Analytic and Geometric Aspects of Stochastic Processes

Martin T. Barlow (University of British Columbia)
Alexander Grigoryan (Imperial College) Elton Hsu, (Northwestern)

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The conference was attended by about 35 participants, including Ph.D. students, postdoctoral fellows, young researchers and international leaders in stochastic analyis and related fields. What follows is an attempt to focus on some of the key topics discussed at the meeting both in the lectures and in the informal meeting rooms.

1. Brownian motion and classical analysis

The original development of this area arises from the link between Brownian motion, the heat equation, and classical potential theory. One should see this link as acting in both directions – that is, both sets of objects are of mathematical interest, and one can exploit these connections to (e.g.) use Brownian motion to study harmonic functions on domains in $\mathbb{R}^d$, or properties of the heat equation to refine estimates on Brownian motion. With the late P.A. Meyer, we rather regret the view (common in the USA at points in the last century) that regards work on stochastic processes as only interesting if it leads to some result in analysis.

Chris Burdzy spoke on several problems inspired or related at the technical level to the "hot spots" conjecture of J. Rauch, which was made in 1974. This was that the second Neumann eignefunction attains its maximum at the boundary for every Euclidean domain. Progress on the conjecture was slow, but it is now known (see [2], [1]) that while it is false in general, it is true for some classes of domains. To study this question by probabilisitic methods, one needs to construct reflecting Brownian motion in the domain D. There are various possible constructions; Burdzy discussed a 'strong' construction via the Skorohod equation, and stated a theorem which shows that strong existence and uniqueness hold if the domain is Lipschitz with the Lipschitz constant less than 1.

He then discussed the question of when the second Neumann eigenvalue is simple: this holds in long and thin domains, in domains with bottlenecks, and in lip domains. Next, he considered the location of the nodal line (zero line) of the second Neumann eigenfunction. Coupling methods give some information about the location of this line: for example, in obtuse triangles. Finally he presented an explicit formula for the Lyapunov exponent for the flow of reflected Brownian motions in a smooth domain.

M. van den Berg spoke on properties of the Wiener sausage $W(t)$ in $\mathbb{R}^3$. This is the random set $B(s) + k$,where $B(.)$ is a Brownian motion , $s \in [0, t]$,and where k runs over a fixed compact set K. This volume of this set is of fundamental importance in physics e.g. in the modelling of polymers, diffusion of matter. In probability theory it provides a simple, non trivial example of a non Markovian process. Moreover its expectation shows up in the calculation of the amount of heat which has emanated from the compact K (kept at temperature 1) into its complement (initial temparature 0). F. Spitzer [11], J.F.Le Gall [10], S. Port and others have obtained the first few terms in the asymptotic series for this expectation as $t \to \infty$. Donsker and Varadhan [7] studied the large deviation properties of the volume and van den Berg , Bolthausen and den Hollander [3] studied the moderate deviations of the volume . The results in [3] were then used to obtain the moderate deviations for the intersection volume of two and three independent Wiener sausages.

In his talk he summarised some properties for the expected volume of n Wiener sausages in Euclidean space $\mathbb{R}^m$. The case where $n = 2$ can be reduced to the case of a single sausage. The planar case for any n has been studied extensively by J. F. Le Gall in his St. Flour lecture notes [10]. The cases where $n > 2$ and $m > 3$ gives finite expectation as $t \to \infty$. His main result was for the case $n = m = 3$: the first three terms in the asymptotic series, of order $\log t$, order 1, and order $(\log t)/t^{1/2}$ were obtained together with a remainder $O(t^{-1/2})$ (which is sharp for the ball). The proof relies on repeated use of the strong Markov property together with a last exit time decomposition .

2. Diffusions, stochastic differential equations and calculus on manifolds

The connection between Markov processes, potential theory and differential equations is much broader than just that between Brownian motion and classical analysis. In this section we describe connections between diffusions and various geometric objects.

