BIRS Workshop
Differential Invariants and
Invariant Differential Equations

July 19-24, 2003

Abstracts
(in alphabetical order by speaker surname)

Speaker: **Stephen Anco** (Brock University)
Title: *Applications of frames to nonlinear wave equations and integrable evolution equations*
Abstract: Geometric frames have played a key role in much recent work in analysis of the Cauchy problem for nonlinear geometrical wave equations, particularly sigma models with Riemannian target spaces. In other recent work, a frame/connection formulation of arclength-preserving curves in Riemannian geometries has been shown to give a geometrical derivation of integrable evolution equations, such as vector mKdV equations, and their associated Hamiltonian operator structures. In this talk I plan to discuss the relationship between these two applications of frames from a common point of view, giving more insight into some further aspects of that work. One result is I will show how this relationship leads to a simple geometrical origin for vector sine-Gordon equations along with their Hamiltonian structure, as integrable evolution equations derived from arclength-preserving curves.

Speaker: **Ian M. Anderson** (Utah State University)
Title: *Nonlinear Superposition Principles for Exterior Differential Systems*
Abstract: In this talk I shall present a general framework, within the theory of exterior differential systems, for generalizing the familiar superposition principle for linear differential equations to nonlinear differential equations. The focus of the talk will be on examples of nonlinear superposition principles for a wide range of partial differential equations and on the group theoretical origins of these superposition principles. These examples suggest a host of open research problems and, in particular, suggest deep new relationships between explicitly integrable exterior differential systems and group actions on jet spaces. Current progress on these research problems will be summarized.

Speaker: **Robert Bryant** (Duke University)
Title: *Path geometry, Finsler metrics and exotic holonomy*
Abstract: I will review some of my recent work in Finsler geometry that has uncovered some interesting tiees with both path geometry and some heretofor overlooked examples of holonomy groups of torsion-free connections. In particular, I will explain how Cartan's generalization of Lie's third theorem allows one to understand the generality of the space of Finsler metrics of constant flag curvature and how the analysis of the geometry of these equations leads to some very interesting Kahler metrics on the complex n-quadric.

Speaker: **Boris Doubrov** (ISLC)
Title: *Generalization of Wilczynski invariants to nonlinear ODEs*
Abstract: E. Wilczynski described in [1] all invariants of linear ODEs $y^{(n)}+p_{n-1}(x)y^{(n-1)}+\cdots+p_0(x)y=0$ considered up to the pseudogroup of transformations $(x,y)\mapsto(\lambda(x),\mu(x)y)$. Using the universal linearization of ordinary differential equations, we show how these invariants can be generalized to invariants of arbitrary ODEs viewed up to contact transformations.

Next, we discuss the role of these invariants in the canonical coframe for ordinary differential equations (see [2]). In particular, we show that if all these invariants vanish, then all other invariants of the canonical coframe become first integrals of the original equation.

Further, using the technique developed in [3], we show how to constuct ODEs with vanishing Wilczynski invariants and present several non-trivial examples.

Finally, we also mention how this invariants appear in the inverse problem of variational calculus for ODEs.

References

[1] E.J. Wilczynski, *Projective differential geometry of curves and ruled surfaces*, Leipzig, Teubner, 1905.

[2] B. Doubrov, B. Komrakov, T. Morimoto, *Equivalence of holonomic differential equations*, Lobachevsky Journal of Mathematics, v.3, 1999, pp.39-71.

[3] R. Bryant, *Two exotic holonomies in dimension four, path geometries, and twistor theory*, Proc. Symp. Pure Math. **53** (1991), pp. 33-88.

Speaker: **Michael Eastwood** (University of Adelaide)
Title: *Higher symmetries of the Laplacian*
Abstract: Which linear differential operators preserve harmonic functions? Even on Euclidean space, this is a deceptively simple question. The answer may be expressed in terms of conformal geometry and the AdS/CFT correspondence.

