Complex Analysis and Complex Geometry

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1 Overview of the Field

Complex analysis and complex geometry form synergy through the geometric ideas used in analysis and analytic tools employed in geometry, and therefore they should be viewed as two aspects of the same subject. The fundamental objects of the theory are complex manifolds and, more generally, complex spaces, holomorphic functions on them, and holomorphic maps between them. Holomorphic functions can be defined in three equivalent ways as complex-differentiable functions, convergent power series, and as solutions of the homogeneous Cauchy-Riemann equation. The threefold nature of differentiability over the complex numbers gives complex analysis its distinctive character and is the ultimate reason why it is linked to so many areas of mathematics.

Plurisubharmonic functions are not as well known to nonexperts as holomorphic functions. They were first explicitly defined in the 1940s, but they had already appeared in attempts to geometrically describe domains of holomorphy at the very beginning of several complex variables in the first decade of the 20th century. Since the 1960s, one of their most important roles has been as weights in a priori estimates for solving the Cauchy-Riemann equation. They are intimately related to the complex Monge-Ampère equation, the second partial differential equation of complex analysis. There is also a potential-theoretic aspect to plurisubharmonic functions, which is the subject of pluripotential theory.

In the early decades of the modern era of the subject, from the 1940s into the 1970s, the notion of a complex space took shape and the geometry of analytic varieties and holomorphic maps was developed. Also, three approaches to solving the Cauchy-Riemann equations were discovered and applied. First came a sheaf-theoretic approach in the 1950s, making heavy use of homological algebra. Hilbert space methods appeared in the early 1960s and integral formulas around 1970 through interaction with partial differential equations and harmonic analysis. The complex Monge-Ampère equation came to the fore in the late 1970s with Yau's solution of the Calabi conjectures and Bedford and Taylor's work on the Dirichlet problem.

Today, as before, complex analysis and complex geometry is a highly interdisciplinary field. The foundational work described above has been followed by a broad range of research at the interfaces with a number of other areas, such as algebraic geometry, differential geometry, dynamical systems, functional analysis, homotopy theory, partial differential equations, and symplectic geometry. Complex analysts and complex geometers share a common toolkit, but find inspiration and open problems in many areas of mathematics.

2 Recent Developments and Open Problems

2.1 Holomorphic foliations

In recent years the theory of holomorphic foliations has become a central topic in geometric complex analysis and complex geometry. An important problem which has attracted the attention of several researchers concerns the existence of Levi-flat hypersurfaces in complex manifolds. Despite many attempts, the problem of the existence of a real analytic compact Levi-flat hypersurface in the complex projective plane still remains open. In higher dimensions this is a well-known result of Lins Neto [19].

Many interesting questions concern the local geometry of Levi-flat hypersurfaces near singular points. D. Burns and X. Gong [5] obtained a local classification of Levi-flat real analytic hypersurfaces near a Morse-type singularity. It is an open problem to obtain a similar classification for a wider class of singularities. Shafikov and Sukhov [22] proved that near nondicritical singularities, the Levi foliation extends as a singular holomorphic web to a neighbourhood of the singular point in the ambient space. Many interesting results and questions concerning the structure of Levi-degeneracy sets of CR manifolds are contained in the work of D. Cerveau and A. Lins Neto [6].

Another aspect of the theory involves the relationship between holomorphic foliations and the complex Monge-Ampére equation. The classical work of E. Bedford and M. Kalka in the late 1970s has had important applications in the recent work of Lempert and L. Vivas [18], and of Y. Rubinstein and S. Zelditch [21]. Lempert's approach (*Bull. Soc. Math. France,* 1981) to the construction of solutions of the complex Monge-Ampère equation was further developed by S. Donaldson [9], and more recently by G. Patrizio and A. Spiro [20], and in the almost complex setting by H. Gaussier and J. Joo [12]. More advanced analytic tools are used by V. Tosatti and B. Weinkove [23].

A series of papers by H. B. Lawson and F. R. Harvey (starting with *Amer. J. Math.,* 2009) is devoted to a geometric approach to the fundamental notion of plurisubharmonicity; they prove deep results on the Monge-Ampère equation on almost complex manifolds. This approach to plurisubharmonicity has already yielded a number of interesting applications, for example, Forstnerič and Drinovec Drnovšek in their presentations at the workshop gave a characterization of the minimal hull of compact sets in $\mathbb{R}^n$.

A very recent and highly active direction of research concerns the ergodic theory of holomorphic foliations, developed by T.-C. Dinh, V. Nguyên, and N. Sibony [8], and by C. Dupont and B. Deroin [7]. They have produced new and original constructions of invariant measures and Green currents associated to holomorphic foliations by analogy with discrete dynamical systems. This promising direction contains various natural and attractive questions. The interaction between holomorphic foliations and pluripotential theory is fundamental and will be a focus of activity in the coming years.

2.2 Elliptic complex geometry and Oka theory

Elliptic complex geometry is concerned with the flexible analytic geometry of complex affine spaces $\mathbb{C}^n$ and similar manifolds, opposite to the rigidity that characterizes hyperbolic complex manifolds. Three important classes of similar manifolds, in order of increasing size, are Stein manifolds with the density property, elliptic manifolds as defined by M. Gromov in a seminal paper of 1989, and Oka manifolds that have only recently emerged from the developments inspired by Gromov's paper.

