MEASURABLE DYNAMICS: THEORY AND APPLICATIONS

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1 Brief overview of the field

The central aim of measurable dynamics is to apply modern mathematical techniques, including measure and probability theory, topology and functional analysis to study the time-evolution of complex evolving systems.

The fact that many simple models in the natural sciences may lead to classically intractable mathematical problems was already observed in the 19th century by H. Poincaré during his investigations into the orbits of celestial bodies. At about the same time, the formal development of thermodynamic theory alerted scientists to a major shift in the mathematical modelling paradigm that was about to take place. Since then, researchers have coined terms like *chaos* and *strange attractor* to describe the perplexing properties observed by Poincaré and others, and we now know that these systems, rather than being isolated curiosities are, in fact, increasingly likely to be encountered once one leaves the familiar territory of standard mathematical models derived from classical Physics, Chemistry or Engineering.

While the origins of the field are rooted in application, the mathematical development in the next century embraced both theoretical and applied approaches. In fact, for the first half of the 20th century, it is fair to say the former dominated as mathematicians struggled to develop new tools to describe the complex systems they were encountering. The celebrated ergodic theorems of Birkhoff and von Neumann, the development of a complete theory of measurable entropy (Rohlin, Kolmogorov, Shannon etc.) and a rudimentary structure theory for such systems (Halmos, von Neumann Hopf, and others) are all examples of powerful theoretical developments on which countless modern applications are built. In some sense the first modern 'application' of measurable dynamics was its role in formalizing the theory of stochastic processes in the first few decades of this century (the work of Kolmogorov, Khinchine and Doob for example).

All of this theoretical development took a sharp turn with the appearance of computing machinery, whereby, one of the most intractable parts of the dynamical model – the repeated, *infinite* iteration of a single mapping applied to a point to produce an orbit, became one of it's most accessible features. In the 60's and 70's there was an explosion of experimental mathematics focused on the use of computers to study dynamical systems. Fractals and other fractional dimensional objects, Julia sets and associated objects, strange attractors and numerous other examples poured into both the scientific and popular literature as the idea of a dynamical systems approach took hold. Slowly it was becoming clear that the exotic behaviour encountered in theoretical studies could be reproduced in extremely simple systems on the computer – the challenge (and opportunity) this presented to theoretical researchers was irresistible and the new phase of theory/application in dynamics took hold.

During the next few decades, modern theoretical results due to Ornstein, Ratner, Bowen, Sinai, Furstenberg and Weiss to name just a few, were finding application in both pure mathematics (differential geometry, number theory, group theory and thermodynamics, for example) and in applied mathematics, (ODE, Kinetic theory, Billiards and other hard-sphere dynamics, population dynamics, mechanical models, finance and so-on) simultaneously.

This balance has continued into the current decade. It is hoped that even a cursory review of the presentations outlined in the next section will make this clear; in particular, that modern dynamical systems in general, and measurable dynamics in particular continues to be a productive mix of theoretical efforts linked with exciting applications both in mathematics and other sciences.

This in large part underlay the motivation for our choice to try to balance the participants between pure and applied mathematicians working in the field. It is apparent that there is a considerable spectrum in terms of paradigm and outlook amongst researchers in the field. We believe the meeting was highly successful and we look forward to having a chance to attend or organize another one soon.

2 Talks given during the workshop

We give a brief synopsis of the talks given at the workshop, in order of presentation. Additional information is contained in the speaker's abstract and/or through the cited web links.

James Yorke (Maryland) gave an entertaining presentation on the dynamics of a 'Taffy-pulling Machine' – a mechanical device with two overlapping arms which is used to stretch and fold a batch of taffy (candy). A mathematical model of this machine produces an interesting diffeomorphism of an open subset of the plane that contains a Plykin-like attractor. Various studies were presented to support the statement, and a 1-dimensional reduction of the model was described. This is joint work with J. Halbert. The talk was video-recorded and appears in the publications directory of the BIRS website. More details are available at
`http://www-chaos.umd.edu/˜yorke/`