Thierry Coulhon talked on "Riesz transform on manifolds, and heat kernel regularity": this is joint work with Pascal Auscher, Xuan Thinh Duong, and Steve Hofmann. The aim was to give a necessary and sufficient condition for the two natural definitions of homogeneous first order L^p Sobolev spaces to coincide on a large class of Riemannian manifolds, for p in an interval (q_0, p_0), where $2 < p_0 \leq \infty$ and q_0 is the conjugate exponent to p_0. On closed manifolds, these definitions are well-known to coincide for all $1 < p < \infty$. For non-compact manifolds, and again for $p_0 = \infty$, a sufficient condition was asked for by Robert Strichartz in 1983 and many partial answers have been given since. The condition proposed was in terms of regularity of the heat kernel: more precisely in terms of integral estimates of its gradient. This allowed him to treat manifolds which satisfy the doubling property and natural heat kernel bounds, as well as one with locally bounded geometry where the bottom of the spectrum of the Laplacian is positive.

Bruce K. Driver talked on joint work with his student Tai Melcher (who also attended the meeting), on "Hypoelliptic heat kernel inequalities". In the last twenty years or more, a fairly complete and very beautiful theory has been developed applying to elliptic operators on Riemannian manifolds. This theory relates properties of the solutions of elliptic and parabolic equations to properties of the Riemannian geometry. These geometric properties are determined by the principal symbol of the underlying elliptic operator. The following is a typical example of this type of result:

Theorem 0.1 (Bakry, Ledoux, Emery,...) *Suppose (M, g) is a complete Riemannian manifold, and ∇ and Δ are the gradient and Laplace-Beltrami operators acting on $C^\infty(M)$. Let $|v| := \sqrt{g(v,v)}$ for all $v \in TM$, Ric denote the Ricci curvature tensor, and k denote a constant. Then the following are equivalent:*

1. $\mathrm{Ric}(\nabla f, \nabla f) \geq -2k|\nabla f|^2$ *(or equivalently $\Gamma_2(f,f) \geq -2k\Gamma_1(f,f)$ for all $f \in C_c^\infty(M)$),*
2. $\left|\nabla e^{t\Delta/2} f\right| \leq e^{kt} e^{t\Delta/2} \left|\nabla f\right|$*, for all $f \in C_c^\infty(M)$ and $t > 0$,*
3. $\left|\nabla e^{t\Delta/2} f\right|^2 \leq e^{2kt} e^{t\Delta/2} \left|\nabla f\right|^2$*, for all $f \in C_c^\infty(M)$ and $t > 0$, and*
4. *there is a function $K(t) > 0$ such that $K(0) = 1$, $\dot{K}(0) =: 2k$ exists, and*
$$|\nabla e^{t\Delta/2} f|^2 \leq K(t) e^{t\Delta/2} \left|\nabla f\right|^2, \tag{1}$$
for all $f \in C_c^\infty(M)$ and $t > 0$.

In his talk, we explored the possible of extension of Theorem 0.1 to hypoelliptic operators of the form
$$L = \sum_{i=1}^n X_i^2, \tag{2}$$
where $\{X_i\}_{i=1}^n$ is a collection of smooth vector fields on M satisfying the Hörmander bracket condition. When L is not elliptic, Theorem 0.1 can no longer hold because, roughly speaking, the "Ricci

curvature" is no longer bounded from below. Nevertheless it is reasonable to ask if inequalities of the form (1) might still hold. To be more precise, let $\nabla := (X_1, \dots, X_n)$, $p \in [1, \infty)$ and $t > 0$ and let $K_p(t)$ be the best constant such that

$$|\nabla e^{tL/2} f|^p \leq K_p(t) e^{tL/2} |\nabla f|^p \text{ for all } f \in C_c^\infty(M). \tag{3}$$

The question then becomes; when is $K_p(t) < \infty$? In this regard the following theorem was demonstrated in the talk.