The group of holomorphic automorphisms of $\mathbb{C}^n$, $n \geq 2$, is an infinite-dimensional group with a very rich structure. It has been intensively studied since a groundbreaking paper of E. Andersén and L. Lempert in 1992. Their work has been extended to Stein manifolds with the density property and applied to a range of embedding problems, in particular to the long-standing conjecture that every open Riemann surface can be embedded into $\mathbb{C}^2$. The flexibility of Oka manifolds is manifested in a tight connection between homotopy theory and complex analysis that has brought D. Quillen's theory of model categories into analysis for the first time in the work of F. Lárusson.

F. Forstnerič formally introduced Oka manifolds and Oka maps in 2009 and 2010, respectively, following his proof that more than a dozen Oka properties that had been under investigation for several years are all equivalent. A 500-page monograph of his, published in 2011 in Springer's Ergebnisse series, gives the first systematic exposition of much of the work that has been done in elliptic complex geometry and Oka theory over the past 25 years.

Recent work in elliptic complex geometry and Oka theory is varied and makes contact with several areas of mathematics. As the foundations of the area become more established, the focus of activity has shifted to applications and interaction with other areas.

Forstnerič and A. Alarcón have successfully applied concepts and methods from Oka theory to the theory of minimal surfaces: see for example [1]. In brand new work [11], Forstnerič and Lárusson have used convex integration theory and the theory of absolute neighbourhood retracts to prove a strong parametric h-principle for conformal minimal immersions and holomorphic null curves from a finitely-connected source. Among open problems in this area is how to treat more general sources; for those, Forstnerič and Lárusson proved a weak parametric h-principle.

F. Kutzschebauch and S. Kaliman continue to pursue a vigorous program of research into algebraic aspects of elliptic complex geometry, particularly the so-called algebraic density property. In very recent work [14], they have produced an effective sufficient condition for the algebraic volume density property and used it to show that certain homogeneous spaces satisfy the condition. The density property is very special, so new examples are of great interest. Kutzschebauch, R. Andrist and P.-M. Poloni have shown that certain Gizatullin surfaces have the algebraic density property [2]. Challenging problems remain open in this area, for example as simple a question as whether $(\mathbb{C}^*)^n$, $n \geq 2$, has the density property.

R. Lärkäng and Lárusson have made the first study of Oka theory for singular targets and in the process made a connection with the theory of affine toric varieties [17]. Their work shows that fundamental results of standard Oka theory break down when the targets are allowed to have mild singularities. To what extent a general Oka theory for singular targets can be developed is a challenging open question.

2.3 Holomorphic mappings

The theory of holomorphic mappings lies at the heart of the development of several complex variables in the second half of the twentieth century. In view of the classical result of Poincaré that the ball in $\mathbb{C}^n$, $n > 1$, is not biholomorphically equivalent to the polydisc, it is a fundamental problem to describe necessary and sufficient conditions for the equivalence of domains and the interaction between the equivalence of domains and their boundaries. Through attempts to better understand these questions, many important tools in complex analysis have been developed and sharpened, including metrics invariant under holomorphic maps, the theory of the Bergman projection and the $\overline{\partial}$-Neumann operator, the higher dimensional reflection principle, and Chern-Moser theory of local invariants of real hypersurfaces, to name a few.

In the spirit of the classical work of Bedford and Pinchuk in the nineties, K. Verma [24] generalizes the description of domains with noncompact automorphism groups in $\mathbb{C}^2$. It remains an open problem to classify domains with noncompact automorphism groups for a more general class of domains, say, pseudoconvex domains of D'Angelo finite type in $\mathbb{C}^n$. Several talks at the workshop directly or indirectly referred to this question.

I. Kossovkiy and R. Shafikov [16] recently gave an example of a family of nonminimal hypersurfaces in $\mathbb{C}^n$ which at a Levi degenerate point are formally CR equivalent but not biholomorphically equivalent, thus disproving a conjecture of Baouendi and Rothschild. Kossovskiy and B. Lamel [15] then constructed examples of smooth CR equivalences between Levi-degenerate hypersurfaces which are not analytic.

One of the open questions in CR geometry is the following: does there exist a compact, strictly pseudoconvex CR manifold in $\mathbb{C}^2$ that does not have any umbilical points? In his talk, P. Ebenfelt introduced a new approach to this problem, using a higher order version of Fefferman's Monge-Ampère operator (joint work with A. Zaitsev [10]).

2.4 Pluripotential theory and its applications

Pluripotential theory involves the study of plurisubharmonic functions on complex spaces. The theory of the complex Monge-Ampère operator on classes of plurisubharmonic functions on domains in $\mathbb{C}^n$ initiated by E. Bedford and B. A. Taylor was further developed by Z. Błocki, U. Cegrell, S. Kołodziej, and others. V. Guedj and A. Zeriahi [13] developed the notion of an ω-plurisubharmonic function on a compact Kähler manifold with a Kähler form ω. More recently, S. Boucksom, P. Eyssidieux, Guedj, and Zeriahi [4], advancing the interplay between pluripotential theory and complex geometry, laid down the foundations of pluripotential theory on compact complex manifolds in a general big cohomology class. A variational approach to solving degenerate Monge-Ampère equations in a big cohomology class on a compact Kähler manifold was developed by Berman, Boucksom, Guedj, and Zeriahi [3]. There are applications to topics as diverse as Kähler-Einstein metrics, Arakelov geometry, and equidistribution of zero sets of random sections of holomorphic line bundles.