Oliver Jenkinson (Queen Mary, University of London) described an interesting and natural partial order on the set of (Borel) invariant measures $\mathcal{M}$ for the doubling map of the circle, $x \mapsto 2x \mod 1$ (equivalently, for the 2-shift). Roughly speaking, a measure $\nu \succ \mu$ if ν is more spread out on $[0, 1]$; the precise definition may be found in [5]. The order is related to an ergodic optimization problem: for a given function f, find $\mu \in \mathcal{M}$ which maximizes $\int f d\mu$. An intriguing connection to classical *Sturmian* measures was noted: Sturmians are the unique maximizing measures for $f =$ characteristic function of a semicircle. Also, amongst periodic measures, Sturmians have the property that they are the only ones combinatorially conjugate to a rotation (either rational or irrational) and hence, not all periodic measures are Sturmian. Oliver's website is at
`http://www.maths.qmul.ac.uk/˜omj/`

Erik Bollt (Clarkson University) investigated the notion of 'almost-conjugate' in the category of one-dimensional maps of the interval. Given two maps T and S, using a fixed point iteration scheme it is possible to construct an map f (which he calls a commutator) such that $f \circ T = S \circ f$ if no constraints such as surjectivity or continuity are enforced. Defects in the commutator, such as lack of injectivity, surjectivity, or continuity are used to give a measure of how different the two maps T and S are. Examples were presented that show how these measures can, in some simple cases, better match the heuristic notion of 'similar' than traditional approaches. This is joint work with J. Skufca. Erik's website is at
`http://people.clarkson.edu/˜bolltem/`

Gerhard Keller (Erlangen-Nürnberg) The acronym GOPY is applied to a set of non-chaotic strange attractor examples due to Grebogi/Ott/Pelikan/Yorke from the mid-1980's. While not chaotic in the normal sense of the term, they necessarily exhibit chaotic-like behaviour and, in particular, have complex attractors and sensitive dependence to initial conditions. Many interesting questions remain open about these systems in general – the speaker gave a sample analysis of the attractor for a model problem developed by Grebogi *et al*. This is joint work with Glendinning and Jäger. The talk was video-recorded and appears in the publications directory of the BIRS website. More details are available at
`http://www.mi.uni-erlangen.de/˜keller/english_index.html`

Judy Kennedy (University of Delaware) presented some problems from economics which, when posed in a dynamical systems language involve identification of the inverse limit (= natural extension) of a dynamical system. From this standard construction, one is able to compute expected utility for the process, and hence, to quantify monetary policy aimed at maximizing future utility. The main example discussed in the talk was the so-called 'cash-in-advance' model. This work is joint with two economists R. Raines and D. Stockman. Judy's website is at
`http://www.math.udel.edu/people/faculty/profile/kennedy.html`

Geon Ho Choe (Korean Advanced Institute for Science and Technology) presented a number of examples from the class of piecewise linear circle homeomorphisms where exact invariant densities could be determined using algebraic calculations. Maple was used extensively, as very few of these calculations are feasible by hand. Quantities of dynamical interest, such as rotation number, are then computed exactly with respect to this invariant density. Professor Choe is author of the book *Computational Ergodic Theory*, Springer Verlag, 2004.

Bryna Kra (Northwestern) gave an overview of the role of the so-called Gowers norms in the recent spectacular results in the application of dynamics to questions concerning arithmetic progressions and other patterns in positive density sets. Gowers norms (and their dynamical generalization by Kra and Host) are used to exploit parallelogram structures in a variety of abstract settings including abelian semigroups and suspensions of such groups to arbitrary sets. An abstract notion of parallelogram structure on a set was given, and characterized. Much more information can be found on Bryna's webpage:
`http://www.math.northwestern.edu/˜kra/`

Wael Bahsoun (Victoria) Traditionally, dynamics has considered actions for closed systems, where the orbit of a point remains in the state space for all time. In some applications, a nonequilibrium model is required where the orbit of a point may eventually leave the system (and for convenience of description, never return). The *escape rate* gives the rate at which mass is lost to the system through this mechanism. A simple model for an open system was presented: an interval map with a hole (in the domain). The main question addressed in this talk was to produce a rigorous numerical scheme that can compute the escape rate for such a system. The algorithm is based on theoretical work of Keller and Liverani on spectral perturbation and Ulam method for discretization of the continous domain; the basic steps in the algorithm were reviewed and a simple example computation presented. From September 2006, Wael is with the Department of Economics at the University of Manchester. `http://www.socialsciences.man.ac.uk/economics`

