Multi-Agent Optimization (4)

II. Deterministic models

Variational Inequality technology

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Linear Complementarity Problem

- find

$$\widehat{z} \geq 0 : \langle \widehat{w}, \widehat{z} \rangle = 0 \text{ and } 0 \leq \widehat{w} = M\widehat{z} + q$$

- approximating problem: find

$$z \geq 0 : \langle w, z \rangle = 0 \quad \text{and} \quad 0 \leq w = M^{\nu} z + q^{\nu} \leq r^{\nu}$$

- suppose

$$M^{\nu} \to M,\ q^{\nu} \to q,\ r^{\nu} \nearrow \infty \text{ and } z^{\nu} \underset{cluster}{\to} \overline{z}$$

- when does it imply

$$\overline{z} \text{ solves LCP? } \ldots \text{ under what conditions?}$$

LCP: The approach

- define $\boxed{K(z,v) = \langle Mz+q, v-z\rangle \text{ on } \mathbb{R}^n_+ \times \mathbb{R}^n_+}$
- $\hat{z}$ solves LCP: $z \geq 0 : \langle w, z\rangle = 0$ and $0 \leq w = Mz+q$
- $\Leftrightarrow \hat{z} \in \arg\max\text{-inf } K \;\&\; K(\hat{z},\bullet) \geq 0$
  - $\hat{z}$ solves LCP: $\inf K(z,\bullet) = -\infty$ unless $Mz+q \geq 0$ ($\hat{z}$ is such a z) with $Mz+q \geq 0, v=0$ is optimal, and
  - $0 = \max_{z\geq 0} -\langle Mz+q, z\rangle$ attained by $\hat{z}$ with $K(\hat{z},\bullet) \geq 0$

  $\hat{z} \in \arg\max\text{-inf } K \;\&\; K(\hat{z},\bullet) \geq 0$

  $\Rightarrow \hat{z} \geq 0, M\hat{z}+q \geq 0 \;\&\; 0$ potential arg max-inf value

  since $K(\hat{z},\bullet) \geq 0 \Rightarrow \langle M\hat{z}+q, \hat{z}\rangle \leq \langle M\hat{z}+q, 0\rangle$ $(v=0)$

  $\Rightarrow \langle M\hat{z}+q, \hat{z}\rangle = 0$, i.e., $\hat{z}$ solves LCP

LCP:Approximate solutions

- $K^{\nu}(z,v) = \left\langle M^{\nu} z + q^{\nu}, v - z \right\rangle$ on $\mathbb{R}^n_+ \times [0, r^{\nu}]^n$
- $z^{\nu} \in \arg\max\text{–}\inf K^{\nu}$ with $K^{\nu}(z^{\nu}, \bullet) \geq 0$
- $\left(M^{\nu} \to M, q^{\nu} \to q, r^{\nu} \nearrow \infty \right) \Rightarrow K^{\nu} \underset{lop}{\longrightarrow} K$

$$P = \left\{ z \in \mathbb{R}^n_+ \,\middle|\, Mz + q \geq 0 \right\},\ P^{\nu} = \left\{ z \in [0, r^{\nu}]^n \,\middle|\, M^{\nu} z + q^{\nu} \geq 0 \right\}$$

$$\boxed{\text{When } P^{\nu} \to P \Rightarrow K^{\nu} \underset{lop}{\longrightarrow} K \text{ ancillary tight}}$$

Nonlinear Complementarity Problem

- find

$$\hat{z} \geq 0 : \langle \hat{w}, \hat{z} \rangle = 0 \text{ and } 0 \leq \hat{w} = M(\hat{z}) + q$$

- approximating problem: find

$$z \geq 0 : \langle w, z \rangle = 0 \quad \text{and} \quad 0 \leq w = M^{\nu}(z) + q^{\nu} \leq r^{\nu}$$

- suppose

$$M^{\nu} \to M,\ q^{\nu} \to q,\ r^{\nu} \nearrow \infty \text{ and } z^{\nu} \underset{cluster}{\to} \bar{z}$$

- define bivariate functions:

$$K(z,v) = \langle M(z) + q, v - z \rangle \quad \text{on} \quad \mathbb{R}^n_+ \times \mathbb{R}^n_+$$

$$K^{\nu}(z,v) = \langle M^{\nu}(z) + q^{\nu}, v - z \rangle \quad \text{on} \quad \mathbb{R}^n_+ \times [0, r^{\nu}]^n$$

