Multi-Agent Optimization

I. Variational Analysis Tools (2)

Variational Analysis: Bivariate Functions

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SADDLE-POINTS

Lagrangians

- $$\max \quad f_0(x)$$
  $$\text{so that } f_i(x) \geq 0, \quad i = 1, \ldots, s$$
  $$f_i(x) = 0, \; i = s+1, \ldots, m.$$
- Lagrangian:
  $$L(x,y) = \begin{cases} f_0(x) + \sum_{i=1}^{m} y_i f_i(x) \text{ if } y_i \geq 0, i = 1, \ldots, s \\ -\infty \quad \text{otherwise} \end{cases}$$
- concave-convex, maxinf framework

Hamiltonians

- Opt-control: $\min_u \Phi\left(x_T\right)+\int_0^T l(t,x_t,u_t)dt,$

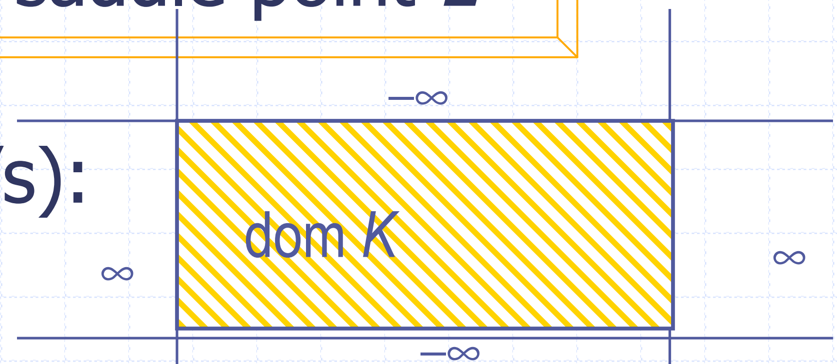
  $\dot{x}_t = f(t,u_t,x_t),\, u_t \in U_t,\, t \in [0,T].$
- Hamiltonian: $H(t,x,y,u) = l(t,u,x) + \langle y, f(t,x,u)\rangle$
- Optimality: $\dot{x}^* = \frac{\partial H^*}{\partial y},\ \dot{y}^* = -\frac{\partial H^*}{\partial x},\ \ y^*(T) = \nabla\Phi(x_T^*)$

  $u_t^* \in \arg\min_{v\in U_t} H(t,x_t^*,v,y_t^*),\ t\in[0,T]$
- Approximation: state x <u>and</u> co-state y

Hypo/Epi-Convergence

- saddle point L^a ~ saddle point L

- Attouch-W. def. (90's):



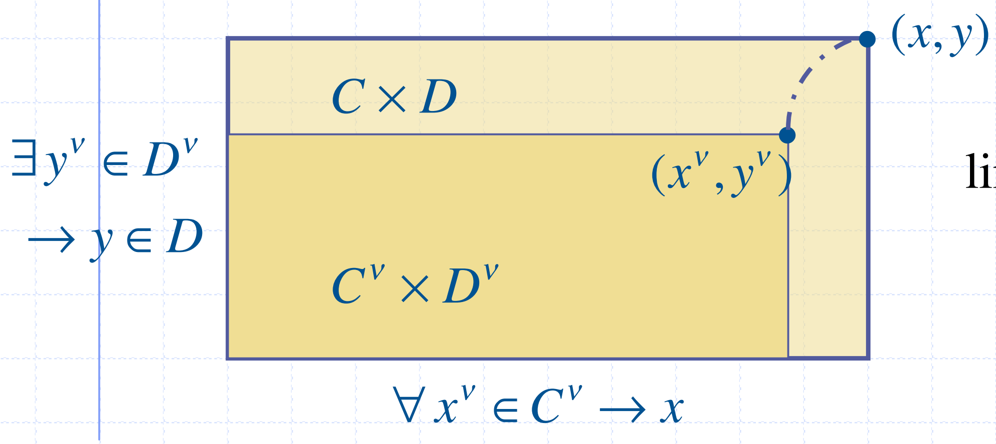
$$\forall\,(x,y)$$

$$\text{(a) } \forall x^\nu \to x, \exists\, y^\nu \to y: \quad \liminf L^\nu(x^\nu, y^\nu) \geq L(x,y)$$

$$\text{(b) } \forall y^\nu \to y, \exists\, x^\nu \to x: \quad \limsup L^\nu(x^\nu, y^\nu) \leq L(x,y)$$

