New trends in arithmetic and geometry of algebraic surfaces

Noam D. Elkies (Harvard University),
Keiji Oguiso (University of Tokyo),
Matthias Schütt (Universität Hannover)

March 12–17, 2017

1 Overview

The interplay of arithmetic and geometry has had a major impact on the recent development of algebraic geometry and number theory. Over the years, algebraic curves have been a driving force, culminating with groundbreaking results such as Faltings' finiteness theorem for rational points on algebraic curves of general type, and the proof of Fermat's Last Theorem by Wiles and Taylor. With curves much better understood, it seems natural to turn to higher-dimensional varieties, starting with surfaces.

Algebraic surfaces have been at the heart of algebraic geometry ever since the Italian school shaped the subject at the beginning of the 20th century. There is little doubt that algebraic surfaces support rich arithmetic structures; precise results on their arithmetic properties, however, are still widely conjectural, despite spectacular recent achievements such as the proof of the Tate conjecture for K3 surfaces in many important settings due to Madapusi Pera, Maulik, and Charles.

The astounding facets of the rich interplay of arithmetic and geometry, which have featured in many recent developments, were a key motivation to propose a workshop at BIRS. While K3 surfaces were a large focus of the workshop, important roles were also reserved for Enriques surfaces and certain aspects of general type surfaces.

2 Recent Developments and Open Problems

Algebraic surfaces are two-dimensional varieties (usually considered to be smooth and projective) over some algebraically closed field. Over the complex numbers, we can identify them with suitable manifolds, to which we can then apply important principles such as GAGA following Serre's classical work. To avoid confusion (arising from the fact that varieties over $\mathbf{C}$ of complex dimension d are real manifolds of dimension $2d$), we emphasize that Riemann *surfaces* correspond to algebraic *curves*, whose properties and structures we take to be fairly well understood (even though in the arithmetic fine print one may argue about this).

Thus turning to algebraic surfaces, we note immediately that they offer much more space for us to manoeuvre. Notably, an algebraic surface admits as subvarieties not only points but also curves, which in fact govern much of the structure of the surfaces. Another crucial difference is fibrations with fibers and base both of dimension one. Fibrations have played a central role in the study and classification of algebraic surfaces, as they come with several critical advantages. To name just two, they provide an important insight into the surface's structure — dividing curves into horizontal and vertical ones, for instance; on the other hand, they allow us to carry over concepts from curve theory, most canonically through the scheme-theoretic concept of the generic fiber.

We next detail some of the most recent developments and describe open problems in the area of algebraic and arithmetic of surfaces, especially those relevant to the present workshop.

2.1 Rational points

Given an algebraic variety X defined over some field K, one fundamental problem is to determine the set $X(K)$ of K-rational points on X. Of course, this problem is most interesting (albeit less geometric) when K is not algebraically closed, especially when K is a number field. Already very basic questions turn out to be often hard to answer, starting with whether $X(K)$ is finite or infinite, which can be made more complicated (and interesting) by removing certain algebraic subvarieties, such as those dominated by rational and abelian varieties. For the workshop, this circle of problems was mostly a very important motivation lurking in the background, but a quick discussion will still set the scene for most of what is to come.

It may be instructive to recall the now classical case of curves, where the picture is clear-cut. Let C, then, be a smooth projective curve defined over some number field K, and let g be the genus of C.

If $g = 0$, then either C has a K-rational point or $C(K) = \emptyset$. In the first case, it quickly follows that C is rational (i.e. isomorphic to $\mathbf{P}^1$ over K), and in particular $C(K)$ is infinite. In the latter case, the same properties hold true over any extension of K where C acquires a rational point.

At the other end of the scale, i.e. if C is of general type (which means $g > 1$), then it was conjectured by Mordell in the 1920's that $C(K)$ would always be finite. After celebrated early contributions, for instance by Siegel, Mordell's conjecture was finally proved by Faltings in the landmark paper [Fa83].

