouched by free probability.

In Voiculescu's inception of bi-free probability [21, 23] it was demonstrated that bi-free independence is well-defined, bi-free probability has a notion of cumulants, and there exists a bi-free partial R-transform. Due to the difficulty of working with reduced free product spaces, Mastnak and Nica in [17] began the process of introducing combinatorial techniques into bi-free probability. The combinatorial approach to bi-free probability reached fruition in [7, 8] where Charlesworth, Nelson, and Skoufranis demonstrated that lattices consisting of permutations of non-crossing partitions produced the correct cumulants to characterize bi-free independence. Since free independence is characterized using non-crossing partitions, many of the combinatorial results from free probability directly had bi-free analogues.

The influx of combinatorial techniques has allowed for a rapid expansion of the theory of bi-free probability, with a flurry of activity going on throughout 2015 and 2016. Here are some recent contributions made in this direction.

- Skoufranis demonstrated that all five natural notions of independence (classical, free, Boolean, monotone, and anti-monotone) can be studied through bi-free independence.
- Freslon and Weber developed bi-free de Finetti theorems to study bi-free independence over tail algebras.
- Gu, Huang, and Mingo developed the theory of bi-free infinitely divisible distributions for bi-partite systems, which was then generalized to arbitrary systems by Gao.
- Skoufranis developed the bi-free analogue of random matrix models by demonstrating that several pairs of matrices tend to bi-free independent distributions and by characterizing the notion of bi-freely independent pairs amalgamated over diagonal matrices.

An overview of combinatorial techniques in bi-free probability was presented at the workshop by Paul Skoufranis, with ample illustrations drawn from his recent paper [18]. These combinatorial techniques will very likely continue to be important in further developing the theory of bi-free probability.

There have also been many advances via analytical techniques.

- Voiculescu developed the bi-free partial S-transform, with Skoufranis later discovering a combinatorial proof.
- Voiculescu also developed the notion of bi-free extremes in the plane, thereby computing the distributions of the maximum and minimum of bi-partite bi-free pairs.
- Dykema and Na obtained the principal function of non-normal bi-free central limit distributions.
- Huang and Wang developed analytical aspects of the bi-free partial R-transform.

At the workshop, the opening talk was given by Dan Voiculescu on his work [22] on bi-free extremes. Another talk on the analytic side of bi-free probability was given by Jiun-Chau Wang, who reported on recent joint work with Hasebe and Huang on analytical aspects of additive and multiplicative bi-free convolution.

Further developments of analytical techniques in bi-free probability are ongoing, and are likely to lead to many interesting avenues of inquiry.

One-sided free probability was known to have intimate connections to other brands of non-commutative probability, and two such connections were extended to the bi-free framework in recent work by Gu and Skoufranis. One of these connections is with the theory of "conditional free independence" (upgraded by Gu and Skoufranis to "conditional bi-free independence"), which is a notion of independence with respect to pairs of states on an algebra of random variables; the framework of conditional bi-free independence widens even more the range of behaviours that can be displayed by two-faced systems of noncommutative random variables, and offers many new examples that are now possible to study. The other connection, which was the topic of the talk given by Yinzheng Gu at the workshop, concerns the relation of free probability to Boolean probability – now upgraded to bi-Boolean probability.

Finally, we mention the interesting point that (unlike the development of one-sided free probability, where the whole theory started from the "alternating centred moments vanish" condition), a description of bi-free indpendence in terms of moments was only found very recently by Ian Charlesworth, who reported on this in his talk at the workshop. It is still unknown to what extent the moment condition of Charlesworth extends to the operator valued setting. Nevertheless, this condition promises to offer a new approach, in future developments, for proving that pairs of algebras are bi-freely independent.

Bi-free probability is a rapidly expanding area of free probability. As there are many interesting questions and potential applications to this theory, bi-free independence will be an active area of research in free probability for the foreseeable future.

2.2 Finite Free Probability

The theory of finite free (polynomial) convolutions is very recent. It started with the paper by Marcus, Spielman, and Srivastava [16], where they established a connection between different polynomial convolutions and addition and multiplication of random matrices, which in the limit is related to free probability. The new feature of this convolution is that instead of looking at distributions of eigenvalues of random matrices, one looks at the (expected) characteristic polynomial of a random matrix.