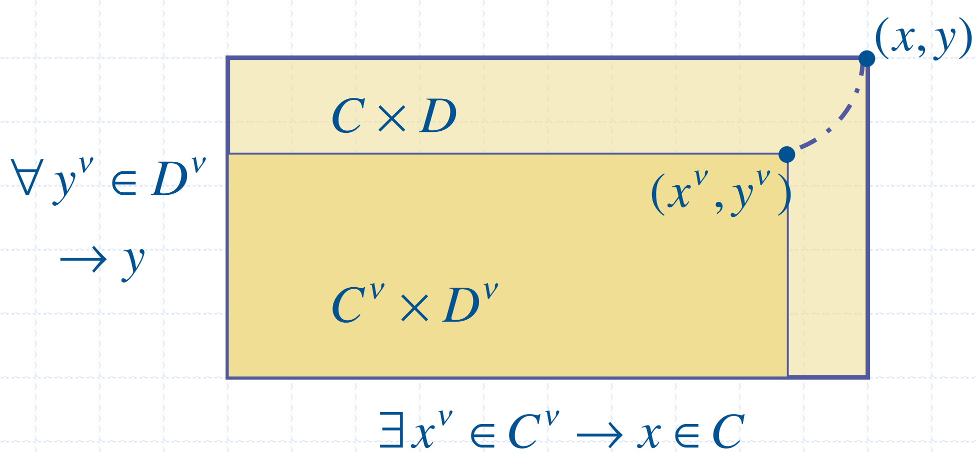
- "problem": hypo/epi-topology not Hausdorff
  …. the pitfall "equivalence classes"

Hypo/Epi convergence - 21st C.



$$\limsup_\nu K^\nu(x^\nu, y^\nu) \le K(x,y) \text{ when } x \in C$$

$$K^\nu(x^\nu, y^\nu) \to -\infty \text{ when } x \notin C$$



$$\liminf_\nu K^\nu(x^\nu, y^\nu) \ge K(x,y) \text{ when } y \in D$$

$$K^\nu(x^\nu, y^\nu) \to \infty \text{ when } x \notin D$$

Hypo/epi: max-inf framework

$$L : C \times D \to \mathbb{R}, \ \left\{L^\nu : C^\nu \times D^\nu \to \mathbb{R}\right\}$$

$$\text{(a) } \forall x^\nu \in C^\nu \to x \in C, \forall y \in D, \exists\, y^\nu \in D^\nu \to y:$$

$$\text{limsup}_\nu L^\nu(x^\nu, y^\nu) \le L(x,y)$$

$$(\text{a}_\infty) \ \forall x^\nu \in C^\nu \to x \notin C, \forall y \in D, \exists\, y^\nu \in D^\nu \to y:$$

$$L^\nu(x^\nu, y^\nu) \to -\infty$$

$$\text{(b) } \forall y^\nu \in D^\nu \to y \in D, \forall x \in C, \exists\, x^\nu \in C^\nu \to x:$$

$$\text{liminf}_\nu L^\nu(x^\nu, y^\nu) \ge L(x,y)$$

$$(\text{a}_\infty) \ \forall y^\nu \in D^\nu \to y \notin D, \forall x \in C, \exists\, x^\nu \in C^\nu \to x:$$

$$L^\nu(x^\nu, y^\nu) \to \infty$$

Hypo/epi: elementary properties

$\boxed{\text{saddle points } L^a \sim \text{saddle points } L}$

hypo/epi $\Rightarrow$ epi-convergence if $L(x,y) = g(y)$

hypo/epi $\Rightarrow$ hypo-convergence if $L(x,y) = f(x)$

but not y-epi-convergence or x-hypo-convergence

Remarks: not a Γ-convergence (De Giorgi)

in ∞-dim. different topologies (σ-hypo/τ-epi)

Hypo/epi-convergence: Properties

$$L^\nu \xrightarrow[h/e]{} L, \quad (x^\nu, y^\nu) \text{ saddle-pts } L^\nu,$$

$$(x^\nu, y^\nu) \xrightarrow[\nu \in N]{} (\bar{x}, \bar{y}) \quad \Rightarrow (\bar{x}, \bar{y}) \text{ saddle-pt of } L$$

$$\text{Moreover, } F(\bar{x}, \bar{y}) = \lim_{\nu \in N} F^\nu(x^\nu, y^\nu).$$

$\{L \text{ concave-convex}\}$ closed under epi/hypo-convergence

hypo/epi-limits: $L(\bullet, y)$ usc, $L(x, \bullet)$ lsc

bijection η: $fv\text{-biv}\left(\mathbb{R}^{\text{n+m}}\right) \leftrightarrow pr\text{-biv}\left(\mathbb{R}^{\text{n+m}}\right)$

Zero-sum games

- Strategies: $x \in X, y \in Y$ - Players: Max-1 & Minie-2

  Payoff: $u_1(x,y) + u_2(x,y) = 0$; set $u = u_1$

  Nash: $x^* \in \arg\max_{x \in X} u(x, y^*)$, $y^* \in \arg\min_{y \in Y} u(x^*, y)$
- Existence: X, Y compact, convex

  $u(\bullet, y)$ concave-usc, $u(x, \bullet)$ convex-lsc

  $\Rightarrow \exists$ optimal strategies (equilibirum)
- Convergence: $u^\nu : X^\nu \times Y^\nu \to \mathbb{R}$ (as for existence),

  $X^\nu \to X, Y^\nu \to Y$ and $u^\nu \underset{h/e}{\to} u \quad \Rightarrow$

  $\exists$ soln's$\left(x^\nu, y^\nu\right) \underset{cluster}{\to} (\bar{x}, \bar{y})$ equilibrium for $u : X \times Y \to \mathbb{R}$

MAX-INF POINTS (maxinf)

or MIN-SUP POINTS

Fixed Points (Brouwer)

C compact convex, $G:C \to C$ continuous

C^ν compact convex, $G^\nu : C^\nu \to C^\nu$ continuous

$\bar{x}^\nu \in C^\nu$ fixed points of G^ν on C^ν

and $\bar{x}^\nu \to \bar{x}$. When is $\bar{x}$ a fixed point of G on C?

