An introduction to Gromov-Witten invariants

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Enumerative algebraic geometry concerns questions about how to count certain objects. A classic example is an old nineteenth century question; given four general lines in three dimensional space, how many lines are there that meet all four.

Many such questions were answered by Hermann Schubert, but his answers depend on rather dubious assertions; for example, the solution to the question above goes as follows. Suppose two of the lines are coplanar; then we can see two lines – the line from the intersection of these lines meeting the other two and the line connecting the intersections of the other two lines with the plane containing the first two. Therefore there are two lines which meet all four lines.

While it turns out that the answer is correct, it is not clear why the answer should be the same when two of the lines are coplanar; indeed other nineteenth century mathematicians attempted to make similar answers, and were frequently wrong (Schubert himself appears to have had remarkable intuition). Therefore we need more sophisticated techniques for answerig such questions. To do this we use cohomology.

For simplicity we compactify everything and ask about lines in projective three space; the space of lines is then a well studied space (the Grassmannian of two-planes in four space, denoted $G(2,4)$). In particular the space is a smooth closed manifold of dimension 4 (one way to count this is that there are three degrees of freedom for the closest point to the origin and then one for the direction of the line).

Cohomology comes in when we want to talk about the intersections with the other four lines; to do this we introduce a space of decorated lines $\mathcal{M}$ which parametrizes a choice of a line in three space together with points x_1, x_2, x_3, x_4 on the line. The choice of each point gives one degree of freedom so $\mathcal{M}$ is eight dimensional. We also have evaluation maps $ev_i : \mathcal{M} \to \mathbb{P}^3$ which pick out the four special points.

Now consider the class of a line $l \in H_1(\mathbb{P}^3; \mathbb{Z})$, and it's Poincaré dual $L \in H^2(\mathbb{P}^3; \mathbb{Z})$. Then the number

$$\langle [\mathcal{M}], ev_1^*(L) \cdot ev_2^*(L) \cdot ev_3^*(L) \cdot ev_4^*(L) \rangle \in H_0(\mathbb{P}^3; \mathbb{Z}) \cong \mathbb{Z}$$

should count the number of lines passing through four general lines (the condition imposed by the cohomology class $ev_1^*(L)$ should be thought of as imposing the condition that the point x_1 should lie in a specified general line). It is possible to do this calculation explicitly, and it does indeed yield Schubert's answer of two.

Gromov-Witten invariants provide a nonlinear version of this procedure. They turn out to be deformation invariants of smooth projective varieties (or complex manifolds). The technical definition is rather difficult, but the basic idea can be outlined as follows.

Fix a smooth complex variety X and a class $\beta \in H_2(X;\mathbb{Z})$. We can define a space $\mathcal{M}_{g,n}(X,\beta)$ which parametrizes triples $\{C, f, (x_1,\ldots,x_n)\}$, where C is a Riemann surface of genus g, $f : C \to X$ is a holomorphic map and the x_i are points of C. An expected dimension of $(1-g)(\dim X - 3) + \beta \cdot c_1(T_X) + n$ can be computed using some version of an index theorem. Then if we choose classes $\gamma_1,\ldots,\gamma_n \in H^*(X;\mathbb{Z})$ so that the sum of the codimensions is the expected dimension of $\mathcal{M}_{g,n}(X,\beta)$, we can define the Gromov-Witten invariant

$$\langle \gamma_1,\ldots,\gamma_n\rangle^X_{g,\beta} = \int_{[\mathcal{M}_{g,n}(X,\beta)]} ev_1^*(\gamma_1)\cdot \cdots \cdot ev_n^*(\gamma_n) \in H_0(\mathcal{M}_{g,n}(X,\beta);\mathbb{Z}) \cong \mathbb{Z}.$$

The Gromov-Witten invariant should then count curves of genus g in X representing the class β and intersecting cycles dual to the gamma classes.

In practice the invariants are more complicated (they can be rational numbers and even negative), but also turn out to have a very rich structure (much of which is inspired by superstring theory) which turns out to be deeply related to the theory of singularities. If X is a Calabi-Yau threefold the invariants are conjectured to be related to certain gauge-theoretic invariants of real three manifolds; my principal research interest is in investigating this connection.