This leaves the case of genus one. If $C(K) \neq \emptyset$, then C is an elliptic curve; this means that $C(K)$ has the structure of an abelian group, which (for any number field K, but also under mild conditions over function fields) is finitely generated. More precisely, $C(K)$ can contain any abelian finite group of length at most two as torsion subgroup (although the precise groups that occur are rather limited by the given field K; for instance, over $\mathbf{Q}$ the torsion subgroups run through all cyclic groups until $\mathbf{Z}/10\mathbf{Z}$ together with $\mathbf{Z}/12\mathbf{Z}$, and the non-cyclic groups $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$ for $n = 1, 2, 3, 4$ by Mazur's famous theorem). As for the rank of $C(K)$, this could also *a priori* be any nonnegative intger r, and in fact it is an open problem whether given K, the rank of $C(K)$ for all elliptic curves C over K is bounded. For instance, the the current record over $\mathbf{Q}$ stands at 28, due to Elkies [Elk07] (whose construction uses elliptic fibrations on K3 surfaces!). The rank can be made arbitrarily large by passing to finite extensions K'/K.

Note, however, that these considerations have taken us far beyond the original question, as we are already investigating the precise structure of $C(K)$ as opposed to its cardinality. Indeed, for the cardinality, the answer is simple: $C(K)$ could be either finite or infinite, but on increasing the base field K by a finite extension K', we can first ensure that C is elliptic (over K') and then endow it with a rational point of infinite order so that $C(K')$ becomes infinite, and indeed dense in the associated Riemann surface. This property is also called *potential density*, and it is of great relevance for us because it is geometric: it does not depend anymore on the field K or the chosen model of C, but only on the isomorphism class of C over $\mathbf{C}$ (given that C may be defined over some number field).

A similar picture persists for algebraic surfaces — as far as we can see. Here the genus is replaced by the Kodaira dimension. Surfaces of Kodaira dimension $\kappa = -\infty$ are well understood as far as the cardinality of the set of rational points, though one can then ask for more subtle properties such as density — or even ask to remove a countable union of rational subvarieties. This leads to Manin's conjectures for Fano varieties, which have seen substantial progress in recent years (in any dimension, in fact). In the opposite direction, for surfaces of general type (i.e. $\kappa = 2$), Lang's conjecture predicts that rational points accumulate on a Zariski-closed proper subset, but this problem still seems hopelessly out of reach. We thus turn to the intermediate case of $\kappa = 0$. Here abelian surfaces can be treated just like elliptic curves, but it is the case of K3 surfaces where new phenomena arise. Notably, potential density is guaranteed by special structures on the surface. Not too surprisingly, an infinite automorphism group suffices for this end; much less trivial is that the same applies to genus one fibrations by a result of Bogomolov and Tschinkel [BT00] (and carries over to Enriques surfaces). This should serve as an indication that one ought to take a closer look at the structure of the curves on a surface (and at fibrations).

2.2 Curves on surfaces

The curves on an algebraic surface S generate a huge group of formal linear combinations, the divisor group $\mathrm{Div}(S)$. As with curves, we can retrieve essential information about the surface S after dividing out by suitable equivalence relations; presently these could be linear equivalence, algebraic equivalence or numerical equivalence. All quotients share the same discrete part — up to torsion, which in fact is eliminated by numerical equivalence so that the resulting group $\mathrm{Num}(S)$ turns out to be an integral lattice with the natural intersection pairing. Its rank $\rho(S)$ is called the *Picard number* of S and is one of the most fundamental invariants of an algebraic surface. We emphasize that the Picard number is in general not preserved by deformations, but in some special instances, as for abelian or K3 surfaces (or irreducible holomorphic symplectic manifolds), there is a nice compatibility with moduli theory.

Despite its importance, the problem of computing the Picard number of a given algebraic surface S remains notoriously difficult. However, if S is defined over some number field K, there is an arithmetic approach, which can serve as a prototypical example for this kind of question. Fix some prime p of K where S has good reduction (denoted by S_p). Then there is an embedding

$$\mathrm{Num}(S) \hookrightarrow \mathrm{Num}(S_p),$$

primitive on the level of lattices. This yields the bound

$$\rho(S) \leq \rho(S_p). \tag{1}$$

It might seem that we have merely reduced one intractable problem to another. But crucially $\mathrm{Num}(S_p)$ lends itself, at least in principle, to explicit computations via the ℓ-adic cycle class map

$$\mathrm{Num}(S_p) \hookrightarrow \mathrm{H}^2_{\acute{e}t}(\bar{S}_p, \mathbf{Q}_\ell(1)).$$

What's more, the embedding is Galois equivariant, so the image lies in the invariant part under Frobenius. In fact, a famous conjecture of Tate states that this should be an equality after tensoring with $\mathbf{Q}_\ell$:

Conjecture (Tate). In the above setting, one always has

$$\mathrm{Num}(S_p) \otimes \mathbf{Q}_\ell = \mathrm{H}^2_{\acute{e}t}(\bar{S}_p, \mathbf{Q}_\ell(1))^{\mathrm{Frob}}.$$

This conjecture, one of the most important and influential in the field, has seen great progress in the last 5 years: it has been proved competely for K3 surfaces. The pioneering work of Artin and Swinnerton-Dyer [ASD73] on homogeneous spaces proved the Tate conjecture for elliptic K3 surfaces; Nygaard [Nyg83a] proved it for ordinary K3 surfaces in any characteristic; and the remaining cases were recently settled by work of Maulik [Mau14], Charles [Cha13], Madapusi Pera [Mad15] (for odd characteristic), and Kim–Madapusi Pera [KM15] (for characteristic 2).

This paves the way for several applications and new directions. First, along the above lines, one can try to compute the action of Frobenius on cohomology to obtain upper bounds as in (1). For parity reasons, these bounds alone on $\rho(S)$ are never sharp if $\rho(S)$ is odd, but this subtlety can be overcome by varying p and examining the lattice structures more closely (see work of van Luijk [vLui07] and Kloosterman [Klo07]). It is another result of Charles [Cha14] that all this information together will be sufficient to derive a sharp upper bound on $\rho(S)$ in any case (at least in theory, and depending on the Hodge conjecture for the self-product $S \times S$). We note that for all of this, we do not need the full action of Frobenius on $\mathrm{H}^2_{\acute{e}t}(\bar{S}_p, \mathbf{Q}_\ell(1))$, only its eigenvalues with multiplicity. These can be computed, for instance, by extensive point counts (and applying the Lefschetz fixed point formula), or through p-adic cohomological methods and p-adic approximation.

We highlight two further applications of the above ideas. The first concerns very special curves on K3 surfaces, namely rational ones. The problem has seen remarkable progress recently, starting from work of Bogomolov-Hassett-Tschinkel [BHT11] and then greatly extended by Li and Liedtke in [LL12]. The main novel technique is to use Picard jumps upon reduction from S to S_p in (1), forced for instance by parity, to detect infinitely many rational curves, first on S_p, but then, using arguments going back to Bogomolov and Mumford, also on the original surface S. In particular, this applies to any K3 surface S over $\bar{\mathbf{Q}}$ with $\rho(S)$ odd or at least 5. The second application draws the connection back to rational points. Wondering whether K3 surfaces over number fields satisfy potential density, the key case to consider seems to be Picard number 1. In order to check this, one may combine the above methods with elementary deformation theory to exhibit explicit K3 surfaces with Picard number one, say over $\mathbf{Q}$, and use them as test cases.

2.3 Good reduction and Honda-Tate

The problem of good reduction lies at the heart of arithmetic geometry, and even more so do integral models, say over integer rings in number fields. Prototypica examples are often provided by moduli spaces; especially for Shimura varieties integrality questions are one of the key issues for a better understanding, both of the varieties and the objects parametrized.

The classical examples are provided by abelian varieties, for which we have a uniform treatment thanks to the existence of Néron models. Indeed, Serre and Tate showed in [ST68] that an abelian variety over a number field has good reduction if and only its ℓ-adic cohomology is unramified (for some ℓ or for all ℓ, and even H^1 suffices). Later Fontaine proved that there cannot be any (positive-dimensional) abelian schemes over $\mathbf{Z}$ [Fo85]. However, already in dimension one, i.e. for elliptic curves, it is not clear yet precisely which quadratic fields support everywhere integral models of some elliptic curve.

A similar picture persists for K3 surfaces (and higher-dimensional Calabi–Yau varieties) where no models over $\mathbf{Z}$ may exist as proven independently by Abrashkin [Ab89] and Fontaine [Fo93]. Surprisingly, there really seem to be no known explicit integral models of K3 surfaces (unless one considers [Ma15] as explicit); but on the theoretical side there has been substantial progress recently, although part of it depends on the arithmetic analogue of parts of the Minimal Model Program. More precisely, assuming the existence of so-called Kulikov models (a strong form of semi-stable reduction which under certain conditions follows from work of Maulik [Mau14]), Liedtke and Matsumoto prove potential good reduction for K3 surfaces over a discrete valuation ring, i.e. unramified Galois representations imply good reduction over some unramified finite extension (which generally cannot be circumvented, see [LM17]). Very recently, there has also been a crystalline version in [CLL17].