The approach $K : C \times C \to \mathbb{R}$, $K^\nu : C^\nu \times C^\nu \to \mathbb{R}$,

set $K(x,y) = \langle G(x) - x, y \rangle$, $K^\nu(x,y) = \langle G^\nu(x) - x, y \rangle$

x fixed point $\Leftrightarrow x \in \arg\max_C (\inf_{y \in C} K(\bullet, y))$

So, $K^\nu \underset{?}{\to} K$ yields convergence of max-inf points?

Variational Inequalities

- $C \subset \mathbb{R}^n$ non-empty, convex
- $G : C \to \mathbb{R}^n$ continuous
- find $\bar{u} \in C$ such that $-G(\bar{u}) \in N_C(\bar{u})$

  where $v \in N_C(\bar{u}) \Leftrightarrow \langle v, u - \bar{u} \rangle \le 0, \ \forall u \in C$
- $C^\nu \to C, \quad G^\nu \colon C^\nu \to \mathbb{R}^n$ continuous
- approx. $u^\nu \in C^\nu$ such that $-G^\nu(u^\nu) \in N_{C^\nu}(u^\nu)$
- Question: $u^\nu \to \bar{u}$? when, how.

V.I.: The approach

- Let $\boxed{K(u,v) = \langle G(u), v-u \rangle}$ on $\operatorname{dom} K = C \times C$
- then $-G(\bar{u}) \in N_C(\bar{u})$ if and only if
- $\exists \hat{u} \in \arg\max\text{–}\inf K$ with $K(\hat{u}, \bullet) \geq 0$
  - $\langle G(\bar{u}), v-\bar{u} \rangle = K(\bar{u}, v) \geq 0, \forall v \in C$

    $\Rightarrow 0 \leq K(\bar{u}, \bullet) \leq \sup_{u \in C} \left(\inf_{v \in C} K(u,v) \right)$
  - $\hat{u} \in \arg\max\text{–}\inf K \;\&\; K(\hat{u}, \bullet) \geq 0 \Rightarrow$

$$\langle -G(\hat{u}), v-\hat{u} \rangle \leq 0, \forall\, v \in C \text{ or } -G(\hat{u}) \in N_C(\hat{u})$$

V.I.:The approach

- $K^{\nu}(u,v) := \left\langle G^{\nu}(u), v-u \right\rangle$, dom $K^{\nu} = C^{\nu} \times C^{\nu}$
- $u^{\nu} \in \arg\max\text{–}\inf K^{\nu}$ with $K^{\nu}(u^{\nu},\bullet) \le 0$
- $$\boxed{K^{\nu} \xrightarrow[\text{"...”}]{} K \text{ and } \ldots}$$
- $\bar{u} \in$ cluster points $\left\{u^{\nu}\right\} \Rightarrow ?\ \bar{u} \in \arg\min\text{–}\sup K$

Non-Cooperative Games

- player: $a \in \mathcal{A}$, payoff: $u_a(x_a, x_{-a}) : \mathbb{R}^N \to \overline{\mathbb{R}}$
- Nash equilibrium: $(\bar{x}_a, a \in \mathcal{A})$ such that

$$\bar{x}_a \in \arg\max u_a(x_a, \bar{x}_{-a}), \;\; \forall a \in \mathcal{A}$$

- Nikaido-Isoda function:

$$N(x, y) = \sum_{a \in \mathcal{A}} u_a(x_a, x_{-a}) - \sum_{a \in \mathcal{A}} u_a(y_a, x_{-a})$$

- $\bar{x} = (\bar{x}_a, a \in \mathcal{A})$ is a Nash equilibrium

$$\Leftrightarrow \bar{x} \in \arg\max\!\text{–}\!\inf N$$

The approach:

- Nikaido-Isoda functions of approximating games

  $$N^\nu(x,y) = \sum_{a\in\mathcal{A}} u_a^\nu(x_a, x_{-a}) - \sum_{a\in\mathcal{A}} u_a^\nu(y_a, x_{-a})$$