To be precise, for a matrix M, let $\chi_M(x) = \det(xI - M)$ be the characteristic polynomial of the matrix M. Then, for $d \times d$ Hermitian matrices A and B with characteristic polynomials p and q, respectively, one defines the finite free additive convolution of p and q to be

$$p(x) \boxplus_d q(x) = \mathbf{E}_Q[\chi_{A+QBQ^T}(x)],$$

where the expectation is taken over orthogonal matrices Q sampled according to the Haar measure.

Similarly, when A and B are positive semidefinite, the finite free multiplicative convolution of p and q is defined to be

$$p(x) \boxtimes_d q(x) = \mathbf{E}_Q[\chi_{AQBQ^T}(x)],$$

where, again, the expectation is taken over random orthogonal matrices Q sampled according to the Haar measure.

Both these convolutions turn out to not depend on the specific choice of A and B, but only on p and q.

The connection with free probability is that, because of the concentration of measure phenomenon for random matrices, as $d \to \infty$, these polynomial convolutions approximate free additive convolution and free multiplicative convolution. This connection is quite remarkable since, as proved in [16], these convolutions have appeared before and there are very explicit formulas to calculate the coefficients of the finite free convolutions of two polynomials.

The original motivation in [16] was to obtain new inequalities between polynomials by using free probabilistic tools. In [16] they proved the following inequalities between the R-transform and the S-transform of the resulting polynomials:

$$R_{p \boxplus_d q}(w) \leq R_p(w) + R_q(w), \qquad S_{p \boxtimes_d q}(w) \leq S_p(w) S_q(w).$$

These inequalities are translated into inequalities of polynomials and were used, in [15], to prove the existence of bipartite Ramanujan graphs on an arbitrary number of vertices, improving on results from [14].

There is not yet much literature on finite free convolution. Indeed, apart from the original work of Marcus, Spielman and Srivastava [16], there are presently only two more papers [1, 13] on the subject.

The purpose of [13] is to make precise the above mentioned connections to free probability. In this paper, the analytic machinery of finite free additive convolution and finite free multiplicative convolution were introduced, in the sense that finite versions of Voiculescu's R transform and S-transform which "linearize" finite convolutions were found. To be precise, there are transforms R^d and S^d such that

$$R^d_{p \boxplus_d q}(w) = R^d_p(w) + R^d_q(w), \qquad S_{p \boxtimes_d q}(w) = S^d_p(w) S^d_q(w).$$

In particular, using the transform R^d, the law of large numbers, central limit theorems and law of rare events have been established for finite free additive convolution. It was an important highlight of the workshop to have Adam Marcus as a speaker presenting these exciting new developments; his presentation prompted many private discussions around further possibilities of continuing this line of research.

On the other hand in [1] the authors gave a combinatorial treatment of finite free additive convolution by introducing cumulants for finite free additive convolution and by deriving moment-cumulant formulas. These cumulants approximate free cumulants, when the degree of the polynomials tends to infinity. This allowed them in [1]

- to give criteria for infinite divisibility;
- to show that there exist $T > 1$ such that for all $t > T$, the polynomial $p^{\boxplus_d t}$ is well defined and real rooted;
- to prove the existence of a counterexample for Cramer's Theorem.

Such examples of properties, that were known before for free additive convolution and that are now shown to already appear in the level of finite free additive convolution, may lead to a better insight on free convolution. These ideas were presented in the talk given at the workshop by Octavio Arizmendi.

There are many open problems and interesting directions related to this topic, which will surely be the subject of future research. Some of them are:

- Combinatorial aspects of finite free multiplicative convolution.
- Free entropy and free Fisher information in the finite setting.
- Extensions of finite free probability, like freeness with amalgamation or second order freeness.
- Multivariate finite convolution or a notion of finite independence.

More specific problems that should serve as leading guides for understanding differences and similarities between finite free convolution and free convolution are the following.

