# Arithmetic Aspects of Algebraic Groups 20w5133

Mikhail Ershov (University of Virginia),
Alex Lubotzky (Hebrew University of Jerusalem),
Dave Morris (University of Lethbridge),
Gopal Prasad (University of Michigan)

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The theory of arithmetic aspects of algebraic groups is rooted in the study of algebraic groups over global fields. (This is described in the classic monograph of V. Platonov and A. Rapinchuk [3].) A few of the major topics are finiteness theorems, rigidity theorems, the Congruence Subgroup Property, and the Kneser-Tits Conjecture. However, the field has expanded to include analogous results about algebraic groups over more general fields of arithmetic nature.

A typical finiteness theorem establishes that only finitely many algebraic groups have certain specified natural properties. For example, it was known classically that there are only finitely many connected, semisimple algebraic $\mathbf{R}$-groups of any given dimension. Such finiteness results over global fields can often be interpreted as the finiteness of the kernel of a homomorphism between certain Galois cohomology groups (cf. [3, Chap. 6]).

Rigidity theorems come in many forms, but an interesting special class of results show (roughly speaking), for certain interesting subgroups $H$ of $G$, that every homomorphism defined on $H$ can be extended to be defined on all of $G$. (See, for example [4].)

If $G$ is an algebraic $\mathbf{Q}$-group, then the group $\Gamma = G_{\mathbf{Z}}$ of all integer points of $G$ has a family of natural finite-index subgroups, called *principal congruence subgroups*. Roughly speaking, the *Congruence Subgroup Property* is the assertion that every finite-index subgroup of $\Gamma$ contains at least one of these principal congruence subgroups [5]. It is still an open problem to determine which arithmetic groups have this property, and this problem has been generalized to certain other situations in which a group has a natural collection of finite-index subgroups. (See, for example, [1].)

It was conjectured more than 60 years ago that if $G$ is an isotropic, simply-connected, almost-simple algebraic group over a field $K$, then the central quotient $G_K/Z(G_K)$ is simple as an abstract group. This is known to be true if $K$ is a local field or global field, but counterexamples over certain other fields are known, and many cases remain open. This is known as the Kneser-Tits Conjecture [2].

The research area of the workshop has close connections to the theory of division algebras. (See, for example, [6].) Slightly more than a third of the talks in the workshop were specifically devoted to Brauer groups or other aspects of this fundamental area: "$SK_1$ triviality for $l$-torsion algebras over $p$-adic curves — a proof sketch" by Nivedita Bhaskhar (University of Southern California), "Polynomials over central division algebras" by Eli Matzri (Bar-Ilan University, Israel), "The unramified Brauer group" by Raman Parimala (Emory University), and "The generic Clifford algebra and its Brauer class" by Charlotte Ure (University of Virginia).

Half of the other talks discussed recent results on finiteness or rigidity: "Rigidity for unirational groups" by Zev Rosengarten (Hebrew University, Israel), "A finiteness theorem for special unitary groups of quaternionic skew-hermitian forms with good reduction" by Srimathy Srinivasan (University of Colorado), and "Superrigidity in rank one" by Matthew Stover (Temple University).

The remaining three talks discussed the Congruence Subgroup Problem, the Kneser-Tits Conjecture, and connections with mathematical logic: "The Congruence Subgroup Problem for automorphism groups" by David El-Chai Ben-Ezra (Hebrew University, Israel), "On the Tits-Weiss conjecture on $U$-operators and the Kneser-Tits conjecture for some groups of type $E_7$ and $E_8$" by Vladimir Chernousov (University of Alberta), and "Mathematical logic and its applications in arithmetics of algebraic groups and beyond" by Jinbo Ren (University of Virginia).

The workshop provided an opportunity for the community to learn about interesting recent work of strong young researchers who might not have had another outlet to obtain international exposure during the pandemic. There was also one keynote address by a leading senior researcher each day (Vladimir Chernousov and Raman Parimala). Each talk was followed by a question period, which was often quite active, and chat rooms were provided for other discussions. Even so, the level of interaction was much less than at a typical workshop, so we are very much looking forward to the in-person workshop that is scheduled to run at BIRS in 2022.

## References

[1] G. A. Fernández-Alcober, A. Garrido, and J. Uria-Albizuri, On the congruence subgroup property for GGS-groups. *Proc. Amer. Math. Soc.* **145** (2017), 3311–3322.

[2] P. Gille, Le problème de Kneser-Tits. Séminaire Bourbaki #983 (2007/2008), 39–81.

[3] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*. Academic Press, Inc., Boston, MA, 1994.

[4] I. Rapinchuk, On abstract homomorphisms of Chevalley groups over the coordinate rings of affine curves, *Transform. Groups* **24** (2019), 1241–1259.

[5] B. Sury, *The congruence subgroup problem*. Texts and Readings in Mathematics, Hindustan Book Agency, New Delhi, 2003.

[6] A. Weil, Algebras with involutions and the classical groups. *J. Indian Math. Soc.* **24** (1960), 589–623.