Eyssidieux, Guedj, and Zeriahi developed an alternative approach to degenerate Monge-Ampére equations in terms of viscosity solutions, and compared viscosity concepts with pluripotential-theoretic ones. The purpose of Zeriahi's talk was to develop a viscosity theory for degenerate complex Monge-Ampère flows on compact Kähler manifolds. This was motivated by studying the time-asymptotic behavior of the Kähler-Ricci flow on mildly singular varieties. Their general theory allows them to generalize results of Song and Tian. Continuing on this geometric theme, Blocki discussed optimal regularity of geodesics in the space of Kähler metrics of a compact Kähler manifold (and on the space of volume forms on a compact Riemannian manifold) as they turn out to be solutions of homogeneous complex Monge-Ampère equations.

With respect to probabilistic results, Bayraktar presented several universality principles related to asymptotic zero distribution of random polynomials (as the degree goes to infinity), and, more generally, for random holomorphic sections of high powers $L^{\otimes n}$ of positive line bundles $L \to X$ for X a projective manifold endowed with a continuous metric. Roughly speaking, under natural assumptions, the asymptotic distribution is independent of the choice of probability law defining the random polynomials or sections.

As already mentioned, Forstnerič and Drinovec Drnovšek gave a characterization of the minimal hull of compact sets K in a minimally convex domain $D \subset \mathbb{R}^n$. This characterization was in the spirit of Poletsky's characterization of the polynomial hull of compact sets in $\mathbb{C}^n$: p belongs to the minimal hull of K with respect to D if and only if one can find a conformal minimal disk in D centered at p with most of its boundary near K. Poletsky raised the question of whether one could construct examples in the spirit of Stolzenberg-Wermer of a K with nontrivial minimal hull but with no minimal disks in its hull. The Poletsky theory relies on the fact that a plurisubharmonic function on a domain D is subharmonic on analytic disks in D. Poletsky's talk aimed at a deeper understanding of the space of analytic disks mapping into D.

3 Presentation Highlights

The presentations given at the workshop can be divided into several themes: geometry of real submanifolds (Drinovec Drnovsěk, Forstnerič, Gupta), geometry and dynamics of holomorphic maps (Arosio, Ebenfelt, Kossovskiy), automorphism groups (Andrist, Kaliman), function theory on complex manifolds (Bayraktar, Brudnyi, Chakrabarti, Kinzebulatov, Shcherbina), analytic discs (Bertrand, Poletsky), geometry of complex manifolds (Lärkäng, Merker, Prezelj, Wulcan), invariant metrics (Gaussier, Zimmer), Kähler metrics and Monge-Ampère equation (Błocki, Zeriahi). In addition, Bedford and Berteloot gave excellent survey talks that described mathematical ideas developed by Sergey Pinchuk who was honoured at this conference.

Speaker: **Rafael B. Andrist** (Bergische Universität Wuppertal)
Title: *The density property for Gizatullin surfaces of type* $[[0, 0, -r_2, -r_3]]$
Abstract: I will give a brief introduction to the density property for Stein manifolds, which is a notion to express that a manifold has "many" holomorphic automorphisms.

Although large classes of Stein manifolds with the density property are known, e.g. most of the homogeneous spaces of Stein Lie groups, they include only very few surfaces, namely $\mathbb{C}^2$, $\mathbb{C} \times \mathbb{C}^*$ and the smooth Danielewski surfaces. The lack of examples of such surfaces is due to the absence of the so-called "compatible pairs" of complete vector fields, which are usually the main tool for proving the density property.

Smooth Gizatullin surfaces provide a good class of candidates for surfaces with the density property, and they include the examples mentioned above. The next natural step is the investigation of Gizatullin surfaces of type $[[0, 0, -r_2, -r_3]],\ r_2, r_3 \geq 2$, which can be described by the equations

$$\begin{cases} yu &= xP(x) \\ xv &= uQ(u) \\ yv &= P(x)Q(u) \end{cases}$$

in $\mathbb{C}^4$ with coordinates (x, y, u, v), where P and Q are polynomials of degree $r_2 - 1$ resp. $r_3 - 1$. We establish the density property for smooth Gizatullin surfaces of this type and describe a dense subgroup of the identity components of their holomorphic automorphism groups.

Joint work with Frank Kutzschebauch and Pierre-Marie Poloni.