- Give qualitative and quantitative aspects of the central limit theorem in finite free additive convolution, such as superconvergence or a Berry-Esseen Theorem.
- Describe properties of free convolution with "gaussian law", γ_d, appearing in the central limit theorem. In particular it is expected that γ_d has no multiple roots.
- Find the minimum $T > 1$ such that for all $t > T$, the polynomial $p^{\boxplus_d t}$ is well defined and real rooted.
- Describe the multiplicity of the roots of $p \boxplus_d q$ in terms of the multiplicity of the roots of p and of q.
- Improve the relation between the R-transform and finite free additive convolution as follows: for $k \in \mathbb{N}$ we expect to have
$$R^d_{p \boxplus_d q}(w) \leq R^{kd}_{p^k}(w) + R^{kd}_{q^k}(w),$$
and similarly for the S-transform and the finite free multiplicative convolution.

2.3 Spectral theory for large random matrices.

One of the most important results in free probability is the asymptotic freeness of large unitarily invariant random matrices, obtained by Voiculescu in his 1991 paper [20]. The result roughly states that if $\{A_1(N), \ldots, A_j(N)\}$ are properly normalized $N \times N$ independent Gaussian random matrices or independent Haar distributed unitary random matrices and $(D_1(N), \ldots, D_k(N))$ is a k-tuple of deterministic $N \times N$ diagonal matrices which converges in distribution to a k-tuple $(d_1, \ldots, d_k)$, then there are noncommutative random variables $a_1, \ldots, a_j$ such that $\{A_1(N), \ldots, A_j(N), (D_1(N), \ldots, D_k(N))\}$ converges in distribution as N tends to infinity to $\{a_1, \ldots, a_j, (d_1, \ldots, d_k)\}$, and the sets $\{a_1\}, \ldots, \{a_j\}, \{d_1, \ldots, d_k\}$ are free. This result has been strengthened numerous times by, among others, Voiculescu, Speicher, Mingo-Speicher, Haagerup-Thorbjørnsen, Male, Collins-Male. One obvious candidate for improvement in the above result is the quality of convergence. The most spectacular first result in this direction is due to Haagerup and

Thorbjørnsen [9]: they show that if $\{A_1(N),\ldots,A_j(N)\}$ are properly normalized independent Gaussian random matrices and $P = P^* \in \mathbb{C}\langle X_1,\ldots,X_j\rangle$ is a selfadjoint polynomial in j non-commuting selfadjoint indeterminates, then $\|P(A_1(N),\ldots,A_j(N))\|$ converges almost surely to $\|P(a_1,\ldots,a_j)\|$, where $a_1,\ldots,a_j$ are free identically distributed semicircular random variables. This kind of convergence is referred to as *strong convergence*, and the strong convergence to a free family of random variables as *strong asymptotic freeness*. Male improved this result to include strongly converging deterministic matrices: if the k-tuple $(D_1(N),\ldots,D_k(N))$ of deterministic selfadjoint $N\times N$ matrices (not necessarily diagonal) converges *strongly* to $(d_1,\ldots,d_k)$ as $N\to\infty$, then so does the family $\{A_1(N),\ldots,A_j(N),(D_1(N),\ldots,D_k(N))\}$ to the family $\{a_1,\ldots,a_j,(d_1,\ldots,d_k)\}$. A further extension to include independent Haar unitary random matrices has been obtained in joint work by Collins and Male.

The recent paper [3], presented by Mireille Capitaine at the workshop, extends the above result of Male to Wigner matrices, i.e. selfadjoint random matrices X whose entries satisfy the conditions that X_{ii}, $1\le i\le N$, $\sqrt{2}\Re X_{il}$, $\sqrt{2}\Im X_{il}$, $1\le i<l\le N$, are all centered, independent, identically distributed, and of variance $1/N$. Gaussian random matrices are a particular case of Wigner matrices. The main result of [3] can be stated as follows.

Theorem: Assume that $\{A_1(N),\ldots,A_j(N)\}$ are independent $N\times N$ Wigner matrices whose entries have finite fourth moments. Let $D_1(N),\ldots,D_k(N)$ be deterministic $N\times N$ matrices and assume that $\{D_1(N),\ldots,D_k(N),D_1(N)^*,\ldots,D_k(N)^*\}$ converges strongly to a $2k$-tuple of bounded random variables. Then $\{A_1(N)\},\ldots,\{A_j(N)\}$, $\{D_1(N),\ldots,D_k(N),D_1(N)^*,\ldots,D_k(N)^*\}$ are strongly asymptotically free.

