Regularity criteria for the "suitable weak solutions" to the Navier-Stokes equations near boundary

S. Gustafson,[*] K. Kang,[†] and T. Tsai[‡]

We study the boundary regularity problem of "suitable weak solutions" to the Navier-Stokes equations

$$\left\{ \begin{array}{c} u_t + \Delta u + (u \cdot \nabla) u - \nabla p = f \\ \text{div u} = 0 \end{array} \right. \qquad \text{in } \; Q_T = \Omega \times (0, T), \tag{1}$$

where Ω is a domain in $\mathbb{R}^3$. Here we mean by "suitable weak solutions", simply speaking, that they solve the Navier-Stokes equations in the sense of distribution and satisfy the local energy inequality.

Caffarelli-Kohn-Nirenberg proved a sufficient condition for boundedness of "suitable weak solutions" in the interior case. To be more precise, their regularity theorem reads as follows: Let u be a "suitable weak solution" of the Navier-Stokes equations and let $z_0 = (x_0, t_0) \in Q_T$. There exists an absolute constant $\epsilon > 0$ such that if

$$\limsup_{r \to 0} \frac{1}{r} \int_{Q_{z_0, r}} |\nabla u(x,t)|^2 dx dt \leq \epsilon, \tag{2}$$

then u is regular in a neighborhood of z_0.

Recently, the interior partial regularity result were extended up to the boundary, that is, the same condition as (2) is sufficient for local Hölder continuity for "suitable weak solutions" to the Navier-Stokes equations near the plane boundary. More precisely, G. Seregin proved that for a "suitable weak solution" u, there exists an absolute constant $\epsilon > 0$ such that if

$$\limsup_{r \to 0} \frac{1}{r} \int_{Q^+_{z_0, r}} |\nabla u(x,t)|^2 dx dt \leq \epsilon, \tag{3}$$

then u is Hölder continuous up to boundary near $z_0 = (x_0, t_0) \in \Gamma \times (0, T)$, where Γ is the plane boundary of Ω and $Q^+_{z_0,r} = Q_{z_0,r} \cap Q_T$. Therefore, "suitable

[*]Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, B.C., Canada V6T 1Z2 (gustaf@math.ubc.ca).

[†]Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, B.C., Canada V6T 1Z2 (kkang@pims.math.ca).

[‡]Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, B.C., Canada V6T 1Z2 (ttsai@math.ubc.ca).

weak solutions" are Hölder continuous up to the boundary away from a closed set $\mathcal{S} \subset \bar{Q_T}$ with $\mathcal{P}^1(\mathcal{S}) = 0$, where $\mathcal{P}^1$ is a one dimensional parabolic Hausdorff measure.

The objective of our project is to present new sufficient conditions for the regularity of "suitable weak solutions" to the Navier-Stokes equations near the plane boundary. Our main result is that instead of the assumption of the condition (3), Hölder continuity of u near boundary can be achieved by the smallness of the scaled mixed $L^{l,m}$-norm of velocity field u. More precisely, there exists an absolute constant $\epsilon > 0$ such that if for any p, q with $3/p+2/q = 2$ and $2 < q < \infty$

$$\limsup_{r \to 0} \frac{1}{r} \left(\int_{t-r^2}^{t} \left(\int_{B^+_{x,r}} |u(y,s)|^p dy \right)^{\frac{q}{p}} ds \right)^{\frac{1}{q}} \leq \epsilon, \tag{4}$$

then u is Hölder continuous up to boundary near $z_0 = (x_0, t_0) \in \Gamma \times (0, T)$. We remark that this regularity criteria is also valid in the interior case.

Previous results along the lines of our approach were obtained for the interior case by Tian and Xin. They showed that a "suitable weak solution" u is regular near an interior point $z_0 \in Q_T$ under the assumption that the scaled L^3-norm of u is small, i.e. $\sup_{r \leq r_0} (1/r^2) \int_{Q_{z_0}} |u|^3 \leq \epsilon$.

Since the condition (4) at an interior point also guarantee the regularity of "suitable weak solutions", our result also improve their result, because, in our case, $p = q = 5/2$ in (4) in case p and q are the same.

The main tools of our analysis are a standard "blow up" method and the decomposition of pressure, which enable us to have the decay property of the scaled Lebesgue norms of velocity and pressure in the case of both interior and boundary. Combining the local estimate of the Stokes system for the pressure, we can have, near boundary, the estimate of pressure for the Navier-Stokes equations, which is the crucial part of our analysis.