Speaker: **Leandro Arosio** (Università di Roma “Tor Vergata”)
Title: *Models for holomorphic self-maps of the unit ball*
Abstract: In order to study the forward or backward iteration of a holomorphic self-map f of a complex manifold X, it is natural to search for a semi-conjugacy of f with some automorphism of a complex manifold. Examples of this approach are given by the Schroeder, Valiron and Abel equation in the unit disc D. Given a holomorphic self-map f of the ball B^q, we show that it is canonically semi-conjugate to an automorphism (called a canonical model) of a possibly lower dimensional ball B^k, and this semi-conjugacy satisfies a universal property. This approach unifies in a common framework recent works of Bracci, Gentili, Poggi-Corradini, Ostapyuk.

This is done performing a time-dependent conjugacy of the autonomous dynamical system defined by f, obtaining in this way a non-autonomous dynamical system admitting a relatively compact forward (resp. backward) orbit, and then proving the existence of a natural complex structure on a suitable quotient of the direct limit (resp. subset of the inverse limit). As a corollary we prove the existence of a holomorphic solution with values in the upper half-plane of the Valiron equation for a hyperbolic holomorphic self-map of B^q.

Speaker: **Turgay Bayraktar** (Syracuse University)
Title: *Universality principles for random polynomials*
Abstract: In this talk, I will present several universality principles concerned with zero distribution of random polynomials or more generally random holomorphic sections of high powers $L^{\otimes n}$ of positive line bundle $L \to X$ defined over a projective manifold endowed with a continuous metric. In one direction, universality phenomenon indicates that under natural assumptions, asymptotic distribution of (appropriately normalized) zeros of random polynomials is independent of the choice of probability law defined on random polynomials. Another form of universality is asymptotic normality of smooth linear statistics of zero currents. Finally, if time permits, I will also describe some recent results on universality of scaling limits of correlations between simultaneous zeros of random polynomials.

Speaker: **Eric Bedford** (Stony Brook University)
Title: *From the Edge-of-the-wedge theorem to the Scaling method*

Speaker: **Francois Berteloot** (Université de Toulouse)
Title: *Rescaling methods in complex analysis*
Abstract: Rescaling methods are very efficient in complex analysis or geometry because they can be combined with the theory of normal families. We will survey some typical examples of such methods and in particular those introduced by Sergey Pinchuk.

Speaker: **Florian Bertrand** (American University of Beirut)
Title: *Riemann-Hilbert problems with singularities*
Abstract: The study of analytic discs attached to a totally real submanifold M of $\mathbb{C}^n$ leads to the consideration of a regular Riemann-Hilbert problem of a special form. Following this approach, Forstneric, and later on Globevnik, characterized the existence and dimension of a family of deformations of a given analytic disc attached to M in terms of certain indices. However, in case M admits some complex tangencies, the indices mentioned above are no longer well-defined and the Forstneric-Globevnik method falls apart. In this talk, I will focus on a class of such singular Riemann-Hilbert problems. We will see that they can be solved by a factorization technique that reduces them to regular Riemann-Hilbert problems with geometric constraints. In particular, we will deduce the existence of stationary type discs attached to finite type hypersurfaces.

Speaker: **Zbigniew Błocki** (Jagiellonian University)
Title: *Geodesics in the space of Kähler metrics and volume forms*
Abstract: We discuss optimal regularity of geodesics in the space of Kähler metrics of a compact Kähler manifold, as well as the space of volume forms on a compact Riemannian manifold. They are solutions

of nonlinear degenerate elliptic equations: homogeneous complex Monge-Ampère equation and Nahm's equation (introduced by Donaldson), respectively. The highest regularity one can expect is $C^{1,1}$.

Speaker: **Alexander Brudnyi** (University of Calgary)
Title: *On the Sundberg approximation theorem*
Abstract: Let H^∞ be the Banach algebra of bounded holomorphic functions on the open unit disk $D \subset \mathbb{C}$. We extend Sundbergs theorem on uniform approximation of functions in BMOA by H^∞ functions to other classes of holomorphic functions on D. In our proofs we use a new characterization of meromorphic functions on D that extend to continuous maps of the maximal ideal space of H^∞ to the Riemann sphere.

Speaker: **Debraj Chakrabarti** (Central Michigan University)
Title: *L^2-cohomology of annuli and Sobolev estimates for the $\overline{\partial}$-problem*
Abstract: We consider the question of L^2-estimates for the $\overline{\partial}$-problem on annuli, a simple but interesting class of non-pseudoconvex domains. We relate this question with W^1-Sobolev estimates on the "hole" of the annulus. We then consider special classes of non-smooth holes for which the questions can be answered. This is joint work with Mei-Chi Shaw and Christine Laurent-Thibaut.

Speaker: **Barbara Drinovec Drnovšek** (University of Ljubljana)
Title: *Minimal hulls and minimally convex domains*
Abstract: Minimal hulls and minimally convex domains were introduced in a series of papers by Harvey and Lawson. They are natural substitutes for polynomial hulls and strictly pseudoconvex domains in the context of minimal surface theory. We present a characterization of the minimal hull of a compact set K in $\mathbb{R}^n$ by sequences of conformal minimal discs whose boundaries converge to K in the measure theoretic sense. We also study some properties of minimally convex domains. This is a report on a joint work with Alarcón, Forstnerič and López.

