Bounds for Restrictions of Laplace Eigenfunctions

Yaiza Canzani (University of North Carolina at Chapel Hill)
Jeffrey Galkowski (Stanford University), John Toth (McGill University)

10/15/2017–10/22/2017

1 Overview of the Field

Let (M, g) be a compact, smooth, Riemannian surface with no boundary, and consider a sequence (ϕ_h) of L^2-normalized Laplace eigenfunctions

$$-h^2 \Delta_g \phi_h = \phi_h.$$

The behavior of ϕ_h, $h \to 0$ and its connection with the geodesic flow on M is the has been the subject of intense study. Recently, there has been an attempt to understand small scale properties of ϕ_h. One way of doing this is to understand properties of the restriction of ϕ_h to various submanifolds. Concentration rates of such restrictions as measured by L^p norms [BGT, HT, Tac], weak limits of $|\phi_h|^2$ [DZ, TZ2], and size and distribution of nodal sets $\{\phi_h = 0\}$ [ET, Jun] have all been extensively studied.

2 Recent Developments and Open Problems

The problem of obtaining lower bounds for $\|\phi_h\|_{L^2(H)}$ is quite challenging and has only been attempted in very specific settings. It has profound applications to the study of the nodal sets of high energy eigenfunctions. In [TZ] where the authors show that if $\Omega \subset \mathbb{R}^2$ is a bounded domain with piece-wise real analytic boundary and (ϕ_h) is a sequence of Neumann eigenfunctions, then there exists $C > 0$ for which $\|\phi_h\|_{L^2(\partial\Omega)} \geq e^{-Ch}$ as $h \to +\infty$. Curves satisfying such exponential lower bound (such as $H = \partial\Omega$), are said to be *good*. This is a concept that arises frequently when bounding from above the number of zeros of ϕ_h along H. Indeed, on real analytic compact surfaces, the goodness condition on H is needed to prove the sharp upper bound $\#\{\phi_h^{-1}(0) \cap H\} = O(h)$, see [CT]. For the purpose of obtaining upper bounds on $\#\{\phi_h^{-1}(0) \cap H\}$, it is proved in [Jun] that horocycles lying inside compact hyperbolic surfaces are good curves. With the same goal, the authors in [ET] proved that if $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with piecewise real analytic boundary and (ϕ_h) is a sequence of quantum ergodic (QE) Neumann eigenfunctions, then any real analytic closed curve with strictly positive geodesic curvature is good. One can view the restriction lower bounds in this proposal as a natural closed manifold analogue of the lower bounds in Theorem 1.3 of [ET]. However, the restriction lower bounds for curved H's and QE eigenfunction sequences (ϕ_λ) give $\|\phi_h\|_{L^2(H)} \geq C$ whereas Theorem 1.3 in [ET] only implies the much weaker goodness estimate $\|\phi_h\|_{L^2(H)} \geq Ce^{-C/h}$ for some $C = C(H, \Omega) > 0$.

The exponential lower bound was improved in [GRS] in the case in which H is a closed horocycle lying inside an arithmetic surface and the eigenfunctions (ϕ_h) are even Maass cusps forms. In this case the authors prove that for every $\epsilon > 0$ there exists C_ϵ so that $\|\phi_h\|_{L^2(H)} \geq C_\epsilon h^{-\epsilon}$ as $h \to +\infty$. An even stronger lower bound was obtained in [BR] for the flat torus. In [BR] the authors prove that for any sequence (ϕ_h) of Laplace eigenfunctions there exists a constant $C > 0$ for which $\|\phi_h\|_{L^2(H)} \geq C$ as $h \to +\infty$, provided H has non-vanishing geodesic curvature.

As for upper bounds, the universal estimates in [BGT] give $\|\phi_h\|_{L^2(H)} = O(h^{1/4})$ when $H \subset M$ is any curve, and $\|\phi_h\|_{L^2(H)} = O(h^{1/6})$ when H has non-vanishing curvature. These upper bounds where slightly improved by a $\log(h)^{-1}$ factor in [Che] for negatively curved surfaces. In some specific cases, the improvement is polynomial in λ. For example, on flat tori in [BR], it is shown that $\|\phi_h\|_{L^2(H)} = O(1)$ when H has non-vanishing geodesic curvature. In [GRS], the authors prove that for any $\epsilon > 0$ there exists

$C_\epsilon > 0$ for which $\|\phi_h\|_{L^2(H)} \leq C_\epsilon h^\epsilon$ when H is a closed horocycle inside an arithmetic surface and the eigenfunctions (ϕ_h) are even Maass cusps forms.

One expects to obtain uniform lower and upper bounds for $\|\phi_h\|_{L^2(H)}$ in cases where the eigenfunctions are equidistributed along H. Indeed, it follows from [TZ, DZ] that if (M, g) has ergodic geodesic flow and H has a 'zero measure of microlocal symmetry', then there exists a density one subsequence (ϕ_{h_j}) of the set of Laplace eigenfunctions for which $\|\phi_{h_j}\|_{L^2(H)} \to C$ as $j \to \infty$. The microlocal asymmetry assumption on H in [TZ] is generic; but it is quite difficult to check and has only been established for geodesic circles, closed geodesics and closed horocycles inside certain hyperbolic surfaces [TZ]. The existence of a limit for $\|\phi_{h_j}\|_{L^2(H)}$ hinges on the assumption that the geodesic flow is ergodic. In particular, this assumption gives the existence of a quantum ergodic sequence from which (ϕ_{h_j}) is built.