Theorem 0.2 (T. Melcher and B. Driver) *Let G be $\mathbb{R}^3$ (equipped with the Heisenberg group multiplication),*

$$X := \partial_x - \frac{1}{2} y \partial_z, \; Y := \partial_y + \frac{1}{2} x \partial_z \text{ and } L := X^2 + Y^2.$$

Then for all $p \in (1, \infty)$,

1. *$K_p(t)$ is independent of t,*
2. *$K_p(t) = K_p < \infty$,*
3. *$K_p > 1$ and in particular, $K_2 \geq 2$.*

Results analogous to Theorem 0.2 will (in Tai Melcher's thesis) be generalized to any nilpotent Lie group with a collection of left invariant vector fields satisfying Hörmander's condition. The case $p = 1$ is still open.

Yves Le-jan talked on research motivated by physics. There have been very few studies of stochastic processes done in a relativistic framework up to now, at least by mathematicians. The idea was to show that some techniques used to define and study stochastic processes on Riemannian manifolds can be tranferred to the framework of general relativity.

He recalled the definition, due to Dudley, of a relativistic diffusion. He then formulated an SDE representation of the solution using the trivial frame bundle on the Minkowski space. This SDE can then be extended to the general relativistic setting in a canonical way. The exemple of the Swatzschild space was studied in more detail, using several barrier functions to show the transience of the process.

Takashi Kumagai talked on "Characterization of sub-Gaussian heat kernel estimates on graphs and measure metric spaces", based on joint work with M.T. Barlow, R.F. Bass, and T. Coulhon. The motivation of the study of sub-Gaussian heat kernel estimates is from analysis on fractals. It is known that the heat kernels for Brownian motions on various "regular" fractals (such as the Sierpinski gasket) enjoy sub-Gaussian estimates. From the estimates, many properties of the processes can be deduced; for instance, law of iterated logarithms, Green kernel estimates etc. So, it is natural and important to ask whether such estimates are stable under perturbations.

He discussed various conditions which are necessary and sufficient for sub-Gaussian heat kernel estimates to hold:

1. A generalized parabolic Harnack inequality,
2. volume doubling + an elliptic Harnack inequality + some hitting time estimate (or some resistance estimate) (due to Grigor'yan-Telcs, [9])
3. volume doubling + a Poincaré inequality + cut-off Sobolev inequality.

It can be proved that (3) is stable under a bounded perturbation of the operator, and under a rough isometry. Under a stronger volume growth condition, a simpler equivalent condition can be given in terms of electrical resistance. As an application, he described quenched heat kernel estimates for simple random walks on the incipient infinite clusters on Galton-Watson branching processes.

3. Jump processes. From an analytic viewpoint, non-local operators arise when one looks at $-(-\Delta)^\alpha$ for $\alpha \in (0, 1)$. These correspond to jump processes, the most familiar being the class of stable processes.

Zhen-Qing Chen talked on “SDEs Driven by Stable Processes”; joint work with R. Bass and K. Burdzy. Stochastic differential equation (SDE) driven by Brownian motion plays a central role in the theory of modern probability and its applications. In the last few years there has been intensive interest in the study of processes with jumps. Much of the motivation has come from mathematical physics and from financial mathematics: in many applications jump processes (such as stable processes) provide more realistic models than continuous processes do. So it is quite natural to study SDE system driven by stable processes.

Given this, it is somewhat surprising that SDE systems with continuous coefficients driven by stable processes have not previously been studied in a systematic fashion. In the first part of his talk, Chen reported on recent progress on the existence of a strong solution, and pathwise uniqueness for 1-dimensional SDEs driven by a symmetric stable processes. In the second part of the talk, he discussed the existence of weak solution and weak uniqueness for systems of SDEs driven by either by n-independent copies of a 1-dimensional symmetric stable processes, or by a symmetric stable processes in R^n. The approach uses the martingale problem method, and requires estimates for pseudodifferential operators with singular state-dependent symbols.