Speaker: **Peter Ebenfelt** (UCSD)
Title: *Stable umbilical points on perturbations of the sphere in $\mathbb{C}^2$*
Abstract: The standard CR structure on the three dimensional sphere can be deformed in such a way that the deformed structures have no (CR) umbilical points. A 1-parameter family of such deformations was essentially discovered by E. Cartan (and later studied by Cap, Isaev, Jacobowitz). The CR manifolds in this family, however, cannot be embedded in $\mathbb{C}^2$. It is an open question whether the unit sphere can be perturbed in $\mathbb{C}^2$ such that no umbilical points remain on the perturbed CR manifolds. In this talk, we shall discuss an approach to this problem, and describe some recent results. One of the results that will be described guarantees stable (in a sense to be made precise in the talk) umbilical points on generic perturbations of almost circular type. This complements a previous result by the speaker and Son Duong on existence of umbilical points on circular three-dimensional CR manifolds.

Speaker: **Franc Forstnerič** (University of Ljubljana)
Title: *The parametric h-principle for minimal surfaces in $\mathbb{R}^n$ and null curves in $\mathbb{C}^n$*
Abstract: Let M be an open Riemann surface. It was proved by Alarcón and Forstnerič that every conformal minimal immersion $M \to \mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve $M \to \mathbb{C}^3$. We prove the following substantially stronger result in this direction: for any $n \geq 3$, the inclusion of the space of real parts of nonflat null holomorphic immersions $M \to \mathbb{C}^n$ into the space of nonflat conformal minimal immersions $M \to \mathbb{R}^n$ satisfies the parametric h-principle with approximation; in particular, it is a weak homotopy equivalence. Analogous results hold for several other related maps. For an open Riemann surface M of finite topological type, we obtain optimal results by showing that the above inclusion and several related maps are inclusions of strong deformation retracts; in particular, they are homotopy equivalences. (Joint work with Finnur Lárusson.)

Speaker: **Hervé Gaussier** (Université Grenoble Alpes)
Title: *Prime ends theory in higher dimension*
Abstract: This is a joint work with Filippo Bracci. We try to extend the Carathéodory prime ends theory in higher dimension, defining the "horosphere boundary" of complete hyperbolic (in the sense of Kobayashi)

manifolds. We prove that a strongly pseudoconvex domain together with its horosphere boundary, endowed with the horosphere topology, is homeomorphic to its Euclidean closure, whereas the horosphere boundary of a polydisc is not even Hausdorff. As an application we study the boundary behaviour of univalent mappings.

Speaker: **Purvi Gupta** (University of Western Ontario)
Title: *Rational density on compact real manifolds*
Abstract: Motivated by the observation that every continuous complex-valued function on the unit circle can be approximated by rational combinations of a single function, we will discuss some conditions under which a manifold M admits N functions whose rational combinations are dense in the space of complex-valued C^k-functions on M. As a result, we will produce an optimal bound on N in terms of the dimension of M. This is joint work with R. Shafikov.

Speaker: **Shulim Kaliman** (University of Miami)
Title: *Algebraic (volume) density property*
Abstract: Let X be a connected affine homogenous space of a linear algebraic group G over $\mathbb{C}$. (1) If X is different from a line or a torus we show that the space of all algebraic vector fields on X coincides with the Lie algebra generated by complete algebraic vector fields on X. (2) Suppose that X has a G-invariant volume form ω. We prove that the space of all divergence-free (with respect to ω) algebraic vector fields on X coincides with the Lie algebra generated by divergence-free complete algebraic vector fields on X (including the cases when X is a line or a torus).

Speaker: **Damir Kinzebulatov** (Indiana University)
Title: *Chern classes of singular metrics on vector bundles*
Abstract: We extend the basic sheaf-theoretic techniques of complex function theory to work within some algebras of holomorphic functions (joint with Alex Brudnyi)

Speaker: **Ilya Kossovskiy** (University of Santa Catharina, Brazil)
Title: *Borel theorem for CR-maps*
Abstract: Following Henri Poincare, numerous results in Dynamics establish the curious phenomenon saying that two smooth objects (e.g., vector fields), which can be transformed into each other by means of a formal power series transformation, can be also transformed into each other by a smooth map. This is a kind of analogue of Borel Theorem on smooth realizations of formal power series. In CR-geometry, similar phenomena hold for real-analytic CR-manifolds, and the usual outcome is that two formally equivalent CR-manifolds are also equivalent holomorphically. However, in our recent work with Shafikov we proved that there exist real-analytic CR-manifolds, which are equivalent formally, but still not holomorphically.

On the other hand, in our more recent work with Lamel and Stolovitch we prove that the following is true: if two 3-dimensional real-analytic CR-manifolds are equivalent formally, then they are C^∞ CR-equivalent. In this talk, I will outline the latter result.

Speaker: **Richard Lärkäng** (Bergische Universität Wuppertal)
Title: *Chern classes of singular metrics on vector bundles*
Abstract: For holomorphic line bundles, it has turned out to be useful to not just consider smooth metrics, but also singular metrics which are not necessarily smooth, and which can degenerate. In relation to vanishing theorems and other properties of the line bundle, one considers plurisubharmonicity properties of the possibly singular metric which correspond to notions of positivity for the line bundle. In particular, having a positive singular metric means that the first Chern form associated to the metric is a closed positive $(1,1)$-current.