In a similar, but somewhat converse direction, one may also ask whether given a candidate zeta function (satisfying all the standard conditions over some finite field imposed by the Weil conjectures, compatibilities etc.), there is a variety within a given class (say abelian or K3) over the fixed finite field with this exact zeta function. Classically this is known for abelian varieties as the theorem of Honda–Tate (cf. [Hon68]). Taelman proved it recently for K3 surfaces in [Tae16], again assuming a strong form of semi-stable reduction and possibly after an extension of the finite field (which, for instance, makes the Galois action on $\mathrm{Num}(S)$ trivial). We point out that it is not known whether the extension can be avoided in general, or under certain conditions; but experiments conducted by Kedlaya and Sutherland provide rich test data for quartic surfaces over $\mathbf{F}_2$ [KS16].

2.4 Automorphisms of Enriques surfaces

Enriques surfaces probably form the most mysterious surfaces of Kodaira dimension zero, because they are governed by their universal covers, namely by K3 surface, and thus by lattice theory, but yet they sometimes show a completely different behaviour. For instance, a very general complex K3 surface has finite automorphism group (or even trivial or of order two), but a general Enriques surface has infinite automorphism group by [BP83]. In fact, Enriques surfaces with finite automorphism group are very rare; over $\mathbf{C}$, they have been classified completely into seven types by Kondō [Kon86]. In (small) positive characteristics, different situations may persist; recently, the odd characteristic case (and part of characteristic two, see below) was solved completely by Martin [Ma17]. In characteristic two, however, the moduli space of Enriques surfaces decomposes into two 10-dimensional components, corresponding to singular (or ordinary) Enriques surfaces on the one hand and classical Enriques surfaces on the other, as was shown by Liedtke in [Li15]. The singular Enriques surfaces still admit a smooth universal K3 cover; in practice this means that they lend themselves to a treatment very similar to all other characteristics, which was exploited with great success in [KK15] and [Ma17]. The classical Enriques surfaces, however, behave rather differently: much of their geometry seems to be closer to the complex counterpart (for instance genus one fibrations still have two ramified fibers), but the universal covers are no longer smooth K3 surfaces, but only K3-like. The Enriques surfaces at the intersection of the singular and the classical component, the so-called supersingular Enriques surfaces, might be considered even more mysterious.

In a different direction, one may ask for finite subgroups of the (usually infinite) automorphism group. For complex K3 surfaces, this problem leads to now classical work of Mukai [Muk88] which classifies finite groups acting symplectically (i.e. leaving the regular 2-form invariant) in terms of the Mathieu group M_{23}.

Recently Mukai and Ohashi started to apply this approach to complex Enriques surfaces and M_{12} in [MO15].

2.5 Other aspects

Of course, there are many other open problems on algebraic surfaces which we cannot mention in great detail here. They range from very classical problems, such as the geography of surfaces of general type, through modern topics such as the ubiquitous moduli theory (especially for K3 surfaces, Enriques surfaces, but also for irreducible holomorphic symplectic manifolds) to the most recent advances in connection with dynamics, with emphasis on rational and K3 surfaces.

3 Presentation Highlights

The workshop featured a great number of very interesting talks. There were numerous experienced leaders as speakers, but we made an effort to schedule a good portion of talks by junior participants, one even before receiving his Master's degree. Throughout the audience was very appreciative and took an active role in the talks, with intensive discussions regularly continuing during the breaks. Below we lay out some of the presentation highlights.

3.1 Anthony Varilly-Alvarado (Rice University): A conjecture on Brauer groups of K3 surfaces

Torsion points govern much of the theory of elliptic curves (as indicated in 2.1). Varilly-Alvarado reported on recent attempts to find a good replacement for torsion points on K3 surfaces, ideally lending itself to uniform bounds such as Merel's 1996 results for elliptic curves [Mer96]. Recent findings suggest that the Brauer groups might be a good candidate yielding analogous statements for K3 surfaces. Growing evidence was lately provided by works of several researchers, including Cadoret–Charles, Orr–Skorobogatov [OS17], and the speaker with collaborators.

3.2 Lenny Taelman (University of Amsterdam): Equivariant Witt groups and zeta functions

Taelman presented work that should have a high impact on the Honda–Tate problem for K3 surfaces (cf. 2.3). The talk itself (which has in the meantime resulted in a joint paper with Bayer-Fluckiger [BFT17]) was almost purely concerned with quadratic forms, but it shed a completely new light on them by working out necessary and sufficient criteria for the following problem: Given a discrete valuation ring R with field of fractions K, consider a symmetric bilinear space V over K. Assume that some group G acts on V by isometries. Does V contain a unimodular lattice stabilized by G?