Renming Song discussed “Potential Theory of Special Subordinators and Subordinate Killed Brownian motions”. The technique of ‘subordination’ was introduced by Bochner [4], and allows the construction of stable processes from a Brownian motion and an independent increasing Lévy process (called a ‘subordinator’). However, subordination of processes in a domain D have only been studied fairly recently.

Let D be a bounded open set in $\mathbb{R}^d$, $d \geq 3$, and let $\Delta|_D$ be the Dirichlet Laplacian in D. This operator is the infinitesimal generator of the semigroup $(P_t^D : t \geq 0)$ corresponding to the process $X^D = (X_t^D : t \geq 0)$, Brownian motion killed upon exiting D. Let $S = (S_t : t \geq 0)$ be an $\alpha/2$-stable subordinator independent of X^D, where $0 < \alpha < 2$, and let $Z_\alpha^D = (Z_\alpha^D(t) : t \geq 0)$ be the process X^D subordinated by S: so that $Z_\alpha^D(t) := X^D(S_t)$. The infinitesimal generator of the semigroup of Z_α^D is the fractional power $-(-\Delta|_D)^{\alpha/2}$ of the negative Dirichlet Laplacian.

The study of the process Z_α^D was initiated in [17]. In [18] the domain of the Dirichlet form of Z_α^D was identified when D is a bounded smooth domain and $\alpha \neq 1$. In [20] and [19], the process Z_α^D was studied in detail and sharp upper and lower bounds on the jumping function and the Green function of Z_α^D were established when D is a bounded $C^{1,1}$ domain. One of the most intriguing aspects of the potential theory of Z_α^D was discovered in [17], and completely described in [16]. Let us introduce another subordinate process, $Z_{2-\alpha}^D$, obtained by subordinating killed Brownian motion X^D by an independent $(1-\alpha/2)$-stable subordinator. Let G^D, G_α^D and $G_{2-\alpha}^D$ denote the potential operators of X^D, Z_α^D and $Z_{2-\alpha}^D$, respectively. Then the following factorization identity holds true:

$$G^D = G_\alpha^D G_{2-\alpha}^D = G_{2-\alpha}^D G_\alpha^D \, . \tag{4}$$

Song discussed applications of this identity to a Harnack inequality, and to the identification of the Martin boundary for Z_α^D.

The Laplace exponent of the $\alpha/2$-stable subordinator is $\phi(\lambda) = \lambda^{\alpha/2}$, $\lambda > 0$. Clearly, $\lambda/\lambda^{\alpha/2} = \lambda^{1-\alpha/2}$ is the Laplace exponent of the $(1-\alpha/2)$-stable subordinator. The existence of a “dual” subordinator of this type is the key for the factorization (4). Song then described a more general family of ‘special subordinators’ for which this kind of duality holds, and showed that the main results of [17] and [16] remain valid when S is only assumed to be a special subordinator.

Masayoshi Takeda talked on “Gaugeability for Symmetric α-Stable processes and and it’s Applications”. Let $\mathbb{M}^\alpha = (\mathbb{P}_x, X_t)$ be a symmetric α-stable process on $\mathbb{R}^d$. Assume that $\mathbb{M}^\alpha$ is transient, and denote by $G(x,y)$ the Green function of $\mathbb{M}^\alpha$. Let μ be a smooth measure and A_t^μ the continuous additive functional corresponding to μ. (If μ has a density f then $A_t^\mu = \in_0^t f(X_s)ds$.) The measure μ is said to be *gaugeable* if

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[\exp(A^\mu(\infty)) \right] < \infty. \tag{5}$$

For Brownian motion on $\mathbb{R}^d$, Zhao [15] introduced the class of Green-tight measures and Chen [5] generalized this to jump processes. Let $K_{d,\alpha}^\infty$ denote this class for the stable process $\mathbb{M}^\alpha$. In [14] and [5] an analytic condition for a measure $\mu \in K^{\infty}_{d,\alpha}$ to be gaugeable was obtained. Let

$$\lambda(\mu) = \inf \left\{ \mathcal{E}^{(\alpha)}(u,u) : \ u \in \mathcal{F}^{(\alpha)}, \int_{\mathbb{R}^d} u^2(x)\mu(dx) = 1 \right\}.$$

Then the gaugeability of μ is equivalent to $\lambda(\mu) > 1$.