Gary Froyland (University of New South Wales) Invariant sets and functions play a central role in the analysis of dynamical systems. In practice, almost invariant sets (or functions) also contain useful information and generically, one expects to have many such objects around. In certain cases, some of these almost invariant objects are also physically interesting and natural. The speaker showed how they can be found by spectral techniques applied to the associated transfer operator. Interesting properties of almost invariant objects include (relatively) slow mixing times and slow rates of correlation decay leading to interesting physical consequences. Examples were presented ranging from simple interval maps to a long-term project the speaker is working on to help model circulation patterns in the Great Southern Ocean. Gary's website is
`http://web.maths.unsw.edu.au/˜froyland/`

Sinan Gunturk (Courant Institute) Gave us a useful introduction to the dynamical ideas underlying a functional approximation method called sigma-delta quantization. This method has applications in half-toning and analog-to-digital conversion. The talk also hinted at an intriguing sequencing problem where two competitors sequentially aim to hit a target which they have identical small unknown probability p of hitting. Sinan's webpage is at
`http://www.cims.nyu.edu/˜gunturk/`

William Ott (Courant Institute) gave a very enjoyable talk on classical notions of recurrence and distality in topological dynamics. The basic definition is *product recurrence*: a point $x \in X$ is product recurrent if it is recurrent and, for every other topological system Y, for every other recurrent $y \in Y$, (x, y) is recurrent for the product system. A classical result identifies this concept with distality for $\mathbb{Z}-$ actions. The relation between these concepts for more general semigroup actions has been investigated by Auslander and Furstenberg. A related notion is *weak* product recurrence, where the test point $y \in Y$ is restricted to the class of uniformly recurrent points. This was shown by the speaker to be not equivalent to distality, even for $\mathbb{Z}$-actions. William's webpage is at
`http://www.cims.nyu.edu/˜ott`

James Meiss (Colorado) brought a visually stunning display of recent computational experiments aimed at uncovering bifurcation of invariant sets in 3-D volume-preserving diffeomorphisms. The setting is a natural development from the area preserving 2-D diffeomorphisms that arise for example in Hamiltonian dynamics. On the other hand, compared to the 2-D situation, the scope for interesting and complicated behaviour is greatly increased. Using a few simple model maps the speaker was able to exhibit the appearance and destruction of invariant tori and to propose various mechanisms that could lead to these complex bifurcations. More stunning graphics and a lot of mathematics can be found at
`http://amath.colorado.edu/faculty/jdm/`

Peter Ashwin (Exeter) A classical example in ergodic theory is the *interval exchange transformation*. An interval (the state-space) is partitioned into finitely many subintervals and the dynamics rearranges these by a permutation. The dynamical properties of interval exchange transformations are well-studied. A multidimensional analogue of the interval exchange is a piecewise isometry from an open, connected domain onto itself. Very little is known in generality about such maps. The speaker discussed a class of such mappings (called pizza maps) on the plane which have an advantage in that the dynamics on $\mathbb{R}^2$ near infinity can be modelled by an interval exchange. Still, on the bounded component a very rich and complex behaviour of escape and attraction may be found. The speaker presented both results from numerical studies and theoretical work. This is a joint project with Arek Goetz. Peter's webpage is
`http://www.secam.ex.ac.uk/~PAshwin/`

Matt Nicol (Houston) The Young Tower construction (L.S. Young, ~1998) provides a convenient, abstract way to construct non-uniformly hyperbolic transformations, or, conversely, to analyze concrete systems with spatially contained non-hyperbolic features (such as indifferent fixed points). This talk discussed the derivation of large deviation estimates on Young Tower maps, that is, estimates on the decay rates of $m\{\frac{1}{N}\sum_{n=0}^{N-1}\phi(T^n) \geq \int \phi dm + \epsilon\}$. It was shown that structural features of the tower control the rate of decay, through both exponential and polynomial classes. A basic question arises from this work: can the exact results on the Tower be reproduced in a concrete intermittent map. Much more about these ideas may be found at
`http://math.uh.edu/~nicol/`