3 Scientific Progress Made

We discussed several approaches to the problem of obtaining lower bounds on the restriction of Laplace eigenfunctions to hypersurfaces H inside a compact Riemannian manifold (M, g). That is we wish to estimate

$$\|\phi_h\|_{L^2(H)} \geq c > 0$$

when

$$(-h^2\Delta_g - 1)\phi_h = 0, \qquad \|\phi_h\|_{L^2} = 1.$$

While we considered several, two such ideas came to the fore. First, the idea of using boundary integral operators together with the semiclassical FIO and Airy calculi and second, that of understanding the geometry of persistent nodal surfaces through propagation of defect measures.

During the discussion, we realized that, in order to make the layer potential approach work, we need control on concentration of eigenfunctions in shrinking (with respect to the spectral parameter h) neighborhoods of H. Unfortunately, such control is not known except in very special circumstances. We plan to investigate such concentration in neighborhoods of tangential rays. In such regions, it is possible that we will be able to obtain estimates using the Airy calculus for boundary layer operators. Calculations with the Friedlander model indicate that we need control on $h^{2/3}$ scales.

The second approach proved more fruitful. By thinking of the surface H as a transparent boundary, we are able to show that defect measures associated to sequences of eigenfunctions, ϕ_h, such that

$$\|\phi_h\|_{L^2(H)} \to 0$$

are jointly invariant under the billiard flow on M with respect to H and the geodesic flow of M. This invariance has many implications on the geometry of H when ϕ_h carries mass in neighborhoods of H. In particular, when the defect measure associated to ϕ_h carries mass transversal to H, this implies that the measure has certain symmetries with respect to H. While we expect this behavior to continue uniformly through tangential rays, we are currently unable to obtain this information. Indeed, it seems that defect measures are not an appropriate tool to probe behavior in this set.

4 Outcome of the Meeting

We plan to write a paper based on our observations of defect measures in the presence of nodal persistence and will continue to investigate when the phenomenon of nodal persistence may occur.

References

[BGT] N. Burq, P. Gerard and N. Tzvetkov. Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds. *Duke Mathematical Journal.* 138.3: 445-486, 2007.

[BR] J. Bourgain and Z. Rudnick. Restriction of toral eigenfunctions to hypersurfaces and nodal sets. *Geometric and Functional Analysis.* 22(4), 878-937, 2012.

[Che] X.Chen. An improvement on eigenfunction restriction estimates for compact boundaryless Riemannian manifolds with nonpositive sectional curvature. *Transactions of the American Mathematical Society.* 367(6), 4019–4039, 2015.

[CHT] H. Christianson, A. Hassell and J.A Toth. Semiclassical control and L^2-restriction bounds for Neumann data along hypersurfaces. *Preprint arXiv:1303.4319v2*, 2013.

[CT] Y. Canzani and J.A Toth. On the local geometry of the nodal sets of Laplace eigenfunctions on compact manifolds. *Preprint.*

[CTZ] H. Christianson, J.A Toth and S. Zelditch. Quantum ergodic restriction for Cauchy data: Interior QUE and restricted QUE. *Mathematical Research Letters*, 2013 (to appear).

[DZ] S. Dyatlov and M. Zworski. Quantum ergodicity for restrictions to hypersurfaces. *Preprint arXiv:1204.0284*, 2012.

[ET] L. El-Hajj and J.A Toth. Intersection bounds for nodal sets of planar Neumann eigenfunctions with interior analytic curves. *Journal of Differential Geometry.* 100.1, 1–53, 2015.

[HT] Hassell, Andrew; Tacy, Melissa. Semiclassical L^p estimates of quasimodes on curved hypersurfaces. *J. Geom. Anal.* 22 (2012), no. 1, 74?89.

[GRS] A. Gosh, A. Reznikov and P. Sarnak. Nodal domains of Maass forms I. *Geometric and Functional Analysis.* 23 (5), 1515–1568, 2013.

[Jun] J. Jung. Zeros of eigenfunctions on hyperbolic surfaces lying on a curve. *Preprint arXiv:1108.2335*, 2011.

[Tac] Tacy, Melissa. Semiclassical Lp estimates of quasimodes on submanifolds. *Comm. Partial Differential Equations, 35 (2010), no. 8, 1538?1562.*

[TZ] J. Toth and S. Zelditch. Counting nodal lines which touch the boundary of an analytic domain. Journal of Differential Geometry*, (81), 649–686.*

[TZ2] J. Toth and S. Zelditch. Quantum ergodic restriction theorems: manifolds without boundary. Geometric and Functional Analysis*, 23(2), 715–775, 2013.*