More recently, singular metrics on holomorphic vector bundles have been considered, Griffiths positivity of a singular metric on a vector bundle is defined in terms of plurisubharmonicity. For a vector bundle with a Griffiths positive singular metric, there is a naturally defined first Chern class which is a closed positive $(1,1)$-current, but there are examples where the full curvature matrix is not of order 0. I will discuss joint work with Hossein Raufi, Jean Ruppenthal and Martin Sera, where we show that one can give a natural meaning to the k:th Chern form $c_k(h)$ of a singular Griffiths positive metric h as a closed (k,k)-current of order 0, as long as h is non-degenerate outside a subvariety of codimension at least k. The proof builds on pluripotential theory, and in particular, one consider in the spirit of Bedford-Taylor products like $(dd^c\varphi)^q \wedge T$, where φ is plurisubharmonic and T is a closed positive (q,q)-current.

Speaker: **Joël Merker** (Université Paris-Sud)
Title: *Ample Examples*
Abstract: Tuan Huynh, Ph.D. student in Orsay, obtained (IMRN 2015) examples of Kobayashi-hyperbolic hypersurfaces $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ of low degree $2n+2$ for $n = 2, 3, 4, 5$, and of degree $\frac{(n+3)^2}{4}$ for $n \geqslant 6$.

Song-Yan Xie, Ph.D. student in Orsay, established in 2015 the ampleness of cotangent bundles (jets of order 1) to generic complete intersections $\mathbb{X}^n \subset \mathbb{P}^{n+c}(\mathbb{C})$ of codimension $c \geqslant n$ with degrees $d_1, \dots, d_c \geqslant (n+c)^{(n+c)^2}$. This result answered fully a conjecture made by Debarre in 2005.

The first part of the talk will present a variation of S. Xie's proof, based on multidimensional resultants, which conducts to an improvement on the degree bound: $d_1, \dots, d_c \geqslant (n+c)^{n+c}$.

In order to reach an effective generic ampleness result about higher order jet bundles, in link with Kobayashi's hyperbolicity conjecture, taking inspiration from Masuda-Noguchi (1996), the second part of the talk will focus on families of hypersurfaces $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ having homogeneous polynomial defining equations of the form:

$$0 = \sum_{\alpha_0+\alpha_1+\cdots+\alpha_{n+1} = \mathsf{fmn}} A_{\alpha_0,\alpha_1,\dots,\alpha_{n+1}}(X) \, \big((X_0)^d\big)^{\alpha_0} \big((X_1)^d\big)^{\alpha_1} \cdots \big((X_{n+1})^d\big)^{\alpha_{n+1}},$$

with *Fermat-Masuda-Noguchi index* $\mathsf{fmn} \geqslant n^2 + n$, and with polynomials $A_\bullet\big(X_0 \colon X_1 \colon \cdots \colon X_{n+1}\big)$ homogeneous of relatively low degree $\deg A_\bullet =: \mathsf{a} \geqslant n$, compared with the dominant degree $d \geqslant n^{n^2}$.

Mainly, some appropriately truncated order-n jet bundles will happen to admit (a wealth of) global holomorphic sections, by means of a new process of forming (huge) Macaulay-type matrices, thanks to an application of Hartogs' theorem, a bit similarly as was performed by Siu-Yeung (Invent. 1996) and by Siu (Invent. 2015).

The end of the talk will conclude by presenting a link between the geometry of complex vector bundles and the first complete effective computations of CR curvatures of CR manifolds up to dimension $\leqslant 5$ performed by Samuel Pocchiola (ex-Ph.D. student in Orsay) and Masoud Sabzevari (Shahrekord).

Speaker: **Jasna Prezelj** (University of Ljubljana and University of Primorska)
Title: *Positivity of metrics*
Abstract: Let $p : Z \to Z$ be a submersion from a complex manifold Z to a 1-convex manifold X with an exceptional set X. Let $E \to Z$ Let $a : X \to Z$ be a holomorphic section. Then there exist a conic neighbourhood U of $a(X \setminus S)$ such that U admits a Kähler metric and a metric on E_U with positive Nakano curvature and with at most polynomial poles over $p^{-1}(S)$.

Speaker: **Evgeny Poletsky** (Syracuse University)
Title: *Homotopic properties of holomorphic mappings*
Abstract: Let W be a domain in a complex manifold M. In 2008 B. Jöricke found a way to extend holomorphic functions from W to another manifold and show that it is the envelope of holomorphy of W and in 2013 F. Lárusson and the speaker used a similar approach to subextend plurisubharmonic functions from W to a complex manifold. To define these manifolds the authors considered the space $A(W, M)$ of analytic disks in M whose boundaries lie in M. The new manifolds were defined as the quotients of this space by equivalence relations, where equivalent analytic disks can be connected by a continuous path or a homotopy in $A(W, M)$.