Vitaly Bergelson (Ohio State) One currently active area in ergodic theory is the study of subsequential limit theorems. The notion of an IP-subset of the integers (and correspondingly, IP convergence) plays a central role, both in establishing such theorems and in generalizing to other semigroup actions the types of results available for the integers. The speaker began with a self-contained introduction to the IP-notions, then moved on to a tour of some of the known results from the *multiple recurrence* literature. Here are a couple of striking results mentioned. Suppose (X, m, T) is weakly mixing. Then

1. Generically, there is IP-rigidity. That is to say there is an IP-subset $n_\alpha \subseteq \mathbb{N}$ and (nontrivial) $f \in L^2$ which is IP-mixing: $f \circ T^{n\alpha} \to f$.

2. (IP-Krengel partition independence) For every finite partition $\mathcal{P}$ and $\epsilon > 0$ there exists a finite partition $\mathcal{P}'$ such that $d(\mathcal{P}, \mathcal{P}') < \epsilon$ and an IP-subset n_α such that $\{T^{-n_\alpha}\mathcal{P}'\}$ is exactly independent.

Other results, surveys, and open questions may be found at
`http://www.math.ohio-state.edu/~vitaly/`

Ryszard Rudnicki (Silesian University) A Markov semigroup $\{P_t\}$ is a generalization of a dynamical system – sufficiently rich to contain, for example, random dynamical systems. For such systems one has the Foguel Alternative: either $\{P_t\}$ is asymptotically stable, or it is sweeping out (mass escapes to 'infinity'). Of particular interest in applications is the case where the semigroup is generated by a partial differential equation. The speaker reviewed two such applications, one in transport theory, the other in a biological model of a gene population which can be used to explain observed properties of maturity-distribution for age profiles. Ryszard's webpage is at
`http://www.impan.gov.pl/User/rudnicki/`

Rua Murray (University of Waikato) Various methods using finite computations to estimate unknown invariant measures have been proposed. Ulam's method (discussed numerous times during the workshop) is one of the most popular and easy to implement, but theoretical problems arise when one tries to validate the method and prove convergence. Rigorous results are known only for a much smaller class of dynamical systems than the class on which numerical experiments would suggest them to hold. Rua in joint work

with Chris Bose, described a completely different approach to the approximation problem, based on convex optimization (a.k.a. the maximum-entropy method). These allow widely valid approximation schemes (they converge in norm under weak assumptions) and for which finite computations, although more delicate than in the case with Ulam's method, are still feasible. There appears to be a great deal of scope for future improvement. Rua's website is
`http://www.math.waikato.ac.nz/˜rua/`

Evelyn Sander (George Mason) discussed bifurcations of low-dimensional dynamical systems giving rise to crises and more specifically explosions: parameter values where chaotic behaviour appears in neighbourhoods that previously contained no recurrent points. A key question is the existence of unstable dimension variability: parameter values for which different points in the attractor have different dimensional unstable manifolds. Evelyn's talk outlined the construction of a three dimensional example exhibiting unstable dimension variability arising from a crisis. Evelyn's web page is at
`http://math.gmu.edu/˜sander/`

Ian Melbourne (Surrey) An interesting model for dynamicists is the billiard flow on the plane outside finitely many convex bodies. This has been proposed as a deterministic model for Brownian motion. Such a map is (Sinai and others) uniformly hyperbolic with singularities and leads to a central limit theorem (CLT), a functional central limit theorem (FCLT) and more generally almost sure invariance principles (ASIP). In joint work with Matt Nicol, the speaker has investigated vector-valued ASIP's for non-Axiom A dynamics, once again using the formal structure of a Young Tower. Ian's website is
`http://www.maths.surrey.ac.uk/people/index.php?display=I.Melbourne`

Arno Berger (Canterbury). Arno discussed the use of shadowing to show that for certain classes of non-autonomous mappings, almost every orbit satisfies the 'first digit property' known as Benford's Law (where the frequency of different initial digits base b is given by a logarithmic distribution). Arno's webpage in New Zealand is
`http://www.math.canterbury.ac.nz/˜abe34/`

3 Open Problems Session

On the evening of Monday, August 7 a problem session was convened and a number of participants presented interesting problems for consideration by workshop participants.