Taelman's result also provides restrictions on the possible characteristic polynomials of Frobenius on the middle cohomology of a smooth projective variety of even dimension over a finite field, thus generalizing a theorem of Elsenhans and Jahnel [EJ15] (in a rather conceptual way).

3.3 Kazuhiro Ito (Kyoto University): On the construction of K3 surfaces over finite fields with given L-function

Ito improved on Taelman's result on Honda-Tate for K3 surfaces [Tae16] (cf. 2.3) by giving a construction of K3 surfaces over finite fields with given L-function, independent of the previously crucial good reduction criterion [It16]. The proof combines Taelman's approach with showing the existence of an elliptic fibration with a section, or of an ample line bundle of low degree. It still involves a potential finite extension of the base field, and a mild condition on the characteristic (either $p \geq 7$, or $p = 5$ plus one of three extra conditions, e.g. $\rho \geq 4$).

3.4 Yuya Matsumoto (Nagoya University): Degeneration of K3 surfaces and automorphisms

Matsumoto's talk provided a good reduction criterion for K3 surfaces (as in 2.3) that works not only for isolated examples but in full families [Ma16]. Again assuming the existence of the so-called Kulikov models, his methods require only the existence of an automorphism (of finite or infinite order) that acts on the regular 2-form by a primitive root of unity of order $m = 5$ or $m \geq 7$. To this end, the author studies the weight filtration of the ℓ-adic cohomology groups. Then the compatibility with the induced action of the automorphism group lets him exclude certain types of degeneration.

3.5 Curtis McMullen (Harvard University): Algebraic integers and surfaces dynamics

McMullen discussed reverse engineering at the interface of algebraic geometry and dynamical systems. In detail, he explained how to create dynamical systems on varieties starting with an algebraic integer (a Salem number, to be precise, which will serve as eigenvalue of some automorphism on cohomology). Explicit dynamical systems of minimal entropy on K3 surfaces, rational surfaces and complex tori were described in detail.

3.6 Simon Brandhorst (Leibniz Universität Hannover): On the dynamical spectrum of projective K3 surfaces

Brandhorst focused on the above mentioned problem of realizing a Salem number $\lambda \in \mathbf{C}$ as entropy of an automorphism of a complex projective K3 surface. He gave a affirmative answer, except that his methods only cover some power of λ. Afterwards, the techniques were extended to Enriques surfaces and 2-dimensional tori, both projective and non-projective [Bra17].

3.7 François Charles (Université Paris-Sud): Arithmetic ampleness and an arithmetic Bertini theorem

Charles discussed properties of certain ample line bundles in arithmetic geometry, aiming for analogues of well-known geometric results in the arithmetic setting [Cha17]. The key case consisted in an ample hermitian line bundle $\mathcal{L}$ on a projective arithmetic variety $\mathcal{X}$ of dimension at least 2 where the proportion of the effective sections of $\mathcal{L}^{\otimes n}$ defining irreducible divisors on $\mathcal{X}$ was shown to tend to 1 as n tends to ∞. Applications included restriction theorems and an arithmetic analogue of the Bertini irreducibility theorem, improving on the results from [CP16] (while at the same time building on them).

3.8 Asher Auel (Yale University): Decomposition of the diagonal and phantom categories on surfaces

Auel surveyed recent developments on the problems of rationality and stable rationality, combining perspectives from unramified cohomology and zero-cycles as well as derived categories and semiorthogonal decompositions. In particular, he reported on joint work with Bernardara (cf. [AB17]), concerning phantom subcategories of the derived category of an algebraic surface. He showed how these can be viewed as a stronger measure of rationality than the existence of a decomposition of the diagonal.

3.9 Jörg Jahnel (University of Siegen): On the distribution of the Picard ranks of the reductions of a K3 surface

Jahnel reported on joint work with Costa and Elsenhans [CEJ16] on the distribution of the Picard ranks of a K3 surface under the reduction map (1) as the prime p varies. To this end, he considered the determinant of the Galois representation on the transcendental cycles. After a Tate twist, this gives either a trivial or a quadratic character. Whenever this character evaluates to -1, he showed that the Picard number jumps. What's more, the character can be related to the discriminant of the K3 surface. This paves the way to a sufficient criterion for the existence of infinitely many rational curves on K3 surfaces not covered by the cases laid out in 2.2.