- $x^\nu \in \arg\max\text{–}\inf N^\nu$, $\bar{x} \in$ cluster points $\{x^\nu\}$

  $$\boxed{N^\nu \xrightarrow[\text{"..."}]{} N \text{ and } \ldots}$$

- $\Rightarrow ?\ \bar{x} \in \arg\max\text{–}\inf N \sim$ equilibrium point

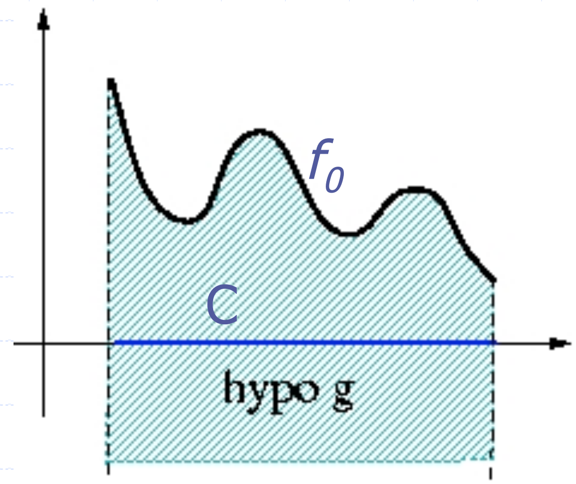
Applications: Convergence and stability

- Saddle-points: Lagrangians, Hamiltonians
- Fixed points
- Solutions of cooperative <u>and</u> non-cooperative games
- Economic-Equilibrium points (Walras)
- Generalized Nash Equilibrium Problems
- Solutions of set-valued inclusions
- Stability of mountain-pass paths

Optimization: Max-Framework

- $\max f(x),\ x \in \mathbb{R}^n$ with

$$f : \mathbb{R}^n \to \overline{\mathbb{R}} = \left[-\infty, \infty\right]$$



- $$f(x) = \begin{cases} f_0(x) \text{ if } x \in C \subset \mathbb{R}^n \\ -\infty \qquad \text{if } x \notin C \end{cases}$$

$$C = \left\{ x \in X \,\middle|\, f_i(x) \leq 0, i \in I_1, f_i(x) = 0, i \in I_2 \right\}$$

Hypo-Convergence (max-framework)

- $\boxed{\text{argmax } f^a \sim \text{ argmax } f}$
- argmax $(f^\nu + g) \rightarrow$ argmax $(f+g)$,
  $\forall$ cont. g $\Rightarrow$ hypo-convergence
- 'new' definition $f : D \to \mathbb{R}, \quad \left\{ f^\nu : D^\nu \to \mathbb{R} \right\}_{\nu \in \mathbb{N}}$

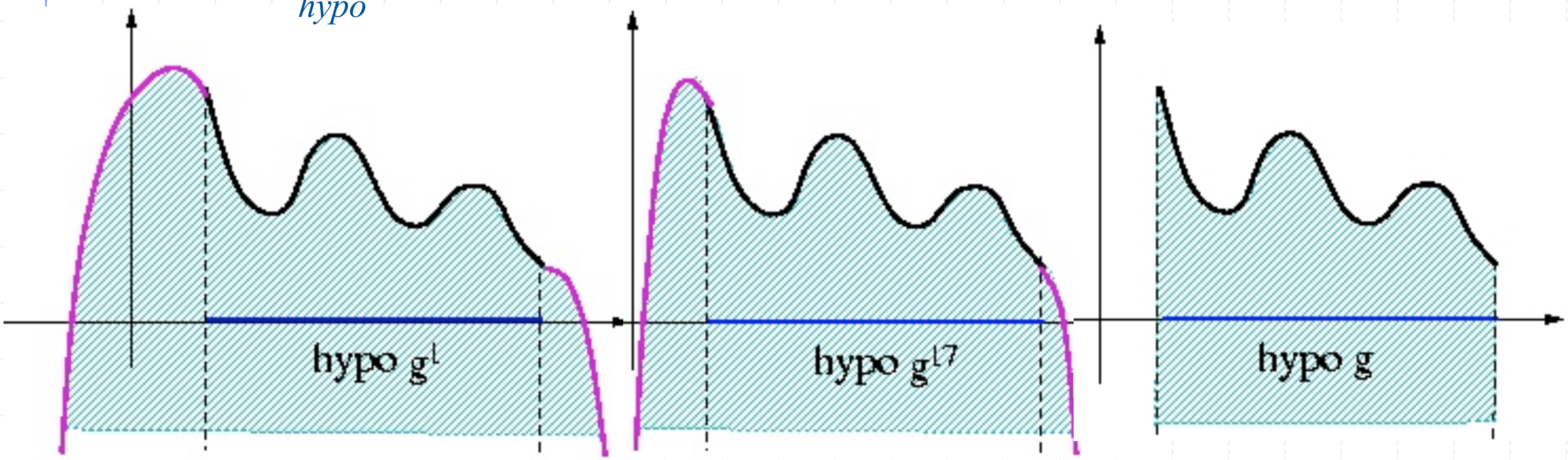
  (a) $\forall x^\nu \in D^\nu \to x \in D, \ \limsup f^\nu(x^\nu) \leq f(x)$