Vitaly Bergelson asked about Ergodic theorem along polynomials and the lazy physicist paradox.

Assume that T_v, $v \in \mathbb{R}$, is a continuous ergodic measure-preserving flow on a probability Lebesgue space. Note that due to the ergodic decomposition, the assumption of ergodicity does not limit the generality of our our discussion. It is not too hard to show that for all but countably many v the measure-preserving transformation $S = T_v$ is totally ergodic (meaning that all the non-zero powers of S are ergodic as well). Consider now the following situation. A physicist fixes first a time unit v (and we assume, without too much loss of generality, that the corresponding $S = T_v$ is totally ergodic) and then performs the sampling of the flow along "quadratic" instances of time, that is, considers the averages

$$A_N = 1/N \sum_{n=0}^{N-1} f(S^{n^2}(x)),$$

where f is, say, a bounded measurable function on X which describes an important physical parameter (so that $f(S^{n^2}x)$ describes the values of the parameter along the trajectory of the point x in X, measured at quadratic instances of time).

According to a theorem due to Bourgain, (which applies to any totally ergodic transformation and any non-trivial polynomial taking on integer values on integers) the physicist will see that despite the increasing gaps between time measurements, the averages A_N will converge (for almost every x in X) to the space average, $\int f$. Note also that if the flow T_v is weakly mixing, then $S = T_v$ is weakly mixing (and hence totally ergodic) for EVERY non-zero v.

Problem 1 (Philosophical). What is the physical meaning of this? Why does nature (in the case of totally ergodic transformations) work so well along the polynomials? Apropos, there are many more "good" sequences of times with similar properties but the sequences of exponential growth, such as 2^n are not "good".

Problem 2 (Mathematical). Assume that the flow T_v is comprised of smooth enough transformations and that the function f is also smooth enough. What can be said (in terms of smoothness of T_v and f) about the speed of convergence of A_N to $\int f$? Can one show that the convergence along the squares n^2 is (in some sense) faster than that along the cubes n^3 ?

Oliver Jenkinson asked for a continuous f with Lebesgue measure as the unique $\times 2$-invariant f-maximizing measure.

The following is Problem 3.9 in [5]

Problem 1 Let $T(x) = 2x \pmod 1$. Explicitly exhibit a continuous function $f : [0,1] \to \mathbb{R}$ such that $\int f(x)\,dx > \int f\,d\mu$ for all T-invariant probability measures μ other than Lebesgue measure.

Remarks:

(a) The strict inequality is key; if the inequality were weak then a constant function would suffice.

(b) It is known that such functions f exist (see [6, Cor. 1]).

(c) By an "explicit" representation of f we have in mind some sort of series expansion, for example a Fourier expansion.

(d) It is known that any such f cannot be too "regular"; for example f cannot be Hölder (see e.g. the discussion in [5, 6]). There are heuristic reasons (see [6]) for expecting such an f to be highly oscillatory.

Since periodic orbit measures are weak-$*$ dense in the set of T-invariant measures, the following weaker version of the above problem is perhaps no easier to solve.

Problem 2 Let $T(x) = 2x \pmod 1$. Explicitly exhibit a continuous function $f : [0,1] \to \mathbb{R}$ such that $\int f(x)\,dx > \frac{1}{n}\sum_{i=0}^{n-1} f\big(T^i(\frac{j}{2^n-1})\big)$ for all $n \geq 1$ and $0 \leq j \leq 2^n - 1$.

Gerhard Keller asked about (Non)minimality of transitive quasiperiodically forced Denjoy circle diffeomorphisms.

Let T be a quasiperiodically forced circle homeomorphism, i.e. a continuous map of the form

$$T : \mathbb{T}^2 \to \mathbb{T}^2\ ,\ (\theta, x) \mapsto (\theta + \omega, T_\theta(x))\ , \tag{1}$$

where $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and where the *fibre maps* T_θ are orientation-preserving circle diffeomorphisms with the derivative DT_θ depending continuously on (θ, x). To ensure all required lifting properties we additionally assume that T is homotopic to the identity on $\mathbb{T}^2$.