Speaker: **E.V. Ferapontov** (Loughborough University)
Title: *On integrability of (2+1)-dimensional quasilinear systems*
Abstract: A (2+1)-dimensional quasilinear system is said to be 'integrable' if it can be decoupled in infinitely many ways into a pair of compatible n-component one-dimensional systems in Riemann invariants. Exact solutions described by these reductions, known as nonlinear interactions of planar simple waves, can be viewed as natural dispersionless analogs of n-gap solutions. It is demonstrated that the requirement of the existence of 'sufficiently many' n-component reductions provides the effective classification criterion. As an example of this approach a complete classification of integrable (2+1)-dimensional systems of conservation laws possessing a convex quadratic entropy is obtained.
Reference:

[1] F. Finkel, D. Gomez-Ullate, A. Gonzalez-Lopez, M.A. Rodriguez and R. Zhdanov, On the Sutherland spin model of B_N type and its associated spin chain, Commun. Math. Phys. 233 (2003) 191.

Speaker: **Federico Finkel** (Universidad Complutense)
Title: *On the Sutherland model of BC_N type and its associated spin chain*

Abstract: In this talk, we survey recent results on the Sutherland spin model of BC_N type and its associated spin chain. We shall see that the complete integrability and exact solvability of this model can be established by relating it to a set of differential-difference operators known as Dunkl operators. These operators — originally introduced by Dunkl in connection with the theory orthogonal polynomials in several variables— leave invariant a flag of finite-dimensional spaces of smooth functions and satisfy some remarkable algebraic properties. We show that the spectrum and a complete set of integrals of motion for the spin chain associated to the BC_N Sutherland model can be derived using the so-called Polychronakos's "freezing trick".
Reference:
[1] F. Finkel, D. Gomez-Ullate, A. Gonzalez-Lopez, M.A. Rodriguez and R. Zhdanov, On the Sutherland spin model of B_N type and its associated spin chain, Commun. Math. Phys. 233 (2003) 191.

Speaker: **Peter E. Hydon** (University of Surrey)
Title: *Difference Forms*
Abstract: Difference equations on rectangular or cubic lattices arise in many applications, most notably in the discretization of continuous systems. In many instances, it is helpful to abandon the idea that the independent variables are points in a continuum; instead, we use only the natural ordering that is inherited from the integers. This leads to an analogue of the exterior calculus, based on difference forms rather than differential forms. There are modified versions of Stokes' Theorem and the de Rham cohomology. We show how these constructions can be interpreted for various finite difference schemes, including collocation methods. We also describe a systematic method of calculating the simplest nontrivial cohomology groups, namely those of rectangular lattices with holes. This method and its extensions should prove useful in image analysis.

Speaker: **Thomas A. Ivey** (College of Charleston)
Title: *Closed, Knotted and Symmetric Solutions of the Vortex Filament Equation*
Abstract: The vortex filament flow is an evolution equation for curves in three-dimensional space which is a geometric counterpart to the nonlinear Schrodinger equation (NLS). After reviewing the passage from a space curve γ to a potential q satisfying NLS (and vice-versa), I will discuss conditions on the spectrum of q that guarantee the closure of γ. For finite-gap potentials, the simple points of the spectrum are the branch points of a hyperelliptic curve which, along with a choice of divisor, determines q.

I will also report on joint work with Annalisa Calini concerning the connection between this algebro-geometric data and the topology and geometry of γ. This is completely understood in genus one, where γ is the centerline of a Kirchhoff elastic rod. Moreover, we can use isoperiodic deformations to construct knotted solutions of higher genus, which suggests a scheme for labeling the knot type by the spectrum. I will also discuss the connection between symmetric spectra and self-intersecting symmetric filaments in genus two and three.

Speaker: **Joel Langer** (Case Western Reserve University)

Title: *Schwarz reflection geometry*
Abstract: Schwarzian reflection of analytic curves in the complex plane is used to define a (formal, infinite dimensional) symmetric space geometry. A continuous limit of iterated Schwarzian reflection yields the geodesic equation, a second order PDE describing conformally invariant evolution of analytic plane curves. Local behavior of geodesics is illustrated by examples and analyzed generally in terms of meromorphic differentials and the Witt algebra.

Speaker: **Boris Kruglikov** (University of Tromsoe)
Title: *Higher Mayer brackets and integration of PDEs via auxiliary integrals*
Abstract: We formulate a new compatibility criterion of overdetermined systems of (nonlinear) partial differential equations. It involves the notion of the Mayer bracket, which is a reduction of the higher Jacobi bracket on the infinite jet-space. Based on it, we propose a new method of reduction of partial differential equations. Namely we search for an additional equation, called an auxiliary integral, which is compatible with the given one. This method generalizes the classical methods of symmetries and intermediate integrals. We apply it to the WDVV equations and the projective differential geometry.

