# The Inverse Polynomial Reconstruction Method for Discontinous Problem

Jae-Hun Jung

April 15, 2004

The numerical approximation of discontinuous functions is one of the most challenging problem of modern numeical analysis. When a function $f(x)$ has a jump discontinuity, any global approximation suffers from Gibbs oscillations. For example, a Fourier approximation of $f(x)$ shows an oscillatory behaviour in the neighborhood of the domain boundaries, $\partial\Omega$ if $f(x)$ is analytic but not periodic in the given domain $\Omega$, and it also shows highly oscillatory behaviour in the neighborhood of any discontinuity if it exists in $\Omega$ [3]. When orthogonal polynomials are used as basis functions, the same Gibbs oscillations are found in the vicinity of the discontinuites. The convergence order of such approximations are only of $O(1)$ in the $L_\infty$ norm although they are pointwise convergent.

Reducing such Gibbs oscillations is an important task when one seeks numerical solutions of time dependent nonlinear PDEs such as nonlinear hyperbolic conservation laws, $u_t + f(u)_x = 0$. Although the initial condition is $C^\infty$, no regular solution can be anticipated in a finite time. Numerical approximations of such solutions are highly oscillatory due to the Gibbs oscillations induced by the nonsmoothness of the function, and such oscillations destroy the stability. The most popular algorithm relies on filtering out the high frequency components and considerably reducing the over or undershoots near the discontinuities. However, the convergence in $L_\infty$ norm still remains $O(1)$ although it is exponential in the region away from the discontinuities. In the early 90's, a method now called Gegenbauer Reconstruction Method was developed to resolve the Gibbs Phenomenon [1] [2]. Gottlieb and coworkers showed that if there exists a set of orthogonal polynomials which satisfies the Gibbs condition [1], one can recover the original spectral convergence from the Fourier information, i.e. $f_N(x)$. It is not trivial to find such polynomials. Gottlieb and coworkers showed that Gegenbauer polynomials, $C_l^\lambda(x)$ which has a weight function $w(x) = (1-x^2)^{\lambda-\frac{1}{2}}$ satisfy the Gibbs condition. Using $C_l^\lambda(x)$ as a basis set, they proved that the reconstruction is spectrally accurate with $N$. However, this spectral convergence is obtained only if the dimension of the Gegenbauer space $m$ and the parameter $\lambda$ satisfy certain conditions.

We recently developed a new reconstruction method, named as an inverse polynomial reconstruction method [4], [5]. This method can use any polynomial set and there is no additional conditions for the spectral convergence. The main idea of this method is that we seek a polynomial such that the residue between the Fourier projection $f_N(x)$ of $f(x)$ and the Fourier projection $f_N^m(x)$ of our seeking polynomial $f^m(x)$ is minimized in the Fourier or the polynomial spaces. By doing this, one can easily find a polynomial representation for which the truncation error, $TE$ and the regularization error $RE$ have the same size in the Fourier space $\mathbf{F}_N$, and this yields spectral convergence of the inverse method. Since there is no condition on $\lambda$, one can easily show that the inverse method is basis independent as long as the basis is a polynomial. Moreover, if $f(x)$ is a polynomial, its inverse reconstruction $f^m(x)$ is exact. By minimizing the residue between $f_N(x)$ and $f_N^m(x)$, the inverse method is reduced to a linear system, $W \cdot g = \hat{f}$ where $W$ is a transformation matrix from Fourier space $\mathbf{F}_N$ to Gegenbauer space $\mathbf{G}_m$, $g$ are the expansion coefficients of $f^m(x)$ and $\hat{f}$ are the Fourier coefficients. Thus the inverse method involves an inversion of $W$.

We will discuss some numerical aspects of the inverse method, such as ill-posedness of $W$, and consistency of $W$ and $\hat{f}$. Numerical examples of the inverse method for some PDEs and image reconstruction will also be presented.

## References


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