# REPORT ON BIRS RESEARCH IN TEAMS, NO. 19RIT273

KAROL KOZIOŁ AND STEFANO MORRA

## 1. Context

Our research stay at BIRS was motivated by the **mod-$p$ and $p$-adic Langlands programs**. These programs constitute a vast web of conjectures which have had tremendous impact on arithmetic algebraic geometry over the past 30 years (e.g., the work of Wiles, Taylor–Wiles, and Breuil–Conrad–Diamond–Taylor on the Taniyama–Shimura–Weil conjecuture, leading to a proof of Fermat's Last Theorem).

More specifically, pursuing the calculations of [BCDT01] and following an early observation of Serre ([Ser87]), Breuil suggested in the early 2000s that certain congruences between modular forms should be explained by a $p$-adic version of the classical Langlands Correspondence. This "$p$-adic correspondence" may be stated loosely as follows:

**Conjecture 1.0.1.** *Let $p$ be a prime number and $F$ be a finite extension of $\mathbf{Q}_p$ (the field of $p$-adic numbers). Then there is a natural bijection*

$$\left\{\begin{array}{c}\textit{certain continuous representations}\\ \mathrm{Gal}(\overline{F}/F)\to\mathbf{GL}_n(\overline{\mathbf{Q}}_p)\end{array}\right\}\overset{?}{\longleftrightarrow}\left\{\begin{array}{c}\textit{certain continuous representations}\\ \textit{of } \mathbf{GL}_n(F) \textit{ with } \overline{\mathbf{Q}}_p\textit{-coefficients}\end{array}\right\}$$

The connotation of "natural" above means that the bijection should be compatible with the classical local Langlands correspondence with $\mathbf{C}$-coefficients, its geometric realization in the cohomology of arithmetic varieties, and so on.

So far, this conjecture has been made precise and proven **only** for the group $\mathbf{GL}_2(\mathbf{Q}_p)$ (by work of Breuil, Colmez, Emerton, Kisin, Paškūnas and others; see [Ber11] and [Bre12] for a survey). Even in the $\mathbf{GL}_2(\mathbf{Q}_p)$ case its applications are far-reaching, and afford us a better understanding of *global* objects, such as the absolute Galois group of $\mathbf{Q}$ and cohomology of algebraic varieties.

The starting point in attacking the above conjecture is to consider first an "inertial" version of the above situation with mod-$p$ coefficients, which is the content of Serre's modularity conjecture [Ser87] and its generalization in the form of the Breuil–Mézard conjecture [BM02, Conjecture 1.1]. The latter predicts the complexity of the special fiber of local Galois deformation rings (with $p$-adic Hodge theoretic conditions) in terms of representation theory of the group $\mathbf{GL}_2(\mathbf{Z}_p)$. Further, the Breuil–Mézard conjecture has now taken on a geometric reformulation, due to Emerton and Gee [EGa], [EGb], which opens up the possibility of importing new tools to handle the problem.

Despite these achievements, we note that $\mathbf{GL}_2$ (or more generally $\mathbf{GL}_n$ for $n \geq 2$) is just an example of reductive groups governing symmetries in nature. One of the most exciting and mysterious aspects in the classical Langlands program is known as **Langlands functoriality**: how can we relate automorphic forms and Galois representations (or, more appropriately, Galois parameters) for *various* reductive groups (such as unitary groups)? In particular, can we describe such parameters in terms of those pertaining to general linear groups, possibly endowed with additional symmetries?

## 2. Background

During our research stay at BIRS we explored the setting above; more precisely, we considered relating Langlands correspondences for $\mathbf{GL}_n$ and those for the unitary group $\mathbf{U}_n$. In this context, we seek a correspondence of the form

$$\left\{ \begin{array}{c} \text{certain continuous representations} \\ \mathrm{Inertia}(\overline{F}/F) \to {}^C\mathbf{U}_n(\overline{\mathbf{F}}_p) \end{array} \right\} \overset{?}{\longleftrightarrow} \left\{ \begin{array}{c} \text{certain continuous representations} \\ \text{of } \mathbf{U}_n(\mathcal{O}_F) \text{ with } \overline{\mathbf{F}}_p\text{-coefficients} \end{array} \right\}$$

(Here, ${}^C\mathbf{U}_n$ is an enhancement of the $L$-group of $\mathbf{U}_n$ called the **$C$-group**, and it is the appropriate object to consider in order to make this correspondence compatible with global and cohomological considerations.)

This modification in terms of unitary groups makes the project interesting in several ways: many of the tools used in the $\mathbf{GL}_n$ setting still apply, but we also have base change techniques at our disposal to relate representations of $\mathbf{U}_n$ to those of $\mathbf{GL}_n$. This hints at a **mod-$p$ principle of Langlands functoriality**. Note that we have already proven a version of correspondence above when $n = 2$ in [KM] and the project we started at BIRS explores the extension of our results to higher rank. Moreover, recent advances in the $p$-adic Langlands correspondence for $\mathbf{GL}_n$ open up the possibility of utilizing novel geometric techniques in the context of unitary groups (namely, moduli stacks of ${}^C\mathbf{U}_n$-valued Galois representations [EGb] and local models of Galois deformation rings [LLHLM]) in order to geometrize the functoriality principle.