The requirement of finite fourth moments can be replaced by a slightly weaker, but more technical condition. The proof involves a fairly broad array of methods and tools coming from both classical and free probability, matrix theory and analytic noncommutative functions theory. The main steps involve a truncation of the entries of the Wigner matrices (idea due to Bai-Yin and Bai-Silverstein), a linearization trick that reduces the study of noncommutative polynomials of arbitrary degree and complex coefficients to the study of linear polynomials with coefficients that are $n\times n$ selfadjoint matrices for a fixed n depending on the polynomial (this idea was first introduced in free probability by the above-mentioned work of Haagerup and Thorbjørnsen [9], but was already well-known in other fields) and an application of some analytic properties of Voiculescu's operator-valued subordination functions.

As byproducts of the proof of the main result, a spectral inclusion property is also obtained. It can be roughly outlined as follows: assume that $\{a_1,\ldots,a_j\}$ are free semicircular random variables and the k-tuple $(d_1(N),\ldots,d_k(N))$ is free from $\{a_1,\ldots,a_j\}$ for all $N\in\mathbb{N}$. Assume also that $(d_1(N),\ldots,d_k(N))$ has the same distribution as the constant selfadjoint matrices $D_1(N),\ldots,D_k(N)\in M_N(\mathbb{C})$. If $P=P^*\in\mathbb{C}\langle X_1,\ldots,X_{j+k}\rangle$ is such that for all N sufficiently large the spectrum of $P(d_1(N),\ldots,d_k(N),a_1,\ldots,a_j)$ does not intersect the interval $[b,c]$, then for any $\delta>0$, the spectrum of $P(D_1(N),\ldots,D_k(N),A_1(N),\ldots,A_j(N))$ does not intersect $[b+\delta,c-\delta]$ almost surely as $N\to\infty$. This yields also a characterization of the outliers generated by spikes of the constant matrices.

In this context, it is worth to point out that it is a quite surprising, and also very exciting, realization in the last couple of years (see, eg, [4]) that free probability is able to address also asymptotic properties of special single eigenvalues; this is surely a direction which will be followed up in the future.

2.4 Traffics and their independence

Random matrices are important in free probability since canonical models of random matrices are free in the large dimension limit. The most basic of these results is that a collection of independent Wigner matrices converges to free semicircular variables, see [20]. Nevertheless, the notion of non-commutative distribution is sometimes too restrictive to treat certain models of large matrices. For any $N\ge1$ let $X_{\ell,N}$, $\ell=1,\ldots,L$, be independent symmetric random matrices with i.i.d. sub-diagonal entries distributed according to the Bernoulli distribution with parameter $\frac{c_\ell}{N}$, $c_\ell>0$. Then $(X_{\ell,N})_{\ell=1,\ldots,L}$ converges toward non-free random variables [12]. If moreover Y_N is a sequence of deterministic matrices, then the possible limiting distributions of $X_{\ell,N}$ and Y_N depend on more than the limiting non-commutative distribution of Y_N. A similar problem appears when considering matrices $Z_{\ell,N}=V_{\ell,N}A_{\ell,N}V_{\ell,N}^*$, $\ell=1,\ldots L$, where the matrices $V_{\ell,N}$ are independent random permutation matrices uniformly chosen. When $A_{\ell,N}$ are diagonal, so is $Z_{\ell,N}$ and so $(Z_{\ell,N})_{\ell=1,\ldots,L}$

cannot be asymptotically free.