In 1983 L. Rudolph introduced quasipositive elements of braid groups that are fundamental groups of the complements to some set W of planes in Cn. He proved that these elements are boundaries of analytic disks in $A(W, \mathbb{C}^n)$ and form a semi- group. The talk will be divided in two parts. In the first part we will discuss general constructions of extensions of Riemann domains and subextensions of plurisubharmonic functions. In the second part we will address the notion of quasipositive elements in general situation and explain why they form a semigroup.

An important question is whether this semigroup is embeddable into the fundamental group. That is equivalent of asking whether two analytic disks are homotopic as analytic disks when their boundaries are equivalent in the fundamental group. A similar problem was studied by M. Gromov and, recently, by F. Forstnerič and his colleagues for homotopies of submanifolds in elliptic manifolds. In our case the ambient manifold is hyperbolic and the answer is not known. In the special case is when W is an analytic variety in M we will show that this problem can be reduced to the problem involving only real disks.

Speaker: **Nikolay Shcherbina** (University of Wuppertal)
Title: *A domain with non-plurisubharmonic squeezing function*
Abstract: We construct a strictly pseudoconvex domain with smooth boundary whose squeezing function is not plurisubharmonic. This is a joint work with J.E. Fornaess.

Speaker: **Alexander Tumanov** (University of Illinois at Urbana-Champaign)
Title: *Symplectic non-squeezing for the discrete nonlinear Schrödinger equation*
Abstract: The celebrated Gromov's non-squeezing theorem of 1985 says that the unit ball B^n in C^n can be symplectically embedded in the "cylinder" $rB^1 \times C^{n-1}$ of radius r only if $r \geq 1$. Hamiltonian differential equations provide examples of symplectic transformations in infinite dimension. Known results on the non-squeezing property in Hilbert spaces cover compact perturbations of linear symplectic transformations and several specific non-linear PDEs, including the periodic Korteweg - de Vries equation and the periodic cubic Schrödinger equation. We prove a new version of the non-squeezing theorem for Hilbert spaces. We apply the result to the discrete nonlinear Schrödinger equation. This work is joint with Alexander Sukhov.

Speaker: **Elizabeth Wulcan** (Chalmers University of Technology)
Title: *Direct images of semi-meromorphic currents*
Abstract: I will discuss a joint work in progress with Mats Andersson, in which we study and develop a calculus for direct images of semi-meromorphic currents. In my talk I will focus on regularity properties of these and in particular show that the sheaf of such currents is stalkwise injective.

Speaker: **Ahmed Zeriahi** (Université Paul Sabatier, Toulouse)
Title: *Weak solutions to degenerate complex Monge-Ampère flows*
Abstract: Studying the (long-term) behaviour of the Kähler-Ricci flow on mildly singular varieties, one is naturally lead to study weak solutions of "degenerate parabolic complex Monge-Ampère equations". The purpose of this work, is to develop a viscosity theory for degenerate complex Monge-Ampère flows on compact Kähler manifolds. The main ingredient is the "parabolic comparison principle" which allows us to prove uniqueness of the solution to a general complex Monge-Ampère flow starting from a singular metric with bounded potentials in a given Kähler class. Then using our previous results on the "degenerate elliptic side" of the complex Monge-Ampère theory, we are able to construct barriers that allow us to prove the existence of a solution to the Cauchy problem by means of the classical method of Perron. Our general theory allows in particular to define and study the behaviour of the (normalized) Kähler-Ricci flow on projective varieties with canonical singularities, generalizing results of Song and Tian. In the case when the variety is Calabi-Yau or of general type, we prove that the Kähler-Ricci flow converges weakly in the sense of currents (strongly at the level of potentials) to the unique Kähler-Einstein metric on the the variety. The case of intermediate Kodaira dimension is more tricky and will be briefly sketched if time permits. This is a joint work with P. Eyssidieux and V. Guedj which will appear in Advances in Math.

Speaker: **Andrew Zimmer** (University of Chicago)
Title: *Characterizing domains by their automorphism group*
Abstract: It is generally believed that (up to biholomorphism) very few domains have a large automorphism group and a nice boundary. For instance the Wong-Rosay Ball theorem says that a strongly pseudoconvex domain with non-compact automorphism group must be bi-holomorphic to the ball. Later, Bedford and Pinchuk proved that a convex domain of finite type and non-compact automorphism group must be bi-holomorphic to a domain defined by a polynomial. I will discuss a recent result which removes the finite type condition from the Bedford-Pinchuk result but at the cost of assuming that the automorphism group is slightly larger than non-compact. In particular, a smoothly bounded convex domain is biholomorphic to a domain defined by a polynomial if and only if an orbit of the automorphism group accumulates on at least two different complex faces of the set. The proof of this result combines rescaling arguments and ideas from the theory of metric spaces of non-positively curvature.

4 Scientific Progress Made

Almost all the talks generated many interesting questions from the audience, related to the results presented in the talks. Some of the questions were about new directions of research, while others pointed to possible connections of the results to other fields of mathematics.

Aside from such questions, there were quite a few longer discussions between groups of participants regarding not only the topics presented in the lectures but also other important open problems. We note here a few such discussions.