  (a_∞) $\forall x^\nu \in D^\nu \to x \notin D, \ f^\nu(x^\nu) \to -\infty$

  (b) $\forall x \in D, \exists x^\nu \to x, \liminf f^\nu(x^\nu) \geq f(x)$

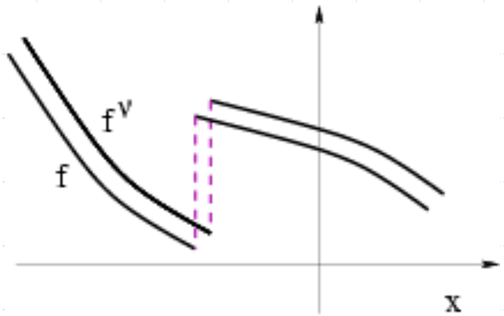
Hypo-Convergence

- $\boxed{\text{argmax } g^a \sim \text{argmax } g}$
- $g^v \xrightarrow[h]{} g \Leftrightarrow -g^v \rightarrow_{epi} -g$
- $g^v \xrightarrow[hypo]{} g \Leftrightarrow \text{hypo } g^v \rightarrow \text{hypo } g$



Hypo-convergence: Properties

- $f^\nu \to_h f \neq f^\nu \to_p f$
- $f^\nu \to_u f \Rightarrow f^\nu \to_h f$



- $f^\nu \underset{hypo}{\to} f, x^\nu \in \arg\max_{D^\nu} f^\nu, x^{\nu_k} \to \bar{x} \in D \Rightarrow \bar{x} \in \arg\max_D f$
- $\bar{x} \in \arg\max_D f \Rightarrow \exists \varepsilon^\nu \searrow 0, x^\nu \in \varepsilon^\nu\text{–}\arg\max_{D^\nu} f^\nu : x^\nu \to \bar{x}$
- $f^\nu \to_h f \Leftrightarrow dl(\text{hypo } f^\nu, \text{hypo } f) \to 0$

  (Γ-convergence)

Tight Hypo-Convergence

- $f_{D^\nu}^\nu \underset{h-tightly}{\longrightarrow} f_D : f_{D^\nu}^\nu \underset{h}{\longrightarrow} f_D \ \& \ \forall \varepsilon > 0, \exists B \text{ compact}:$

  $$\forall \nu > \nu_\varepsilon : \sup_{B \cap D^\nu} f^\nu \leq \sup_{D^\nu} f^\nu - \varepsilon$$

- THM: $f_{D^\nu}^\nu \underset{h-tightly}{\longrightarrow} f_D, \sup_D f \in \mathbb{R} \Rightarrow \sup_{D^\nu} f^\nu \to \sup_D f$

  also: $x^\nu \in \arg\max f_{D^\nu}^\nu, x^{\nu_k} \to \bar{x} \in D \Rightarrow \bar{x} \in \arg\max f_D$

Lopsided convergence

max-inf framework

$$K : C \times D \to \mathbb{R},\ \left\{K^\nu : C^\nu \times D^\nu \to \mathbb{R}\right\}_{\nu \in \mathbb{N}}$$

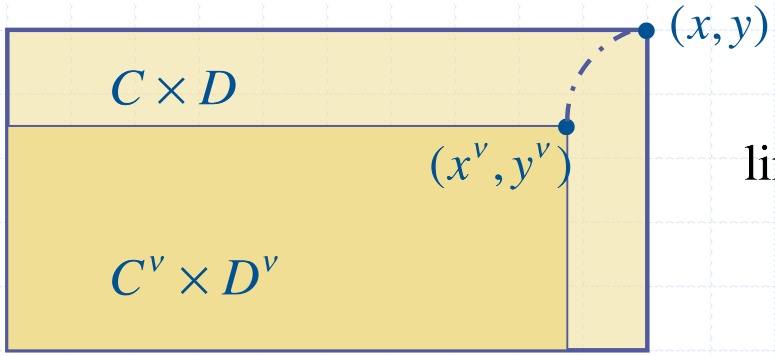
$$\boxed{\arg\max\text{-}\inf K^a \simeq \arg\max\text{-}\inf K}$$

$$\bar{x} \in \arg\ \max\text{-}\inf K \Leftrightarrow \bar{x} \in \arg\max(\inf_y K(x,y))$$