Arrow-Debreu model

- Economy: pure exchange of goods
- agents: $i \in I, \quad |I|$ finite
- consumption by agent *i*: x_i, free disposal
- endowment: e_i, strict survivability $(\in \operatorname{int} X_i)$
- utility: $u_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, concave, insatiability
- survival set: $X_i = \operatorname{dom} u_i$, convex, not necessarily closed
- exchange at market prices: $p \in \Delta$, unit simplex
- budget constraint: $\langle p, x_i \rangle \leq \langle p, e_i \rangle$

Market Clearing & Equilibrium

agent's problem: $\bar{x}_i(p) \in \arg\max \left\{ u_i(x_i) \middle| \langle p, x_i - e_i \rangle \le 0 \right\}$

market clearing: $\sum_{\mathrm{i}} \left(e_i - \bar{x}_i(p) \right) = s(p) \ge 0$

equilibirum price: $\bar{p}$ such that $s(\bar{p}) \ge 0$

Equilibrium with utility scaling: market clearing with

$s_l(\bar{p}) = 0$ when $\bar{p}_l > 0$ and $\exists \bar{\lambda}_i$ (utility scale factor) so that

$$\bar{x}_i \in \arg\max \left\{ u_i(x_i) - \bar{\lambda}_i \langle \bar{p}, x_i - e_i \rangle \right\}$$

$$\bar{\lambda}_i \ge 0, \langle \bar{p}, x_i - e_i \rangle \le 0 \text{ if } \bar{\lambda}_i = 0$$

$$\langle \bar{p}, x_i - e_i \rangle = 0 \text{ if } \bar{\lambda}_i > 0$$

Variational representation

$$\text{V.I.: } \bar{z} \in C, -G(\bar{z}) \in N_C(\bar{z}) = \left\{ v \middle| \langle v, z-\bar{z} \rangle \le 0, \forall z \in C \right\}$$

existence of solutions: C convex compact, G continuous

$$\text{define} \quad C = \Delta \times \left[\prod_{i\in I} X_i \right] \times \left[\prod_{i\in I} \mathbb{R}_+ \right]$$

$$G : C \to \mathbb{R}^n \times \left[\prod_{i\in I} \mathbb{R}^n \right] \times \left[\prod_{i\in I} \mathbb{R} \right]$$

$$G(p;\ldots,x_i,\ldots;\ldots,\lambda_i,\ldots)$$

$$= \left(\sum\nolimits_i [e_i - x_i];\ldots,\lambda_i p - \nabla u_i(x_i),\ldots;\ldots,p[e_i - x_i],\ldots \right)$$

V.I. & Equilibrium

$\exists\left(\bar{p},(\bar{x}_i),(\bar{\lambda}_{\mathrm{i}})\right)$ an equilibirum with utility scaling

$$\Updownarrow$$

$$G(p,(x_i),(\lambda_i)) = \left[\sum\nolimits_i (e_i - c_i);\left(\lambda_i p - \nabla u_i(x_i)\right);\left\langle p, e_i - x_i\right\rangle\right]$$

$$C = \Delta \times \left(\prod\nolimits_i X_i\right) \times \left(\prod\nolimits_i \mathbb{R}_+\right)$$

$\boxed{-G(\bar{p},(\bar{x}_i),(\bar{\lambda}_i)) \in N_C(\bar{p},(\bar{x}_i),(\bar{\lambda}_i))}$ has a solution

C (unfortunately) is unbounded

bounding *D*: "solvable" V.I.

from C to $\hat{C}$ bounded with

explicit bounds derived via duality

(global bound for X_i: κ_i depends on 'var'(u_a))

$$-G(\bar{p},(\bar{x}_i),(\bar{\lambda}_i)) \in N_{\hat{C}}(\bar{p},(\tilde{x}_i),(\bar{\lambda}_i))$$

$$\hat{C} = \Delta \times \left(\prod_i \hat{X}_i\right) \times \left(\prod_i [0,\kappa_i]\right)$$

Polyhedral case: efficient algorithmic procedures

Limiting iterative scheme

Theorem. When $\kappa^\nu \geq$ threshold, $\left(p^\nu,(x_i^\nu)\right)$ furnish a classical equilibrium w.r.t. X_i and u_i but possibly with different endowments $\mathrm{e}_\mathrm{i}^\nu \geq e_i, \mathrm{e}_\mathrm{i}^\nu \to e_i$. The sequence of nearby classical equilibria $\left(p^\nu,(x_i^\nu)\right)$ is bounded and every cluster point $\left(\bar{p},(\bar{x}_i)\right)$ furnishes a virtual equilibirum. When only one virtual equilibirum exists, it's the limit of the sequence.