Takeda gave three applications of this fact: to the diffentiability of spectral functions, the ultracontractivity of Schrödinger semigroups, and the behaviour of branching symmetric α-stable processes.

4. Infinite dimensional analysis

Maria Gordina talk used stochastic differential equations (SDEs) in infinite-dimensional spaces to construct and study heat kernel measures (a noncommutative analogue of Gaussian or Wiener measure) on the infinite-dimensional manifolds. In general these infinite-dimensional groups are not locally compact, and therefore do not have an analogue of the Lebesgue (volume) measure. The motivation comes from several fields. Infinite-dimensional spaces such as loop groups and path spaces appear in physics, for example, in quantum field theory and string theory. One goal is to formalize some of the notions used in physics, such as Gaussian measures on certain infinite-dimensional spaces.

Her talk described the construction of the Gaussian measures on certain groups of infinite matrices, and gave some analytical properties of these measures. In addition, she presented new results on Riemannian geometry of these groups, which show that these groups are drastically different from their finite-dimensional analogs.

Shigeki Aida's interest is in analysis in infinite dimensional spaces and the interplay between the analysis and the geometry on such infinite dimensional spaces. In particular, the analysis on loop spaces is natural object but the basic property of the differential operator on the spaces are not well understood. For example, it is not clear when the Dirichlet forms on loop spaces satisfies a Poincaré's inequality, or when the Dirichlet forms satisfy log-Sobolev inequalities. As the terminology suggests, 'weak Poincaré inequalities' (WPIs) are weaker than Poincaré's inequalities, but nevertheless this property is stronger than irreducibility. WPIs hold on the loop spaces over simply connected compact manifolds in general. These inequalities contain a function which describes the degree of the ergodicity of the diffusion semi-group. However, explicit estimates on this function are not known in general.

In his talk, he proved WPIs on domains in Wiener spaces which are inverse images of open sets in $\mathbb{R}$ by continuous functions of Brownian rough paths. First, WPIs are established for ball like sets in the sense of rough paths and next the results are extended to "connected domains". This result is applicable to Dirichlet forms on loop spaces and connected open subsets of path spaces over compact Riemannian manifolds by using Lyons' continuity theorem of the solution of SDE. We still need more to obtain the estimate on the function in WPI in the case of loop spaces.

Xuemei Li talked on 'Asymptotics of Exponential Barycentres of mass transported by a random flow on Cartan Hadamard manifold.' This is joint work with M. Arnaudon. They considered the motion of a mass moving according to the law of a random flow. This can be used to model the motion of passive tracers in a fluid, e.g. the spread of oil spilt in an ocean. Such motion can be assumed to obey a stochastic flow where particles at nearby points are correlated. The evolution of pollution clouds in the atmosphere or a gas of independent particles, on the other hand, can be described as blocks of masses moving according to the laws of independent stochastic flows. We study the dynamics of masses transported by stochastic flows by investigating the motion of its centre of mass. As the media in which the liquid travels is not necessarily homogeneous or flat it makes sense to work on a non linear space, e.g. on a manifold diffeomorphic to the flat space but with different geometric structure.

The talk considered the mass pushed forward by a random flow in the sense of Kunita. The state spaces under consideration are Cartan Hadamard manifolds. Under suitable conditions on the flow and on the initial measure, the Barycentre can be shown to be a semi-martingale and is described by a stochastic differential equation. They showed that under suitable conditions an unstable flow satisfying the law of large numbers pushes the exponential barycentre of a discrete mass to the Busemann Barycentre of the limiting measure on the visibility boundary.

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