Lopsided convergence: definition

$\exists y^\nu \in D^\nu$

$\to y \in D$



$\forall x^\nu \in C^\nu \to x$

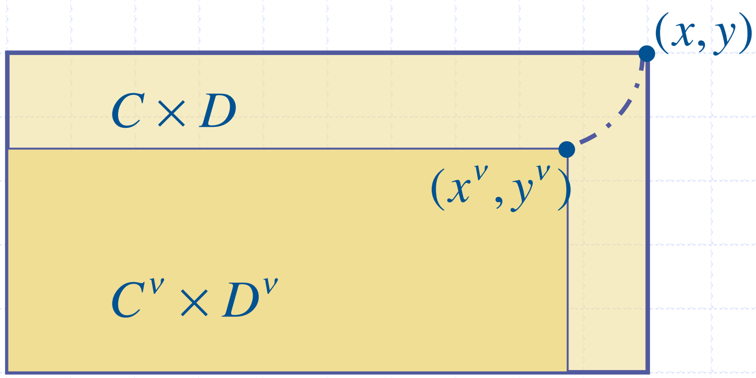
$$\limsup_\nu K^\nu(x^\nu, y^\nu) \le K(x,y) \text{ when } x \in C$$

$$K^\nu(x^\nu, y^\nu) \to -\infty \text{ when } x \notin C$$

like hypo/epi convergence

$\forall y^\nu \in D^\nu$

$\to y$



$\exists x^\nu \in C^\nu \to x \in C$

$$\liminf_\nu K^\nu(x^\nu, y^\nu) \ge K(x,y) \text{ when } y \in D$$

$$K^\nu(x^\nu, y^\nu) \to \infty \text{ when } x \notin D$$

<u>unlike</u> hypo/epi convergence

Lopsided convergence

$$\boxed{\text{argmax-inf } K^a \sim \text{argmax-inf } K}$$

$$\bar{x} \in \arg\ \text{max-inf}\, K \Leftrightarrow \bar{x} \in \arg\max(\inf_y K(x,y))$$

definition: $K : C \times D \to \mathbb{R},\ \left\{K^\nu : C^\nu \times D^\nu \to \mathbb{R}\right\}_{\nu\in\mathbb{N}}$

(a) $\forall x^\nu \in C^\nu \to x \in C, \forall y \in D\, \exists y^\nu \in D \to y :$

$$\limsup K^\nu(x^\nu, y^\nu) \le K(x,y)$$

$(\mathrm{a}_\infty) \forall x^\nu \in C^\nu \to x \notin C, \forall y\ \ \exists y^\nu \in D^\nu \to y : K^\nu(x^\nu, y^\nu) \to -\infty$

(b) $\forall x \in C, \exists x^\nu \to x, \forall y^\nu \in D^\nu$ and $y^\nu \to y :$

$$\limsup K^\nu(x^\nu, y^\nu) \le K(x,y) \ \ \text{if } y \in D$$

$$K^\nu(x^\nu, y^\nu) \to \infty \quad \text{if } y \notin D$$

Lopsided: elementary properties

$$\boxed{\text{argmax-inf } K^a \sim \text{argmax-inf } K}$$

lopsided $\Rightarrow$ epi-convergence if $K(x,y) = g(y)$

lopsided $\Rightarrow$ hypo-convergence if $K(x,y) = f(x)$

Remarks: not a Γ-convergence,
in ∞-dim. different topologies

Ancillary tight lop-convergence

- $K^\nu_{C^\nu \times D^\nu} \underset{ancil.-tight}{\longrightarrow} K_{C\times D}$ if $K^\nu_{C^\nu \times D^\nu} \underset{lop}{\longrightarrow} K_{C\times D}$ &

  (b) $\forall x \in C, \exists x^\nu \to x, \forall y^\nu \in D^\nu$ and $y^\nu \to y$:

  $$\liminf K^\nu(x^\nu, y^\nu) \ge K(x,y) \quad \text{if } y \in D$$

  $$K^\nu(x^\nu, y^\nu) \to \infty \quad \text{if } y \notin D$$

  but also $\forall \varepsilon > 0$, $\exists B_\varepsilon$ compact (depends on $x^\nu \to x$):

  $$\inf_{B_\varepsilon \cap D^\nu} K^\nu(x^\nu, \bullet) \le \inf_{D^\nu} K^\nu(x^\nu, \bullet) + \varepsilon, \ \forall \nu \ge \nu_\varepsilon$$

- THM. $K^\nu_{C^\nu \times D^\nu} \underset{lop}{\longrightarrow} K_{C\times D}$ ancillary tight, $\bar{x}$ cluster point of

  $\{x^\nu \in \arg\max\text{–}\inf K^\nu_{C^\nu \times D^\nu}\}_{\nu\in\mathbb{N}} \Rightarrow \bar{x} \in \arg\max\text{–}\inf K_{C\times D}$

Proof ....

- $K^{\nu}_{C^{\nu}\times D^{\nu}} \underset{lop}{\longrightarrow} K_{C\times D}$ ancillary tight

  Let $g^{\nu} = \inf_{y\in D^{\nu}} K^{\nu}(\bullet, y), \quad g = \inf_{y\in D} K(\bullet, y).$

  $$\Rightarrow g^{\nu} \underset{hypo}{\longrightarrow} g \quad \text{when} \quad \begin{cases} C^{\nu}_g = \left\{x\in C^{\nu} \,\middle|\, g^{\nu}(x) > -\infty\right\} \\ C_g = \left\{x\in C \,\middle|\, g(x) > -\infty\right\} \end{cases} \neq \emptyset$$