Speaker: **Elizabeth Mansfield** (University of Kent at Canterbury)
Coauthor: Peter Hydon (University of Surrey)
Title: *Difference Forms – from local to global*
Abstract: Many physical systems are modelled on spheres, tori (obtained when periodic BCs are imposed), or in a space with "holes" (such as fluid flow around a surface or body). Finite difference systems approximating such models need to have the same global qualities as the underlying model for a correct qualitative fit. This talk explores analogues for difference forms of some well known results in the de Rham theory. In particular we show an analogue of the celebrated Weil proof that the Cech and de Rham cohomologies coincide for spaces with a good cover. Thus, we have a computational mechanism for calculating difference cohomology classes. Even more importantly though, we can see that if the combinatorics of the covers of the underlying smooth and lattice spaces are the same, then the global features of systems of equations defined on them will match.

This talk is a companion talk to Peter Hydon's which preceeds it. It can be understood on its own but will rely on results given in Peter Hydon's talk and will be informed by the issues discussed there.

Speaker: **Gloria Mari-Beffa** (University of Wisconsin)
Title: *Poisson structures associated to the geometry of curves in manifolds with a conformal structure.*
Abstract: In this talk we will describe an invariant moving frame along a curve in a manifold with a conformal structure. We will use the frame to show that there exists a Poisson bracket defined on the space of conformal differential invariants of curves. A conformally invariant flow of curves naturally induces an evolution on the differential invariants of the flow. A conformally invariant flow of curves naturally induces an evolution on the differential invariants of the flow. We will give the explicit conditions that guarantee that the evolution of the invariants is Hamiltonian with

respect to the bracket. If time allows it, we will comment on the implications for completely integrable systems.

Speaker: **Peter J. Olver** (University of Minnesota)
Title: *Moving Frames for Pseudogroups*
Abstract: This talk is the first of two devoted to new computational algorithms for infinite-dimensional Lie pseudo-group actions; the second part will be presented by Juha Pohjanpelto.

In my talk, I will review the basic ideas behind the equivariant theory of moving frames and applications to the classification of differential invariants, the invariant bicomplex, equivalence and symmetry problems, and the calculus of variations. I will then present a new, direct construction of invariant Maurer–Cartan forms for infinite-dimensional pseudo-groups, which is a prerequisite for the development of the moving frame theory to be presented in Juha's talk.

Speaker: **Juha Pohjanpelto** (Oregon State University)
Title: *Differential Invariants for Pseudogroup Actions on Submanifolds*
In this talk I will report on my ongoing joint work with Peter Olver on developing systematic and constructive algorithms for the identification of various invariants for the action of infinite dimensional pseudogroups. I will focus on the action of a pseudogroup $\mathcal{G}$ the submanifolds of a given manifold. In this context the groupoid structure of $\mathcal{G}$ naturally gives rise to moving frames as equivariant sections of the associated action groupoid. I will survey some important applications of moving frames, in particular, an invariantization process by which one can systematically produce a complete set of differential invariants and an invariant coframe for the action of $\mathcal{G}$ on the jets of submanifolds, as well as recurrence relations amongst these invariant quantities. Moreover, I will describe a practical algorithm based on Taylor series expansions for constructing various invariants for the action of $\mathcal{G}$.

Speaker: **Greg Reid** (University of Western Ontario)
Title: *Symmetries and Topological Methods in Numerical Jet Geometry*
Recent work has initiated a study of Numerical Jet Geometry, or the study of jet spaces under deformations. The symbolic analysis of exact systems of PDE, using tools from differential algebra (radical differential ideals, primary decompositions) has made much progress in recent times. However such methods lack the properties of continuity and stability for approximate systems. Topological and continuous tools are more suited for approximate systems. Specifically we represent the jet components (or submanifolds) by generic points on each component, using recent developments in Numeric Algebraic Geometry. These points are cut out by intersection with random linear subspaces and computed using Homotopy continuation methods, the number of such points being the degree of a component. Specific applications discussed include:

- application to computer vision
- determination of approximate symmetries of PDE

Speaker: **Jan A. Sanders** (Vrije Universiteit)
(work with Jing Ping Wang)
Title: *Cartan geometry and integrability: the conformal case*

Abstract: We show that if we write down the structure equations for the flow of the *parallel* connection of a curve embedded in a flat n-dimensional conformal manifold leads to two compatible Hamiltonian operators. The corresponding integrable scalar-vector equation is known from earlier classification results. These results are similar to those we obtained in the Riemannian case, implying that the method employed is well suited for the analysis of the connection between geometry and integrability.