A. Rashkovskii gave an informal seminar talk to N. Levenberg and N. Shcherbina on his recent study of local geodesics for plurisubharmonic functions (posted on arXiv). Rather than working in the space of Kähler metrics on compact, complex manifolds, he works on Cegrell classes of plurisubharmonic functions on bounded hyperconvex domains in $\mathbb{C}^n$. Rashkovskii will visit Levenberg in fall 2016 to continue these discussions. Shcherbina also mentioned the possibility of a future visit.

T. Bayraktar, T. Bloom and Levenberg had daily discussions on possible extensions of Bayraktar's results on asymptotic zero distribution of random polynomial mappings in $\mathbb{C}^n$. Bayraktar and Levenberg will organize a special session at an AMS meeting in spring 2017 on randomness in complex geometry.

L. Arosio and F. Lárusson discussed a new research project aimed at relating Oka theory and dynamics. Their discussions resulted in a proof, a few weeks after the workshop, of a new characterisation of ellipticity for Stein manifolds in dynamical terms.

I. Kossovskiy and R. Shafikov discussed connections between singular Levi-flat hypersurfaces and the theory of singular ODEs. This approach may lead to a classification of Levi-flat singularities in the spirit of Burns and Gong [5].

F. Berteloot, H. Gaussier, A. Sukhov and A. Zimmer discussed the relations between hyperbolicity in the sense of Kobayashi and hyperbolicity in the sense of Gromov. It is a striking recent achievement in complex geometry that these fundamental notions are closely related for a wide class of complex manifolds with boundary (such as bounded convex domains of finite type). Many open questions remain. These discussions will lead to the organization of a workshop or summer school concerning the interplay between complex hyperbolic theory, Gromov's theory of hyperbolic spaces, the theory of quasiconformal structures and related topics.

One usually cannot expect major theorems to be proved during a five-day workshop. However, the organizers are confident that the ideas generated by the talks and by the many discussions that took place during the week will lead to important progress in some of the many interrelated topics covered by the conference.

5 Outcome of the Meeting

Although a single workshop cannot do justice to the breadth and depth of contemporary complex analysis and complex geometry, the organizers believe it was beneficial to bring together a group of experts from diverse subfields to discuss recent results and work in progress and to share ideas on open questions. We chose a coherent collection of interrelated topics for the workshop, representing some of the most vibrant developments in the subject today.

The workshop covered a wide variety of topics of modern research in complex analysis and geometry. There were 39 participants, ranging from leading experts to graduate students (2 in total) and recent PhDs (7 in total). Among the participants were 5 female mathematicians.

The program consisted of 26 talks, each 45 minutes long, with a break of at least 15 minutes in between them. Seven of the talks were by recent PhDs. There were three full days, when the presentations ended by 5pm, while the remaining two days consisted of morning sessions. This allowed ample time for questions and discussions. Many of the talks exposed very recent important results. Quite a few of the talks reported on significant work in progress. Many of the talks gave rise to substantial discussions.

In conclusion, the participants found the workshop exciting and stimulating. The excellent facilities provided at BIRS, together with the inspiring scenery, helped to make the workshop a success.

6 List of participants

1. Andrist, Rafael, Bergische Universität Wuppertal
2. Arosio, Leandro, Università di Roma 2

3. Bayraktar, Turgay, Syracuse University
4. Bedford, Eric, Stony Brook University
5. Berteloot, Francois, Université de Toulouse
6. Bertrand, Florian, American University of Beirut
7. Błocki, Zbigniew, Jagiellonian University
8. Bloom, Tom, University of Toronto
9. Brudnyi, Alex, University of Calgary
10. Chakrabarti, Debraj, Central Michigan University
11. Drinovec Drnovšek, Barbara, University of Ljubljana
12. Ebenfelt, Peter, University of California at San Diego
13. Edigarian, Armen, Jagiellonian University
14. Forstnerič, Franc, University of Ljubljana
15. Gaussier, Herve, Université Grenoble Alpes
16. Gupta, Purvi, University of Western Ontario
17. Kaliman, Shulim, University of Miami
18. Kinzebulatov, Damir, Université Laval
19. Kossovskiy, Ilya, University of Santa Catharina, Brazil
20. Larkang, Richard, University of Wuppertal
21. Lárusson, Finnur, University of Adelaide
22. Levenberg, Norman, Indiana University
23. Magnússon, Benedikt, University of Iceland
24. Merker, Joel, Université Paris-Sud
25. Mitrea, Octavian, University of Western Ontario
26. Poletsky, Evgeny, Syracuse University
27. Prezelj, Jasna, University of Ljubljana and University of Primorska
28. Ramos-Peon, Alexandre, University of Bern
29. Rashkovskii, Alexander, Stavanger University
30. Ritter, Tyson, University of Oslo
31. Shafikov, Rasul, University of Western Ontario
32. Shcherbina, Nikolay, University of Wuppertal
33. Sukhov, Alexandre, Université des Sciences et Technologies de Lille
34. Tumanov, Alexander, University of Illinois at Urbana-Champaign
35. Vivas, Liz, Ohio State University

36. Winkelmann, Jörg, Ruhr-Universität Bochum
37. Wulcan, Elizabeth, Chalmers University of Technology
38. Zeriahi, Ahmed, Université Paul Sabatier (Toulouse)
39. Zimmer, Andrew, University of Chicago

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