- then apply

  $$g^{\nu}_{C^{\nu}} \underset{hypo}{\longrightarrow} g_C,\ x^{\nu}\in \arg\max_{C^{\nu}} g^{\nu},\ x^{\nu_k} \to \bar{x}\in C \Rightarrow \bar{x}\in \arg\max_C g$$

Ky Fan functions & Inequality

- $K : C \times C \to \mathbb{R}$ Ky Fan function if

  (a) $\forall$ y: $x \mapsto K(x,y)$ usc

  (b) $\forall$ x: $y \mapsto K(x,y)$ convex

- K Ky Fan fcn, dom $K = C \times C$, C compact

  $$\Rightarrow \arg\max\text{–}\inf K \neq \emptyset$$

  if $K(x,x) \geq 0$ on dom K, $\bar{x} \in \arg\max\text{–}\inf K$

  $$\Rightarrow \inf_y K(\bar{x}, y) \geq 0.$$

Extending Ky Fan's inequality

- $K^\nu \underset{lop}{\longrightarrow} K$ ancillary tight with $C^\nu \to C$,
- K^ν Ky Fan $\Rightarrow K$ Ky Fan
  & $\forall \nu : \arg\max\text{–}\inf K^\nu \neq \varnothing$
  if $\bar{x} \in$ cluster-pts $\{\arg\max\text{–}\inf K^\nu\}$
  $\Rightarrow \bar{x} \in \arg\max\text{–}\inf K$ & $K(\bar{x}, \bullet) \geq 0$
- Ky Fan fcns closed under lopsided

  saddle fcns closed under h/e-convergence

  usc fcns closed under hypo-convergence

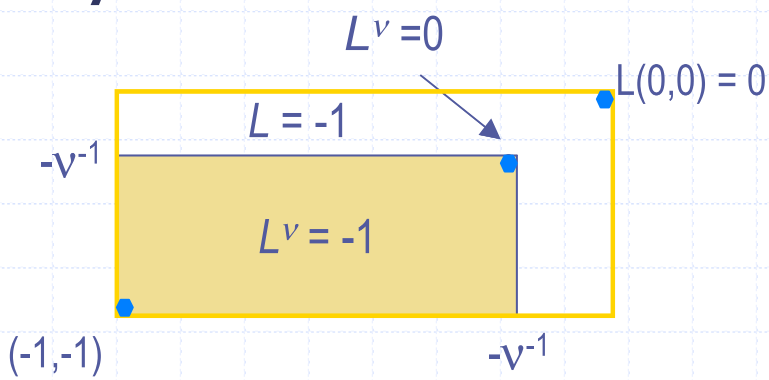
Lopsided & hypo/epi-convergence

- lopsided $\Longrightarrow$ hypo/epi-convergence
- not conversely

$\forall x^\nu \to 0,$

$\exists y^\nu \to 0 \ldots$

hypo/epi



$\exists x^\nu \to 0,$

$\forall y^\nu \to 0 \ldots$

lopsided

$$(x,y) = (0,0), y^\nu = -\nu^{-1} \Rightarrow x^\nu = -\nu^{-1} : \liminf K^\nu(x^\nu, y^\nu) \geq 0$$

$$\text{but with } \hat{y}^\nu = -\nu^{-1}/2 : \liminf K^\nu(x^\nu, \hat{y}^\nu) = -1 < 0 \text{ !}$$

Lopsided & hypo/epi-convergence

- $L^\nu \underset{h/e}{\longrightarrow} L$, convex-concave $\Rightarrow L^\nu \underset{lop}{\longrightarrow} L$

first condition … identical

for the second condition: given (x, y), let

$$u \in \partial_x L(x,y), -v \in \partial_y(-L(x,y))$$

$$L^\nu \underset{h/e}{\longrightarrow} L \Rightarrow L^\nu - \langle u, \bullet \rangle + \langle v, \bullet \rangle \underset{h/e}{\longrightarrow} L - \langle u, \bullet \rangle + \langle v, \bullet \rangle$$

(x, y) saddle point of $L - \langle u, \bullet \rangle + \langle v, \bullet \rangle$

$\exists (x^\nu, y^\nu)$ saddle-pts $\to (x, y)$ & $\{x^\nu\}_{\nu\in\mathbb{N}}$ is \`lop-sequence' $\to x$

APPLICATIONS

Fixed Points (Brouwer)

C compact convex, $G : C \to C$ continuous

find $\bar{x} \in C$: $G(\bar{x}) = \bar{x}$, fixed point

set $K(x, y) = \langle G(x) - x, y \rangle$

Ky Fan Inequality applies (usc,convex, K(x,x) ≥ 0):

$\exists \bar{x}$ such that $\langle G(\bar{x}) - \bar{x}, y \rangle \le 0, \forall y \in C$