Speaker: **Harvey Segur** (University of Colorado)
Title: *Partial differential equations with multi-symplectic structure*
Abstract: This is a work in progress. T. Bridges and others have observed that some Hamiltonian systems of partial differential equations are Hamiltonian either in time, or in some spatial directions. This extra structure can have practical value in stability calculations, as we demonstrate in a concrete example (the nonlinear Schrodinger equation in two or more dimensions). This extra Hamiltonian structure is sometimes present, and sometimes not. We have not figured out why.

Speaker: **Kaleem Siddiqi** (McGill University)
Title: *Flux Invariants for Shape*
Abstract: We consider the average outward flux through a Jordan curve of the gradient vector field of the Euclidean distance function to the boundary of a 2D shape. Using an alternate form of the divergence theorem, we show that in the limit as the area of the region enclosed by such a curve shrinks to zero, this measure has very different behaviours at medial points than at non-medial ones, providing a theoretical justification for its use in our Hamilton-Jacobi skeletonization algorithm. We then specialize to the case of shrinking circular neighborhoods and show that the average outward flux measure also reveals the object angle at skeletal points. Hence, formulae for obtaining the boundary curves, their curvatures, and other geometric quantities of interest, can be written in terms of the average outward flux limit values at skeletal points. Thus this measure can be viewed as a Euclidean invariant for shape description: it can be used to both detect the skeleton from the Euclidean distance function, as well as to explicitly reconstruct the boundary from it. We illustrate our results with several numerical simulations. Joint work with Pavel Dimitrov, School of Computer Science, McGill University and James N. Damon, Department of Mathematics, UNC Chapel Hill.

Speaker: **Keti Tenenblat** (Universidade de Brasilia)
Title: *On Differential Systems Describing Surfaces of Constant Curvature*
The geometric notion of a differential system describing surfaces of constant nonzero Gaussian curvature is introduced. The nonlinear Schrödinger equation (NLS) with $\kappa = 1$ and $\kappa = -1$ are shown to describe a family of spherical surfaces (s.s.) and pseudospherical surfaces (p.s.s.) respectively. The Schrödinger flow of maps into S^2 (the HF model) and its generalized version, the Landau-Lifschitz equation, are shown to describe spherical surfaces. The Schrödinger flow of maps into H^2 (the M-HF model) provides another example of a system describing pseudo-spherical surfaces. New differential systems describing surfaces of nonzero constant Gaussian curvature are obtained. Furthermore, we give a characterization of evolution systems which

describe surfaces of nonzero constant Gaussian curvature. In particular, we determine all differential systems of type

$$\begin{cases} u_t = -v_{xx} + H_{11}(u,v)u_x + H_{12}(u,v)v_x + H_{13}(u,v) \\ v_t = u_{xx} + H_{21}(u,v)u_x + H_{22}(u,v)v_x + H_{23}(u,v), \end{cases}$$

which describe η-pseudospherical or η-spherical surfaces. As an application, we obtain 4-parameter family of such systems for a complex valued function $q = u + iv$ given by

$$iq_t + q_{xx} \pm i\gamma(|q|^2 q)_x - i\alpha q_x \pm \sigma |q|^2 q - \beta q = 0,$$

where $\sigma \geq 0$ if $\gamma = 0$. Particular cases of this family, obtained by the vanishing of the parameters, are the linear equations, the NLS equation, the derivative nonlinear Schrödinger equation (DNLS) and the mixed NLS-DNLS equation.

Speaker: **P. Winternitz** (Centre de recherches mathematiques)
Title: *Lie symmetries, Lagrangians, Conservation Laws and Solutions of Difference Equations*
Abstract: The main part of the lecture will be devoted to Lie point symmetries of difference schemes involving one dependent and one independent variable. The symmetries will act on the equation and on the lattice. We will show that if the lattice is suitably adapted, difference schemes will have essentially the same symmetries as differential equations obtained as their continuous limits. We will show how symmetries can be used to classify difference schemes, to decrease the order of the scheme, and to obtain exact solutions. Variational symmetries are particularly useful in this context. Consequences for numerical analysis will be discussed, as well as generalizations to multivariable difference schemes. Finally we will show that if we wish to obtain interesting symmetry results for difference equations on fixed lattices, we must go beyond the concept of point symmetries.