Proof. Lopsided-convergence applied to V.I.$^\nu$ $\square$

Consumption & Production

Consumers: choose $x_i \in X_i$

Producers: choose $y_j \in Y_j$

Endowment and shares: $e_i + \sum_j \theta_{ij} y_j, \theta_{ij} \geq 0, \sum_i \theta_{ij} = 1$

Walrasian Equilibrium: as earlier (adjusted endowments)

and $\bar{y}_j \in \arg\max \left[\left\langle \bar{p}, y_j \right\rangle \middle| y_j \in Y_j \right]$

$\Rightarrow$ V.I. functional type: $-G(\bar{z}) \in \partial f(\bar{z}) \Leftrightarrow \bar{z} \in M(\bar{z})$

$M = (I + \alpha \partial f)^{-1} \circ (I - \alpha G)$, for any $\alpha > 0$

Path Solver … (M.Ferris *et al*)

$$-G(\bar{z}) \in N_C(\bar{z}), \quad \bar{z} = (\bar{p}, (\bar{x}_i), (\bar{\lambda}_i))$$

$$C = \Delta \times \left(\prod_i X_i\right) \times \left(\prod_i \mathbb{R}_+\right) = \{z \mid Az \geq b\}$$

Complementarity problem:

$$-G(z) = A^T y, \;\; y \geq 0, \;\; Az - b \perp y$$

with $K = \mathbb{R}^N \times \mathbb{R}_+^M$:

$$\boxed{(z,y) \in K, \;\; H(z,y) \in -K^*, \;\; (z,y) \perp H(z,y)}$$

$$H(z,y) = \begin{bmatrix} G(z) + A^T y \\ Az \end{bmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Equivalent nonsmooth mapping

- $0 = H(\mathrm{prj}_{\mathrm{K}}(z, y)) + (z, y) - \mathrm{prj}_{\mathrm{K}}(z, y)$
- with simplified *K*:

  (CP) $0 \le x \perp F(x) \ge 0$ Complementarity Problem

  (NS) $0 = F(x_+) + x - x_+$ Nonlinear system

- $\bar{x}$ sol'n (CP) $\Rightarrow$ $\tilde{x}$ sol'n (NS):

  $$\tilde{x}_i = \bar{x}_i \text{ if } F_i(\bar{x}) = 0, \; \tilde{x}_i = -F_i(\bar{x}_i) \text{ if } F_i(\bar{x}) > 0$$

  $\tilde{x}$ sol'n (NS) $\Rightarrow \tilde{x}_+$ sol'n (CP):

  $$\tilde{x}_+ \ge 0, F(\tilde{x}_+) = \tilde{x}_+ - \tilde{x} \ge 0 \;\&\; \tilde{x}_+ \perp \tilde{x}_+ - \tilde{x}$$