It was in order to remedy this point that Male [11] introduced another non-commutative framework: the traffic spaces, which come with an associated notion of distribution and of independence. Traffics are non-commutative variables, with additional structure, given by a generalization of polynomials called graph polynomials. In particular, given a collection A_ℓ, $\ell = 1, \dots, L$, of N by N random matrices, the traffic distribution of $\mathbf{A}_N = (A_\ell)_{\ell=1,\dots,L}$ encodes the data, for any finite connected graph $T = (V, E)$ and any map γ from E to $\{1, \dots, L\}$, of the observable

$$\tau_{\mathbf{A}_N}(T, \gamma) = \mathbb{E}\Big[\frac{1}{N} \sum_{\phi: V \to \{1,\dots,N\}} \prod_{e=(v,w)\in E} A_{\gamma(e)}\big(\phi(w), \phi(v)\big)\Big].$$

For instance let T be a simple cycle, with edges $e_p = (v_p, v_{p+1})$, $p = 1, \dots, n$ where $v_{n+1} = v_1$ and $v_1, \dots, v_n$ are pairwise disjoint. Then $\tau_{\mathbf{A}_N}(T, \gamma)$ is the normalized trace $\mathbb{E}\,\frac{1}{N}\mathrm{Tr}$ of the monomial $A_{\gamma(1)} \cdots A_{\gamma(n)}$. For another example, if T denotes a graph consisting of two simple cycles that are connected at one vertex, then $\tau_{\mathbf{A}_N}(T, \gamma)$ is the normalized trace of the the entry-wise product of monomials. The convergence in traffic distribution of $\mathbf{A}_N$ is the pointwise convergence of $\tau_{\mathbf{A}_N}$.

Traffic distributions come together with a notion of independence, which is more complicated to introduce since it involves non-algebraic (combinatorial) formulas. However, this notion of independence allows one to describe uniquely the joint traffic distribution of several families of variables, in terms of the separate distributions of those families. Traffic-independent variables can be freely independent, but in general we obtain a different relation.

Traffic independence applies to a large class of random matrices:

- Independent matrices with i.i.d. entries that converge in non-commutative distribution (for instance when the entries are truncated heavy tailed variables) are asymptotically traffic-independent [12]. Their limiting distribution is characterized by a kind of Schwinger-Dyson system of equations.

- Traffic is the good notion at the intersection of free probability and random graph theory. For sparse random graphs (when the degree of a generic vertex is bounded), the traffic convergence of the adjacency matrix of a graph is equivalent to the so-called local weak convergence of Benjamini-Schramm [11]. Independent adjacency matrices are asymptotically traffic-independent, but not asymptotically freely independent in general.

- A class of large graphs with large degree was studied in joint work of Male and Peché. In particular adjacency matrices of independent regular graphs uniformly chosen with growing degree are asymptotically freely independent.

- In 2015, Gabriel introduced the concepts of $\mathcal{P}$-algebras and of $\mathcal{P}$-independence, which slightly generalized traffic independence, and he defined a notion of cumulant for traffic distributions. His work implies a result of convergence for the additive and multiplicative matricial Lévy processes invariant in law by conjugation by permutation matrices, and then for random walk on the symmetric group.

- Cébron, Dahlqvist and Male studied in [5] canonical constructions associated to traffic spaces. In particular they proved the positivity of the free product of distributions. They also proved that every non-commutative probability space can be endowed with a structure of traffic space. Benson Au and Camille Male are presently studying the non-commutative structure of this canonical construction; both of them presented their current work in talks at the workshop.

2.5 Free quantum groups

The study of symmetries has always been a central topic in the history of mathematics. Since about two hundred years ago, symmetries are mostly modelled by actions of groups. However, modern mathematics requires an extension of the symmetry concept to highly non-commutative situations. This was the birth of quantum groups in the 1980's due to the pioneering work of Drinfeld and Jimbo in the purely algebraic setting, and Woronowicz in the topological one. The latter one is more relevant for this workshop. His approach to quantum groups is based on the concept of "non-commutative function algebras" by Gelfand-Naimark, using C^*-algebras as underlying algebras. Main features of compact quantum groups are:

- Every compact group is a compact quantum group, but the converse is not true; hence compact quantum groups are honest generalizations of compact groups.
- Every compact quantum group possesses a Haar state. This is an analogue of the well-known result that every compact group admits a Haar measure. Hence, Haar integration is possible both on compact groups as well as on compact quantum groups.
- We have a quantum version of Schur-Weyl or Tannaka-Krein duality. Given a compact matrix quantum group, its space of finite-dimensional unitary representations is a tensor category of a certain kind; conversely, to any such tensor category we may find a universal compact matrix quantum group whose representation theory is exactly given by this tensor category.