Let $\hat{T} : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ be a lift of T. Then the quantities

$$\rho_{\hat{T}} := \lim_{n\to\infty} \frac{1}{n}(\widehat{T}^n_\theta(\hat{x}) - \hat{x})\ , \quad \rho_T := \rho_{\hat{T}} \mod 1 \tag{2}$$

exist and are independent of θ, $\hat{x}$ and the choice of the lift $\hat{T} : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$. They are called the fibrewise rotation numbers of $\hat{T}$ and of T, respectively. (This result is due to Herman ([3]), an alternative proof can be found in [9].)

Suppose from now on that T satisfies the following Denjoy condition:

$$\int_{\mathbb{T}^1} \mathrm{var}(\log DT_\theta)\,d\theta < \infty\ .$$

The following is known [4, Theorem 4.4]:

Theorem: If ρ_T is irrational, then $T : \mathbb{T}^2 \to \mathbb{T}^2$ is topologically transitive.

Problem: In this situation, is it true that T is necessarily topologically minimal?

It is also known that, if such a T is non-minimal, each minimal invariant subset $M \subset \mathbb{T}^2$ is highly disconnected in the sense that each connected component of M is contained in a single fibre $\pi^{-1}(\theta)$ [4, Theorem 4.5].

Example: A concrete example where, to the best of my knowledge, the answer to the above question is not known is the *critical Harper map* where T_θ is given by

$$T_\theta(x) = \frac{-1}{x + 2\cos(2\pi\theta)}\ .$$

If this map has a nontrivial minimal subset, then it should like the figure below reproduced from [4].

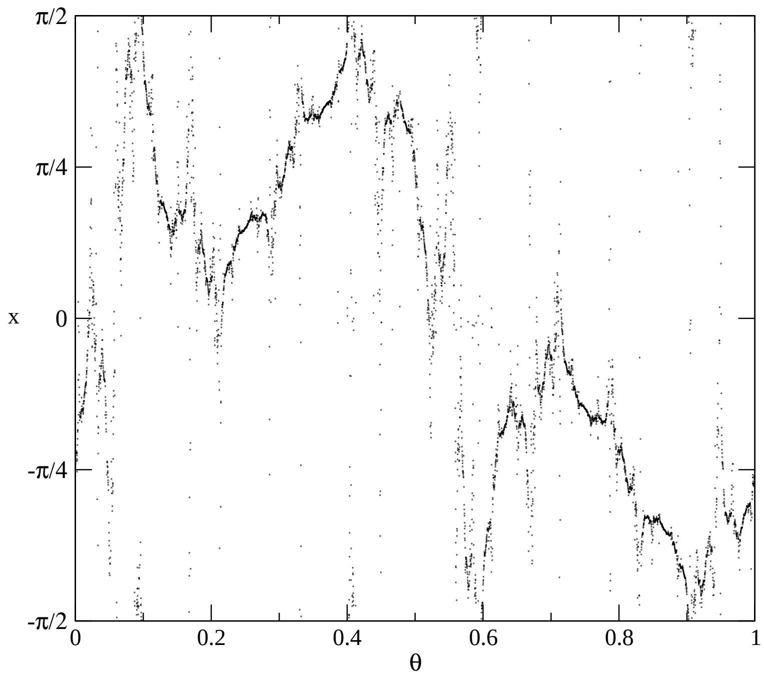

Figure 1: Numerical reconstruction of the invariant measure support for the critical Harper map.

Ian Melbourne asked a question about intermingled attractors.

Let $f : M \to M$ be a C^∞ diffeomorphism on a compact manifold M. We say that f has k intermingled attractors $A_1, \dots, A_k$ if the A_j are closed f-invariant topologically transitive sets and the basins of attraction $B_j = \{x \in M : \omega(x) = A_j\}$ satisfy

(i) $\mathrm{Leb}\,(M - \{A_1 \cup \cdots \cup A_k\}) = 0$,

(ii) $\mathrm{Leb}\,(A_j \cap U) > 0$ for all nonempty open subsets $U \subset M$ and all $j = 1, \dots k$.