Speaker: **Thomas Wolf** (Brock University)
Title: *Two classifications of integrable systems based on higher order symmetries and first integrals*
Abstract: The existence of higher order symmetries or first integrals as a main tool in a classification programme of integrable systems leads in its simplest form to large bi-linear algebraic conditions for unknown coefficients in a homogeneous ansatz. In the talk two classification programmes are described.

In the first classification linear Poisson brackets on e(3) typical of rigid body dynamics are considered. All quadratic Hamiltonians of Kowalevski type having additional first integral of third or fourth degree are found and quantum analogs of these Hamiltonians are given. This is work done together with Olya Efimovskaya.

In the second programme several classes of systems of evolution equations with one or two vector unknowns have been considered together with Vladimir Sokolov. Especially rich results gave the classification of systems with one vector and one scalar unknown together with Takayuki Tsuchida. For these classes all equations having the simplest higher symmetry are given together with some of their relations, like Miura-type transformations.

If time permits comments are made about computer algebra programs written to solve the bi-linear systems of conditions.

Speaker: **Keizo Yamaguchi** (Hokkaido University)
Title: *Pseudo-projective systems of higher order*
Abstract: The aim of this talk is to give an overview on the geometry of ordinary differential equations or more generally systems of (partial) differential equations of finite type and to discuss the invariants of these systems.

Historically, geometric study of differential equations, especially ordinary differential equations, was initiated by Sophus Lie [Lie91]. For linear ordinary differential equations, Laguerre and Forsyth studied the differential invariants of these equations by transforming them to the canonical forms (cf. [Wil06]).

For higher order equations, after Lie, the classification of the second order ordinary differential equations by point transformations was achieved by Tresse [Tre96] and E. Cartan [Car24] studied the case when the equation is associated with paths in projective geometry by his method of the equivalence. The third order equations were studied by S.S.Chern [Ch50], following the method of E.Cartan (cf. [SY98]). Then N.Tanaka [Tan82] studied the equivalence problem for the system of second order ordinary differential equations by point transformations and formulated this geometry in terms of the pseudo-product structures. Furthermore he constructed normal Cartan connections on these systems and utilized the connections to the normal form problem and the integration problem of these systems [Tan79] [Tan89].

In this talk we adopt the point of view initiated by N.Tanaka. Explicitly let us consider a system of higher order differential equations of finite type of the following form :

$$\frac{\partial^k y^\alpha}{\partial x_{i_1}\cdots\partial x_{i_k}} = F^\alpha_{i_1\cdots i_k}(x_1,\dots,x_n,y^1,\cdots,y^m,\dots,p_i^\beta,\dots,p^\beta_{j_1\cdots j_{k-1}})$$
$$(1 \leqq \alpha \leqq m, 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n),$$

where $p^\beta_{i_1\cdots i_l} = \frac{\partial^l y^\beta}{\partial x_{i_1}\cdots\partial x_{i_l}}$. Namely let us consider a system R of k-th order equations such that every k-th derivative is expressed in terms of the derivatives of the lower order. If we regard R as a submanifold of the k-jet space J^k with coordinates $(x_1,\dots,x_n,y^1,\cdots,y^m,\dots,p_i^\beta,\dots,p^\beta_{j_1\cdots j_k})$, then R is diffeomorphic to J^{k-1}. Moreover R specifies an n-dimensional subspace $E(x)$ of $C^{k-1}(x)$ at each point x of J^{k-1}, where C^{k-1} is the **canonical differential system** on J^{k-1} . Thus R defines a differential system E on J^{k-1} such that $C^{k-1} = E \oplus F$, where $F = \mathrm{Ker}(\pi^{k-1}_{k-2})_*$ and $\pi^{k-1}_{k-2} : J^{k-1} \to J^{k-2}$ is the projection. E is completely integrable when the system R is integrable. The triplet $(J^{k-1}; E, F)$ is called the **pseudo-projective structure** associated with R.

Geometry of Pseudo-projective system $(J^{k-1}; E, F)$ is nothing but the geometry of systems R of k-th order equations under point or contact tranformations. We will dicuss the invariants of these systems utilizing Cartan connections constructed by N.Tanaka and T.Morimoto for these systems.

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