3.10 Toshiyuki Katsura (Hosei University): Classification of Enriques surfaces with finite automorphism groups in characteristic 2

Katsura's talk concerned the problem of finite automorphism groups of those Enriques surfaces in characteristic two whose universal cover is not a (smooth) K3 surface; that is, classical or supersingular Enriques surfaces – exactly those which were not covered in 2.4. In joint work with Kondō and Martin [KKM17], they use genus one fibrations (much like Kondō did in [Kon86], but now also including quasi-elliptic fibrations) and the classification of conductrices from [ESB04] to establish the full classification of the configurations of smooth rational curves (which happen to form a finite set) on Enriques surfaces with finite automorphism group. In particular, they provide examples for each configuration, some supporting different finite groups of automorphisms. This leaves open only the determination of the underlying moduli spaces.

4 Outcome of the Meeting

The workshop brought together experts from all over the world working on algebraic surfaces, ranging from leaders in the field to rising stars and newcomers. It featured exciting talks about the latest developments, which were extremely well received and led to lively discussions. The workshop strengthened existing interactions and at the same time initiated several new collaborations and directions.

References

[Ab89] Abrashkin, V. A.: *Modular representations of the Galois group of a local field, and a generalization of the Shafarevich conjecture*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), 1135–1182.

[ASD73] Artin, M., Swinnerton-Dyer, H.: The Shafarevich-Tate conjecture for pencils of elliptic curves on $K3$ surfaces, Invent. Math. **20** (1973), 249–266.

[AB17] Auel, A., Bernardara, B.: *Cycles, derived categories, and rationality*, Surveys on Recent Developments in Algebraic Geometry, Proceedings of Symposia in Pure Mathematics **95**, Amer. Math. Soc., Providence, RI, 2017, 199–266.

[BP83] Barth, W., Peters, C.: *Automorphisms of Enriques surfaces*, Invent. Math. **73** (1983), 383–411.

[BFT17] Bayer-Fluckiger, E., Taelman, L.: *Automorphisms of even unimodular lattices and equivariant Witt groups*, preprint (2017), arXiv: 1708.05540.

[BHT11] Bogomolov, F. A., Hassett, B., Tschinkel, Y.: *Constructing rational curves on K3 surfaces*, Duke Math. J. **157** (2011), 535–550.

[BT00] Bogomolov, F. A., Tschinkel, Y.: *Density of rational points on elliptic $K3$ surfaces*, Asian J. Math. **4** (2000), no. 2, 351–368.

[Bra17] Brandhorst, S.: *On the stable dynamical spectrum of complex surfaces*, preprint (2017), arXiv: 1708.06169.

[Cha13] Charles, F.: *The Tate conjecture for K3 surfaces over finite fields*, Invent. Math. **194** (2013), 119–145. Erratum: Invent. Math. **202** (2015), 481–485.

[Cha14] Charles, F.: *On the Picard number of K3 surfaces over number fields*, Algebra & Number Theory **8** (2014), 1–17.

[Cha17] Charles, F.: *Arithmetic ampleness and an arithmetic Bertini theorem*, preprint (2017), arXiv: 1703.02481.

[CP16] Charles, F, Poonen, B.: *Bertini irreducibility theorems over finite fields*, J. Amer. Math. Soc. **29** (2016), 81–94.

[CLL17] Chiarellotto, B., Lazda, C., Liedtke, C.: *A Néron-Ogg-Shafarevich critersion for K3 Surfaces*, preprint (2017), arXiv: 1701.02945.

[CEJ16] Costa, E., Elsenhans, A.-S., Jahnel, J.: *On the distribution of the Picard ranks of the reductions of a K3 surface*, preprint (2016), arXiv: 1610.07823.

[ESB04] Ekedahl, T., Shepherd-Barron, N. I.: *On exceptional Enriques surfaces*, preprint (2004), arXiv: math/0405510.

[Elk07] Elkies, N. D.: *Three lectures on elliptic surfaces and curves of high rank*, preprint (2007), arXiv: 0709.2908.

[EJ15] Elsenhans, A.-S., Jahnel, J.: *On the characteristic polynomial of the Frobenius on étale cohomology*, Duke Math. J. **164** (2015), 2161–2184.