PATH Solver: $x = (z, y),\ x_+ = \mathrm{prj}_K(x, y)$

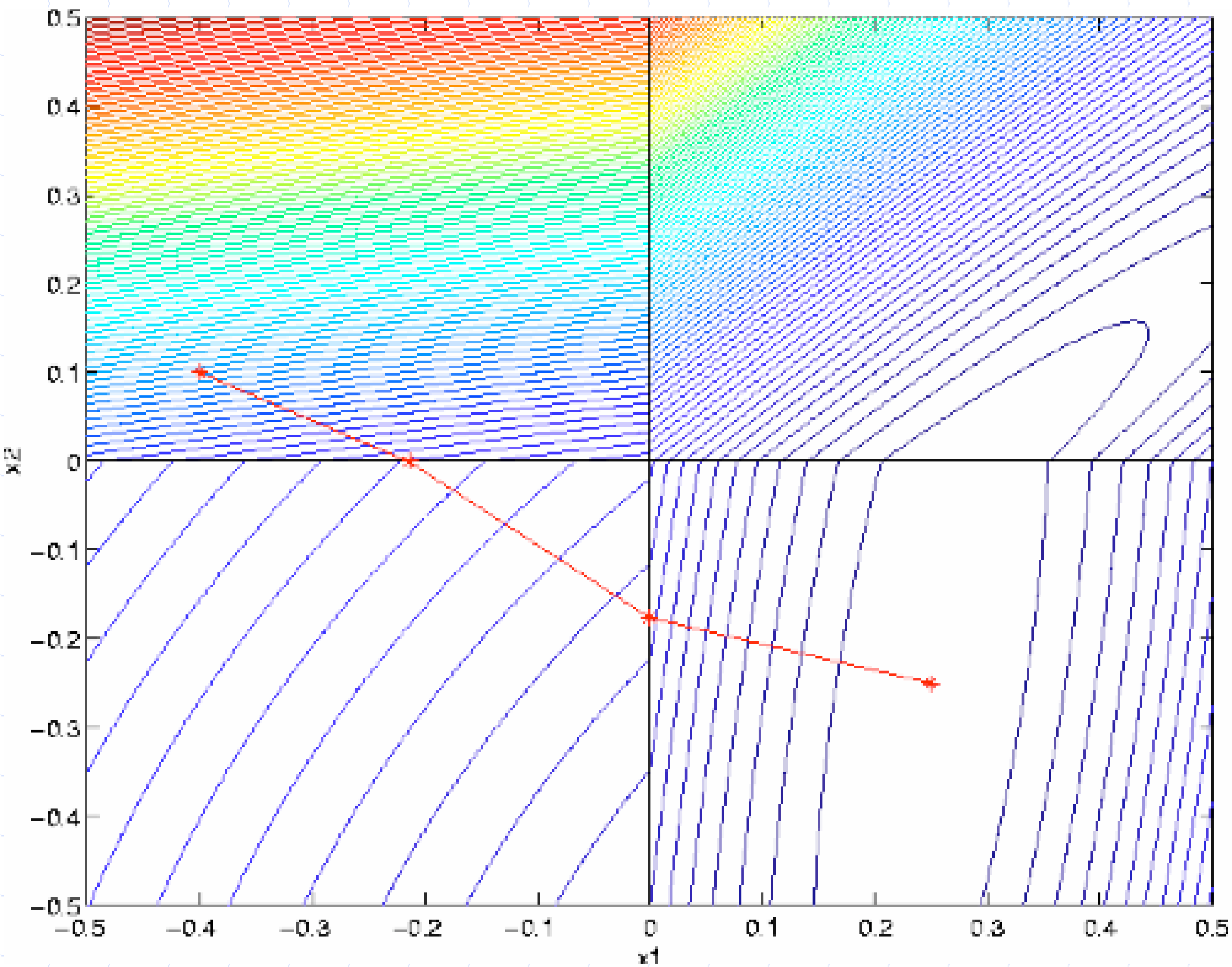
- PATH: Newton method based on nonsmooth normal mapping:

$$H(x_+) + x - x_+$$

- Newton point: solution of piecewise linearization:

$$H(x_+^k) + \left\langle \nabla H(x_+^k), x_+ - x_+^k \right\rangle + x - x_+ = 0$$

The “Newton” step



A dynamic problem

Agent's problems:

$$\max \quad u_i^0(x^0) + u_i^1(x^1)$$

$$\text{so that } \left\langle p^0, x^0 + T_i^0 y \right\rangle \leq \left\langle p^0, e^0 \right\rangle$$

$$\left\langle p^1, x^1 \right\rangle \leq \left\langle p^1, e^1 + T_i^1 y \right\rangle$$

$$x^0 \in X^0,\ \ x^1 \in X^1, y \geq 0$$

y_i activity levels (savings, production technology, ...)

$T_i^0 y_i$ input goods, $T_i^1 y_i$ output goods

using PATH Solver

- Economy: (5 goods)
  - Skilled & unskilled workers
  - Businesses: Basic goods & leisure
  - Banker: bonds (riskless), 2 stocks
- 2-stages,   solved under # of scenarios (280)
- utilities: CES-functions (gen. Cobb-Douglas)
  - Utility in stage 2 assigned to financial instruments
  - Financial instruments only used for transfer in stage 1
- used for calibration (-> stochastic model)

  numerically: `blink’ (5000 iterations).