Second duals of measure algebras

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1 Overview of the Field

Let A be a Banach algebra. Then there are two natural products on the second dual A'' of A arising from left and right translations by elements of A; they are called the *Arens products*; we denote these products by $\Box$ and $\Diamond$, respectively. For definitions and discussions of these products, see [2], [4], and [5], for example. We briefly recall the definitions. For $a \in A$, $\lambda \in A'$, and $\Phi \in A''$, define $\lambda \cdot a$ and $a \cdot \lambda$ in A' by

$$\langle b,\, \lambda \cdot a\rangle = \langle ab,\, \lambda\rangle\,, \quad \langle b, a \cdot \lambda\rangle = \langle ba,\, \lambda\rangle \quad (b \in A)\,,$$

and then define $\lambda \cdot \Phi \in A$ and $\Phi \cdot \lambda \in A'$ by

$$\langle a,\, \lambda \cdot \Phi\rangle = \langle \Phi,\, a \cdot \lambda\rangle\,, \quad \langle a,\, \Phi \cdot \lambda\rangle = \langle \Phi,\, \lambda \cdot a\rangle \quad (a \in A)\,.$$

Finally, for $\Phi, \Psi \in A''$, define $\langle \Phi \,\Box\, \Psi,\, \lambda\rangle = \langle \Phi,\, \Psi \cdot \lambda\rangle \;(\lambda \in A')$, and similarly for $\Diamond$. The *left topological centre* of A'' is defined by

$$\mathfrak{Z}^{(\ell)}(A'') = \{\Phi \in A'' : \Phi \,\Box\, \Psi = \Phi \,\Diamond\, \Psi \;\; (\Psi \in A'')\}\,,$$

and similarly for the *right topological centre* $\mathfrak{Z}^{(r)}(A'')$. The algebra A is *Arens regular* if $\mathfrak{Z}^{(\ell)}(A'') = \mathfrak{Z}^{(r)}(A'') = A''$ and *strongly Arens irregular* if $\mathfrak{Z}^{(\ell)}(A'') = \mathfrak{Z}^{(r)}(A'') = A$. For example, every C^*-algebra is Arens regular [2].

There has been a great deal of study of these two algebras, especially in the case where A is the group algebra $L^1(G)$ for a locally compact group G. Results on the second dual algebras of $L^1(G)$ are given in [1], [7], [16], [17], [18], and [19], for example; a full proof that $L^1(G)$ is always strongly Arens irregular was first given in the case where G is compact in [16], and then in the general locally compact case in [18].

More recently, the three participants [5] have studied the second dual of a semigroup algebra; here S is a semigroup, and our Banach algebra is $A = (\ell^{\,1}(S), \star)$. We see that the second dual A'' can be identified with the space $M(\beta S)$ of complex-valued, regular Borel measures on βS, the Stone–Cech compactification of S. It can be shown that $(\beta S, \Box)$ is itself a subsemigroup of $(M(\beta S), \Box)$; properties of the latter algebra are intimately related to those of the semigroup $(\beta S, \Box)$, a subtle and much-studied mathematical object, even in the case where S is the obvious semigroup $(\mathbb{N}, +)$ [14].

Let G be a locally compact group. The measure algebra $M(G)$ of G has also been much studied (see [13], [23], [2], for example). This algebra is the multiplier algebra of the group algebra $L^1(G)$. Even in the case where G is the circle group $\mathbb{T}$, the Banach algebra $M(G)$ is very complicated; its character space is 'much larger' than the dual group $\mathbb{Z}$ of $\mathbb{T}$ [12].

2 Recent Developments and Open Problems

Let A be a Banach algebra which is strongly Arens irregular, and let V be a subset of A''. Then V is *determining for the topological centre* if $\Phi \in A$ for each $\Phi \in A''$ such that $\Phi \,\Box\, \Psi = \Phi \,\Diamond\, \Psi \ \ (\Psi \in V)$. Recently it has become clear that various 'small' subsets of A'' are determining for the topological centre in the case of some of the above algebras.

For example, it is shown in [5, Chapter 12] that, in the case where S is an infinite, weakly cancellative and nearly right cancellative semigroup (which includes the case where S is a group), there is a subset V of A'' of cardinality 2 that is determining for the topological centre of $\ell^{\,1}(S)''$. Independently, a similar results has recently been proved by Filali and Salmi [8]. An extension of these results to weighted convolution algebras is contained in [4]; for example, it is proved that, if ω is a weight on a countable, infinite group G such that ω is diagonally bounded by c on an infinite subset of G (in the sense of [4], etc.), and if $n \in \mathbb{N}$ with $n > c$, then there is a subset of $\ell^{\,1}(G)''$ of cardinality n that is determining for the topological centre.

We now consider which subsets of A'' are determining for the topological centre in the case where $A = L^1(G)$ for a locally compact (non-discrete) group G. The spectrum Φ of $L^\infty(G)$ is naturally a subset of the space A''. A theorem contained within [16] shows (in our terminology) that Φ is determining for the topological centre of A'' whenever G is a compact group. A different approach to this topological centre problem has been recently given by Neufang in [21]. Here it is shown that a certain family of Hahn–Banach extensions of the elements of G, regarded as characters on $LUC(G)$, are determining for the topological centre of A''.

The question whether or not the Banach algebra $M(G)$ is strongly Arens irregular for each locally compact group was raised in [9]. Some related results are given in [10], where it is shown that, in the case where G is comapct, $M(G)''$ uniquely determines G.. The main question was resolved positively for non-compact groups G satisfying certain cardinality conditions by Neufang in [22]. Our proposal stated that we planned to study the Banach algebra $M(G)$, and in particular seek to show that $M(G)$ is strongly Arens irregular for each compact group G.

3 Presentation Highlights

Since this was a workshop for three people assembled for 'Research in teams', there were no formal presentations.

4 Scientific Progress Made

We made progress in two related areas.

First, let $A = L^1(G)$ for a locally compact group G, and let Φ be the spectrum of L^∞. We now know that Φ is determining for the topological centre of A'' for each locally compact group G. This appears to give a shorter proof of the fact that A is always strongly Arens irregular than was known before. Indeed, we have proved that various subsets of Φ are determining for the topological centre, but we cannot yet say exactly which subsets of Φ have this property.

Second, consider the measure algebra $M(G)$. We see that the second dual of $M(G)$ is naturally presented as a space $M(\widetilde{G})$ of measures on a certain hyperstonean space $\widetilde{G}$, and we have characterized $M(G)$ as the space of normal measures in $M(\widetilde{G})$, essentially as in [6]. Our approach to these matters seems to be somewhat different from and more direct than that of earlier work, and allows us to identify easily important subsets of $\widetilde{G}$. We are studying which subsets of $\widetilde{G}$ are semigroups with respect to the map $\Box$. We have various partial results; in particular, we have identified various subsets which are semigroups, and, by using the *spine* of a group algebra (see [23] and [15]), we have proved that, for many compact groups G, $\widetilde{G}$ is not a semigroup; this was previously known [20] for all non-compact groups G.

5 Outcome of the Meeting

The three participants have written a draft paper on the matters described above; we expect to submit it for publication when more complete results are achieved.

Lau will visit the other two authors in England (supported by a grant from the London Mathematical Society) in November 2007; we hope to make further progress on this paper during the visit.

Dales will be a *PIMS Distinguished Visiting Professor* in Edmonton in November/December 2007; we hope to continue our joint work at that time.

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