In the past few decades, the investigation of compact quantum groups has developed into a rapidly growing field of mathematics with many links to other domains. The link to free probability is mainly given by compact matrix quantum groups of combinatorial type. Their construction relies on the Tannaka-Krein duality. The main idea is to identify a certain set of combinatorial objects equipped with operations resembling those of a tensor category; then to associate a tensor category to it; and then to obtain a compact quantum group by the Tannaka-Krein Theorem. The examples of combinatorial type discussed in the workshop were the following.

1. The so called Banica-Speicher quantum groups (also called easy quantum groups), were introduced in 2009 in [2]. Given a partition p of the set $\{1, \ldots, k+l\}$ into disjoint subsets (called blocks), we may associate a linear map $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ to it by sending a basis vector $e_{i_1} \otimes \cdots \otimes e_{i_k}$ to the sum over all $e_{i_{k+1}} \otimes \cdots \otimes e_{i_{k+l}}$ such that the indices $(i_1, \ldots, i_{k+l})$ match with the partition p in a suitable way (i.e. $i_s = i_t$, if s and t are in the same block of p). If a set of partitions is closed under certain natural operations, the linear span of the associated maps T_p forms a tensor category in Woronowicz's sense, and we obtain a compact matrix quantum group, a so called Banica-Speicher quantum group. The definition of Banica and Speicher (requiring self-adjoint entries of the matrices) has been extended by Tarrago and Weber to unitary easy quantum groups (non-selfadjoint case).

   Easy quantum groups are linked to free probability theory by de Finetti theorems: Köstler and Speicher proved in [10] a characterization of free independence by distributional invariance under the quantum symmetric group S_n^+. This is parallel to the classical de Finetti theorem, where independence is characterized by distributional invariance under the symmetric group S_n. This result has been extended also to other quantum groups, see also the case of Boolean independence.

2. Spatial partition quantum groups were introduced by Cébron and Weber in 2016 in [6]. The idea is similar to the one of Banica-Speicher quantum groups, but the underlying objects are three-dimensional partitions rather than two-dimensional ones. This allows for finding new examples of quantum subgroups of Wang's free orthogonal quantum group; there are new kinds of products of quantum groups coming from new products of categories of partitions; and there is a quantum group interpretation of certain categories of partitions which do neither contain the pair partition nor the identity partition.

3. Partially commutative quantum groups have been introduced by Speicher and Weber in 2016 in [19]. They fit with the mixtures of classical and free independence by Młotkowski and Speicher-Wysoczański. The main idea of partial commutation is the following. Given a symmetric matrix $\varepsilon = (\varepsilon_{ij})_{i,j=1}^n$ with $\varepsilon_{ij} \in \{0, 1\}$ and $\varepsilon_{ii} = 0$, two coordinates x_i and x_j shall commute if and only if $\varepsilon_{ij} = 1$. Imposing such partial commutation relations has a long history in various contexts, for example, on the level of groups partial commutation relations have been studied extensively under names such as "right angled Artin groups", "free partially commutative groups", or "graph groups"; on the level of monoids "Cartier-Foata monoids", "trace monoids" are common notions; there is also the general notion of a "graph product of groups" introduced by Green in the 1990's; recently, a corresponding version of a graph product for von Neumann algebras was introduced and investigated by Caspers and others.

   A quantum probabilistic version of the idea of imposing partial commutation relations is given by the notion of Λ-freeness. This concept was defined by Młotkowski in 2004 and revived and refined by Wysoczański and Speicher in 2016. This mixture of classical and free independences goes as follows. Let $\varepsilon = (\varepsilon_{ij})$ be a symmetric matrix as above. If variables $x_1, \ldots, x_n$ are ε-independent, then:

- x_i and x_j are free in the case $\varepsilon_{ij} = 0$
- x_i and x_j are independent in the case $\varepsilon_{ij} = 1$ (in particular, $x_i x_j = x_j x_i$ in this situation).