Similarly, we can speak of countably many intermingled attractors.

For $k = 2$, there are three different constructions with $\dim M = 3$: Kan 1994 ($M = T^2 \times [0,1]$), Fayad 2003 ($M = T^3$), Melbourne & Windsor 2005, ($M = T^3$). For each $k = 3, 4, \dots, \infty$, there is a 4-dimensional construction due to Melbourne & Windsor 2005 ($M = T^2 \times S^2$).

Problem Can the dimension of M in the above constructions be reduced?

Ian Melbourne and Vitaly Bergelson asked a question about weak mixing versus mixing: For measure-preserving transformations with the weak topology it follows from Halmos 1944 and Rokhlin 1948, that generically such transformations are weak mixing but not mixing. In the smooth category, it is possible to construct examples that are weak mixing but not mixing, but genericity is certainly false (mixing Axiom A diffeomorphisms form a nonempty open set of C^r diffeomorphisms for any $r \geq 1$).

In fact, the following anti-Halmos-Rokhlin situation is plausible: Consider C^r diffeomorphisms on a compact boundaryless manifold M. Perhaps there exists an r_0 (say $r_0 = 3$, or $r_0 = 2 + \epsilon$, etc) such that for any $r \geq r_0$, typical C^r diffeomorphisms $f : M \to M$ have the property that if A is a weakly mixing locally maximal ω-limit set for f then A is mixing. (Here, typical could be open-dense, generic, or prevalent.)

Problem: Prove or disprove.

Anthony Quas noted that for sofic $\mathbb{Z}$-shifts there is always a finite-to-one extension to a subshift of finite type. Once consequence is that the topological entropy of this extension is equal to that of the sofic. The finite-to-one ness fails for some $\mathbb{Z}^d$ actions but it is an open question as to whether every $\mathbb{Z}^2$ sofic admits an extension to a subshift of equal entropy.

4 Outcome of the Meeting

This workshop was designed to connect people and research areas across the sprawling, modern discipline of measurable dynamics. The extent to which we succeeded will only be evident some time in the future and even then may be difficult to quantify. However, the organizers are quite satisfied that the goal of creating such connections was bearing fruit already during the few days of the workshop. In addition to the positive impressions we received during our stay at BIRS we received many email comments from participants after the conference. We reproduce a few of these (both praise and constructive criticism) as representative.

- Let me thank the organizers for an excellent workshop. I think that part of the success is due to the cleverly executed implementation of the idea of bringing together representatives of different flavors of dynamics. I personally learned a lot and got plenty of new ideas which will be useful not only in my research but also in my advising activities. We should have more such workshops!
- The most interesting aspect of the meeting (for me) was bringing together people from applied and pure dynamics for the conference - this is something that should be happenning more. I would have liked to even see more applied people, to find out what they are interested in. The length of talks was optimal, and I found the problem session useful. Perhaps for future conferences it would be interesting to have someone take notes for the problem session and post them on the web.
- Thanks (also) for having me at the workshop which I found excellent and enjoyable indeed. As I said already last week, I shall be more than happy to help organizing future events.
- First of all, thanks to you and all the organizers. It was a fantastic conference. I would say as a “new guy” that the group meals and the scheduling of many breaks and social activities was great for me as far as meeting new people and getting conversations going.
- My only small complaint is food related: they served precious few green vegetables other than green peas, which I despise.
- I probably didn’t explicitly mention it, but the conference was the most enjoyable I’ve been to for some time, so thanks for the invite!
- I found the format of the meeting highly conducive to scientific discussion and discovery. Each day included a good number of talks while providing ample time for informal discussion. Bryna, Anthony and Ronnie addressed some of the open problems that I stated at the conclusion of my talk. In general, discussion of both technical challenges and future directions permeated the meeting. I found the problem session highly useful. This idea should be implemented more generally for mathematics conferences.
- The meeting covered a fantastic breadth of subject matter; clearly a great deal of thought had gone into the organization. The BIRS facilities were great, with the natural informality guaranteeing plenty of constructive mathematical interaction. The scheduling was particularly good: the talks were a nice length, and having a few full days with six lectures, and a few days with 3 or 4 lectures but plenty of mingling time made for a very good pace.

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