[Fa83] Faltings, G.: *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. 73, 349–366 (1983).

[Fo85] Fontaine, J.-M.: *Il n'y a pas de variété abélienne sur* $\mathbf{Z}$, Invent. Math. **81** (1985), 515–538.

[Fo93] Fontaine, J.-M.: *Schémas propres et lisses sur* $\mathbf{Z}$, Proceedings of the Indo-French Conference on Geometry (Bombay, 1989), 43–56, Hindustan Book Agency (1993).

[Hon68] Honda, T.: *Isogeny classes of abelian varieties over finite fields*, J. Math. Soc. Japan **20** (1968), 83–95.

[It16] Ito, K.: *Unconditional construction of K3 surfaces over finite fields with given L-function in large characteristic*, preprint (2016), arXiv: 1612.05382.

[KK15] Katsura, T., Kondō, S.: *On Enriques surfaces in characteristic 2 with a finite group of automorphisms*, to appear in J. Alg. Geom., preprint (2015), arXiv:1512.06923.

[KKM17] Katsura, T., Kondō, S., Martin, G.: *Classification of Enriques surfaces with finite automorphism group in characteristic 2*, preprint (2017), arXiv: 1703.09609.

[KS16] Kedlaya, K., Sutherland, A. V.: *A census of zeta functions of quartic K3 surfaces over* $\mathbf{F}_2$, LMS Journal of Computing **19**, special issue A (2016), 1–11.

[KM15] Kim, W., Madapusi Pera, K.: *2-adic integral canonical models and the Tate conjecture in characteristic 2* preprint (2015), arXiv: 1512.02540.

[Klo07] Kloosterman, R.: *Elliptic K3 surfaces with geometric Mordell-Weil rank 15*, Canad. Math. Bull. **50** (2007), no. 2, 215–226.

[Kon86] Kondō, S.: *Enriques surfaces with finite automorphism groups*, Japan. J. Math. **12** (1986), 191–282.

[LL12] Li, J., Liedtke, C.: *Rational Curves on K3 Surfaces*, Invent. Math. **188** (2012), 713–727.

[Li15] Liedtke, C.: *Arithmetic Moduli and Lifting of Enriques Surfaces*, J. reine angew. Math. **706** (2015), 35–65.

[LM17] Liedtke, C., Matsumoto, Y.: *Good Reduction of K3 Surfaces*, to appear in Compos. Math., preprint (2017), arXiv: 1411.4797v3.

[vLui07] van Luijk, R.: *K3 surfaces with Picard number one and infinitely many rational points*, Algebra Number Theory **1** (2007), no. 1, 1–15.

[Mad15] Madapusi Pera, K.: *The Tate conjecture for K3 surfaces in odd characteristic* Invent. Math. **201** (2015), 625–668.

[Ma17] Martin, G.: *Enriques surfaces with finite automorphism group in positive characteristic*, preprint (2017), arXiv: 1703.08419.

[Ma15] Matsumoto, Y.: *On good reduction of some K3 surfaces related to abelian surfaces*, Tohoku Math. J. **67** (2015), 83–104.

[Ma16] Matsumoto, Y.: *Degeneration of K3 surfaces with non-symplectic automorphisms*, preprint (2016), arXiv: 1612.07569.

[Mau14] Maulik, D.: *Supersingular K3 surfaces for large primes*, Duke Math. J. **163** (2014), 2357–2425.

[Mer96] Merel, L.: *Bornes pour la torsion des courbes elliptiques sur les corps de nombres*, Invent. Math. **124** (1996), 437–449.

[Muk88] Mukai, S.: *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. **94** (1988), 183–221.

[MO15] Mukai, S., Ohashi, H.: *Finite groups of automorphisms of Enriques surfaces and the Mathieu group* M_{12}, preprint (2015), arXiv: 1410.7535v2.

[Nyg83a] Nygaard, N. O.: *The Tate conjecture for ordinary K3 surfaces over finite fields*, Invent. Math. **74** (1983), no. 2, 213–237.

[OS17] Orr, M., Skorobogatov, A. N.: *Finiteness theorems for K3 surfaces and abelian varieties of CM type* preprint (2016), arXiv: 1704.01647.

[ST68] Serre, J.-P., Tate, J.: *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492–517.

[Tae16] Taelman, L.: *K3 surfaces over finite fields with given L-function*, Algebra & Number Theory **10** (2016), 1133–1146.