If all entries of ε are zero, we obtain Voiculescu's free independence; if all non-diagonal entries of ε are one, we obtain classical independence. The notion of ε-independence arises naturally in the context of graph products of groups by considering subgroups with respect to the canonical trace state.

The partially commutative quantum groups defined by Speicher and Weber in [19] relate to Λ-freeness in a de Finetti sense, hence characterizing the distributional symmetries related to Λ-freeness. However, the study of these quantum groups reveals many other aspects. For instance:

- they act maximally on noncommutative spheres with partial commutation relations for the coordinates;
- they provide new quantum versions of the orthogonal group;
- they contain Bichon's quantum automorphism groups of graphs as quantum versions of the symmetric group.

A survey of the ideas mentioned above was presented in the talk given at the workshop by Moritz Weber. Some further recent developments concerning Λ-freeness were presented by Frédéric Patras, who reported a recent work on this topic done jointly with Ebrahimi-Fard and Speicher.

2.6 New developments around older concepts

2.6.1 Second order freeness

Second order freeness was introduced by Mingo and Speicher around 2003, and has had some impact on describing global fluctuations of random matrices. Second order freeness should be seen as a refinement of the question about global eigenvalue distributions of random matrices. (The later can be described by the usual, or first order, free probability theory.) Whereas we have by now a quite advanced and satisfactory combinatorial theory of second order freeness (by Collins, Mingo, Sniady, Speicher), we have no good grasp on positivity problems in this context. In particular, we do not know a satisfying answer to the relevant "moment problem" here, i.e., to the question which fluctuations can really arise in a random matrix context. Building an analytic theory of the Cauchy transforms for this theory will be an important step. As a long term goal one also hopes to find an operator-valued version of second order freeness, which would be of relevance for dealing with fluctuations of more general classes of random matrices. A first promising step in this direction was reported at the workshop by Mario Diaz.

2.6.2 Free stochastic analysis

Free stochastic analysis, the foundation of which was laid quite early in the development of free probability, has become a very active area in recent years. On one hand, it can be seen as the large N limit of stochastic analysis on $N \times N$ matrices – and many new results on making this rigorous for various quantities have been derived by Cébron, Kemp, and Ulrich, following the basic work of Biane from the 90s. On the other hand, working directly with the free version provides powerful tools via stochastic differential equations to investigate the structure of operator algebras which are given as the large N limit of random matrix models. We mention here just recent break-throughs by Guionnet and Shlyakhtenko and by Dabrowski, which resulted, amongst others, in the solution of the 20 year old problem on the isomorphism of the q-deformed free group factors. In another direction, work of Kemp, Nourdin, Peccati, and Speicher transferred theorems from classical stochastic analysis to the free setting, thus giving criteria for the convergence of a sequence of variables, under some constraints (for example living in a fixed chaos with respect to free Brownian motion) to specific limits, like the semicircular distribution. These results rely on a mixture of diagrammatic and analytic considerations. However, quantitative versions of these convergence results are not yet well understood and we are, for example, in need of a better understanding of adequate notions of distances between non-commutative distributions. Refinements of the technical tools that are being used (like free Malliavin calculus) will also be instrumental for making more precise statements about the regularity of the involved non-commutative distributions. First progress on such questions, namely the absence of atoms for various distributions, was achieved recently by Shlyakhtenko and Skoufranis and by Mai, Speicher, and Weber. Another interesting progress on such questions, on improving the free logarithmic Sobolev inequality, was reported at the workshop by Brent Nelson.

3 Outcome of the Meeting

The topics highlighted above (which cover not all, but a substantial part of the workshop programme) indicate very clearly that free probability theory, having reached a quite mature status in some of its original directions, is rejuvenating itself with the emergence of new problems, ideas, and concepts.

Free probability is a very active area, with many unsolved problems ahead, as well as various recent new exciting developments. This meeting brought together various mathematical backgrounds – in particular, analytic and combinatorial – with an emphasis on the connections. The meeting was very timely and useful, and will surely have a strong impact on the further developments of free probability and related subjects.

The organisers were in particular pleased to observe the large number of young people interested in the subject; many of them have already made substantial contributions (as witnessed by the large number of talks given by junior researchers) and will surely continue to advance the subject.

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