$\Rightarrow \bar{x}$ is a fixed point of G

Convergence of fixed points

$$G^\nu : C^\nu \to C^\nu \text{ continuous, } C^\nu \text{ convex}$$

$$C^\nu \to C \Rightarrow C^\nu \text{compact,} \quad G^\nu \underset{cont}{\longrightarrow} G$$

$$\text{i.e., } \forall x^\nu \to x : G^\nu(x^\nu) \to G(x)$$

$$\Rightarrow K^\nu \underset{lop}{\longrightarrow} K,\ K(x,y) = \langle G(x) - x, y \rangle$$

Hence $\exists x^\nu$ fixed points of G^ν on C^ν

$\to \bar{x}$ fixed point of G on C

Extension: C just convex

Set-valued Inclusions

- $S^\nu, S : \mathbb{R}^d \to cl/cvx\text{–}sets(\mathbb{R}^n)$, non-empty
- Inclusions: $S^\nu(u) \ni a^\nu,\ S(u) \ni 0, \quad a^\nu \to 0$
- $\sigma^\nu(\bullet, u) =$ supp. fcn $S^\nu(u)$, $\sigma(\bullet, u)$ of $S(u)$
- $\forall u : \sigma^\nu(\bullet, u), \sigma(\bullet, u)$ convex
- S^ν, S continuous, compact-valued (sufficient)
- $\Leftrightarrow \sigma^\nu(\bullet, u^k) \to_{epi} \sigma^\nu(\bullet, \bar{u})$ as $u^k \to \bar{u}$; also σ
- $\Leftrightarrow \sigma^\nu(\bullet, u^k) \to_p \sigma^\nu(\bullet, \bar{u}) \Rightarrow \sigma^\nu(x, \bullet),\ \sigma(x, \bullet)$ usc
- $K^\nu = \sigma^\nu - \langle a^\nu, \bullet \rangle, K = \sigma \ \because$ Ky Fan fcns

Variational Inequalities

- $C \subset \mathbb{R}^n$ nonempty, convex, compact
- $G : C \to \mathbb{R}^m$ continuous, ($m=n$)
- find $\bar{u} \in C$ such that $-G(\bar{u}) \in N_C(\bar{u})$

  where $v \in N_C(\bar{u}) \Leftrightarrow \langle v, u - \bar{u} \rangle \le 0, \forall u \in C$
- with $K(u,v) = \langle G(u), v - u \rangle$ on $\operatorname{dom} K = C \times C$

  $\Rightarrow$ K is a Ky Fan function, K(u,u) ≥ 0.

Find

$$\boxed{\bar{u} \in \arg\max\text{–}\inf K(\bullet,\bullet) \text{ so that } K(\bar{u},\bullet) \ge 0}$$

Convergence: V.I. + extension

$$C^\nu \to C \Rightarrow C^\nu \text{ compact } \nu \geq \bar{\nu},\ G^\nu \text{ continuous}$$

$$G^\nu \to_{cont} G : G^\nu(x^\nu) \to G(x),\ \forall x^\nu \in C^\nu \to x$$

$$K^\nu(u,v) = \left\langle G^\nu(u), v - u \right\rangle \text{ on dom } K^\nu = C^\nu \times C^\nu$$

i.e., Ky Fan functions

- lop-converge ancillary tight to *K*, i.e. any cluster point of the solutions of the approximating V.I. is sol'n of limit V.I.

(sufficient conditions)

Nash Equilibrium points

- $\bar{x}_a \in \arg\max u_a(x_a, \bar{x}_{-a}), \ \forall a \in \mathcal{A}$
- Nash function:

  $$N(x,y) = \sum_{a \in \mathcal{A}} u_a(x_a, x_{-a}) - \sum_{a \in \mathcal{A}} u_a(y_a, x_{-a})$$

- u_a usc & $u_a(x_a, \bullet)$ lsc; $u_a(\bullet, x_{-a})$ concave

  $\Rightarrow N$ a Ky Fan function & $N(x,x) \geq 0$

- $u_a^\nu \to_{cont} u_a$ & dom $u_a^\nu \to$ dom u_a compact

  $\Rightarrow N^\nu \to N$ ancillary tight

  $\Rightarrow$ Nash equilibr.$^\nu$ $\to$ Nash equilibr. (cluster)

Walras Equilibrium points

- $\forall a \in \mathcal{A}:\ d_a(p) = \arg\max \left\{ u_a(x_a) \,\middle|\, \langle p, x_a \rangle \le \langle p, e_a \rangle \right\}$
- $s(p) = \sum_a (e_a - d_a(p))$ excess supply
- find $\bar{p} \in \Delta$ (unit simplex) so that $s(\bar{p}) \ge 0$
- Walrasian: $W(p,q) = \langle q, s(p) \rangle$ Ky Fan fcn
- $\boxed{\bar{p} \in \arg\max\text{–}\inf W \Leftrightarrow s(\bar{p}) \ge 0}$
- conditions: $e_a \in \text{int dom } u_a$, "globally compact"
- Convergence: $u_a^\nu \to_{hypo} u_a,\ e_a^\nu \to e_a \Rightarrow$

  W^ν lop-converge ancillary tight to W