In order to properly approach the setup above, we require some input from the Langlands correspondence with $\mathbf{C}$-coefficients. Namely, when dealing with groups other that $\mathbf{GL}_n$, local Galois parameters correspond to "$L$-packets" on the representation theory side. These are finite sets of smooth $\mathbf{U}_n(F)$-representations characterized by a character identity, and are relevant in the theory of automorphic base change and functoriality over $\mathbf{C}$. In particular, as inertial Galois parameters are the key ingredients used to construct "Breuil–Mézard cycles" on the aforementioned local models, it is now natural to consider *packets* of Bushnell–Kutzko types, i.e., the smooth, tame $\mathbf{U}_n(\mathcal{O}_F)$-representations appearing in the constituents of an $L$-packet, and how these inertial packets interact with base change to $\mathbf{GL}_n$. Note that for representations of finite groups of Lie type there is an independent notion of base change, the so called *Shintani lift*, which is given in purely representation theoretic terms. In particular, Shintani lifting is naturally well-suited for combinatorics à la Breuil–Mézard. (It is indeed a conjecture, verified in low-rank cases [AL05], that the Shintani lift is compatible with automorphic base change.)

Note that the interaction of packets of Bushnell–Kutzko types and automorphic base change gives us the intriguing opportunity to formulate a geometric version of the functoriality principle in terms of the Breuil–Mézard conjecture and Emerton–Gee stacks: we expect the natural morphism between $C$-dual groups ${}^C\mathbf{U}_n \to {}^C\mathbf{GL}_n$ to induce morphisms $M(\lambda)^{\nabla_\tau}_{\mathbf{U}_n} \to M(\mathrm{BC}(\lambda))^{\nabla_{\mathrm{BC}(\tau)}}_{\mathbf{GL}_n}$ between local models of [LLHLM] and the stacks $\overline{\mathcal{X}}_{\mathbf{U}_n} \to \overline{\mathcal{X}}_{\mathbf{GL}_n}$ of [EGb]. In particular, part of our project would consist in understanding how the notions of Breuil–Mézard cycles interact with the map of cycles induced by the morphisms above.

## 3. Results

The main focus of our research activity at BIRS consisted in understanding the combinatorics of $L$-packets and how the Bushnell–Kutzko types interact with base change. Our starting point was the paper of DeBacker–Reeder [DR09], which contains an exhaustive study of $L$-packets associated to *tame regular semisimple elliptic* Galois parameters $\phi$, for a wide class of $p$-adic Lie groups (including our $\mathbf{U}_n$). These $L$-packets are constructed by inducing certain Deligne–Lusztig representations

$\sigma(\phi)$ (of certain varying compact subgroups), and the ellipticity condition on $\phi$ ensures that $\sigma(\phi)$ is cuspidal. Consequently, the DeBacker–Reeder construction produces, starting from $\sigma(\phi)$, a "supercuspidal" representation $\pi$ in the $L$-packet $\Pi_\phi$ associated to $\phi$. (It is conjectured -and likely- that the DeBacker–Reeder construction does indeed give the local Langlands correspondence for $\mathbf{U}_n$ with $\mathbf{C}$-coefficients.)

Given this construction, we examined the tame $\mathbf{U}_n(\mathcal{O}_F)$-representations contained in the $L$-packet $\Pi_\phi$. Making a detailed study of the construction of [DR09], we sketched a strategy (and verified it in several low-rank cases) for the proof of the following fact: if $\phi$ is as above, then the packet $\Pi_\phi$ contains a *unique* tame $\mathbf{U}_n(\mathcal{O}_F)$-representation $\sigma$. Moreover, if we let $\mathrm{BC}(\Pi_\phi)$ denote the automorphic base change of $\Pi_\phi$ to $\mathbf{GL}_n(E)$ (where $E$ denotes the quadratic unramified extension of $F$), then $\mathrm{BC}(\Pi_\phi)$ contains a unique tame $\mathbf{GL}_n(\mathcal{O}_E)$-representation, which is precisely the Shintani lift of $\sigma$. We hope to flesh out the ideas of this argument soon.

Aside from the above result, we moreover outlined a strategy to remove the assumption on ellipticity for the inertial Galois parameter, which goes as follows. We checked that a tame regular semisimple Galois parameter $\phi$ actually factors through (the dual group of) a Levi subgroup $M$ of $\mathbf{U}_n(F)$, and decomposes as a direct sum of *elliptic* tame regular semisimple Galois parameter for the Levi blocks. In particular we can construct, via [DR09], an $L$-packet of smooth $M$-representations, which we then parabolically induce to $\mathbf{U}_n(F)$.

The relation between these parabolically induced representations and the $L$-packet of $\phi$ is not immediate. Indeed, the parabolic induction of supercuspidal representation of $M$ need not be irreducible, and the classification of tame regular semisimple Galois parameters given by [SZ18] requires the knowledge of Langlands quotients. Because of this, more work will be required in order to precisely formulate and prove the necessary results.

The "local automorphic" results obtained above are an excellent starting point towards the proofs of Serre conjectures and geometric Breuil–Mézard conjectures for unitary groups. The other ingredients required (such as the relevant calculations in $p$-adic Hodge theory and additional results from the global theory of automorphic forms on unitary groups) constitute some of the other (substantial) pieces, and we plan to pursue these directions further in the future.

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Department of Mathematics, 4843 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, USA
*E-mail address*: `kkoziol@umich.edu`

Laboratoire d'Analyse, Géométrie, Algèbre, 99 avenue Jean Baptiste Clément, 93430 Villetaneuse, France
*E-mail address*: `morra@math.univ-paris13.fr`