Multi-Agent Optimization (6)

III. Stochastic Models

Stochastic: Numerical procedure(s)

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Pure Exchange: Walras

agent's problem: Agents: $\mathrm{i} \in \mathcal{I} \quad |\mathcal{I}|$ *finite* "large"

$$\bar{x}_i \in \arg\max u_i(x_i) \text{ so that } \langle p, x_i \rangle \leq \langle p, e_i \rangle, \;\; x_i \in X_i$$

e_i: endowment of agent i, $e_i \in \operatorname{int} X_i$

u_i: utility of agent i, concave, usc

$$u_i : X_i \to \mathbb{R}, \quad X_i \subset \mathbb{R}^n \text{ (survival set) convex}$$

market clearing: $s(p) = \sum_{\mathrm{i} \in \mathcal{I}} (e_i - \bar{x}_i)$ excess supply

equilibrium price: $\bar{p} \in \Delta$ such that $s(\bar{p}) \geq 0$

Δ unit simplex

The Walrasian

$$\boxed{W(p,q) = \langle q, s(p) \rangle,\ \ W : \Delta \times \Delta \to \mathbb{R}}$$

$\bar{p}$ equilibirum price (Ky Fan Inequality)

$$\Leftrightarrow \bar{p} \in \arg\max_p (\inf_q W(p,q))\ \&\ s(\bar{p}) \geq 0$$

Properties of W :

continuous in p ($e_i \in \operatorname{int} C_i$, '$i$-inf-compact') usc

linear in q, Δ compact, convex

$W(p,p) \geq 0, \forall p \in \Delta$

i.e., W is a Ky Fan function

Numerical Approaches

- Augmented Walrasian:

$\bar{p} \in$ argmax-inf W

$\cong$ saddle point $(\bar{p}, \bar{q})$ of $\tilde{W}_r$

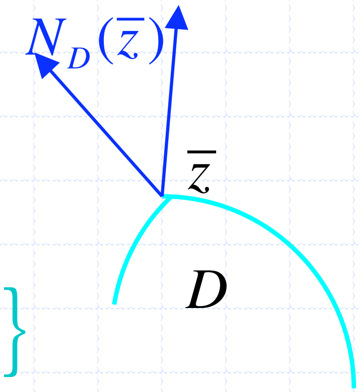
$$\tilde{W}_r(p,q) = \inf_z \left\{ W(p,z) \,\middle|\, \|z-q\| \le r \right\},$$

$\|\bullet\|$ an appropriate norm ($|\bullet|_\infty$ e.g.)

Variational Inequality

$$\max_x u_i(x_i) \text{ so that } \langle p, x_i \rangle \le \langle p, e_i \rangle,\ x_i \in C_i$$

$$\textstyle\sum_i (e_i - c_i) = s(p) \ge 0.$$



$$\Updownarrow \qquad N_D(\bar{z}) = \left\{ v \,\middle|\, \langle v, z - \bar{z} \rangle \le 0, \forall z \in D \right\}$$

$$G(p,(x_i),(\lambda_i)) = \left[\textstyle\sum_i (e_i - x_i); \left(\lambda_i p - \nabla u_i(x_i) \right); \langle p, e_i - x_i \rangle \right]$$

$$D = \Delta \times \left(\textstyle\prod_i C_i \right) \times \left(\textstyle\prod_i \mathbb{R}_+ \right)$$

$$\boxed{-G(\bar{p},(\bar{x}_i),(\bar{\lambda}_i)) \in N_D(\bar{p},(\bar{x}_i),(\bar{\lambda}_i))}$$

$$D \text{ unbounded} \rightarrow \hat{D} \text{ bounded}$$

Stochastic Equilibrium Model

Pure Exchange model
with Input/Output activities

Agent-i problem-stochastic

$$\max_{x_i^0, y_i \in \mathbb{R}^n, x_{i,\bullet}^1 \in \mathcal{M}} u_i^0(x_i^0) + E_i\left\{u_i^1(\xi, x_{i,\xi}^1)\right\}$$

$$\text{so that } \left\langle p^0, x_i^0 + T_i^0 y_i \right\rangle \leq \left\langle p^0, e_i^0 \right\rangle$$

$$\left\langle p_\xi^1, x_{i,\xi}^1 \right\rangle \leq \left\langle p_\xi^1, e_{i,\xi}^1 + T_{i,\xi}^1 y_i \right\rangle, \forall \xi \in \Xi$$

$$x_i^0 \in X_i^0,\ \ x_{i,\xi}^1 \in X_{i,\xi}^1,\ \ \forall \xi \in \Xi$$

- $E_i\{.\}$ rational expectation w.r.t. i-beliefs

Stochastic program with recourse: 2-stage
Well-developed solution procedures
Well-developed “approximation theory”

Simplest-classical assumptions

Ξ finite (support)

$u_i^0 : X_i^0 \to \mathbb{R},\ \forall \xi \in \Xi,\ u_i^1(\xi,\bullet) : X_{i,\xi}^1 \to \mathbb{R}$ concave

continuous. (numerics: differentiable)

$T_i^0, T_{i,\xi}^1$: input-ouput matrices

(production, investment, etc.)

$X_i^0, X_{i,\xi}^1$: closed, convex, non-empty interior

$e_i^0 \in \operatorname{int} X_i^0,\ e_{i,\xi}^1 \in \operatorname{int} X_{i,\xi}^1$ for all ξ

Market Clearing

Agents: $i \in \mathcal{I}$, $|\mathcal{I}|$ *finite* "large"

$$\left(\overline{x}_i^0, \overline{y}_i, \left\{\overline{x}_{i,\xi}^1\right\}_{\xi \in \Xi}\right) \in \arg\max\{\text{agent-}i \text{ problem}\}$$

excess supply:

$$\sum_{i \in \mathcal{I}} \left(e_i^0 - (\overline{x}_i^0 + T_i^0 \overline{y}_i)\right) = s^0(p^0, \{p_\xi^1\}_{\{\xi \in \Xi\}}) \geq 0$$

$$\forall \xi \in \Xi :$$

$$\sum_{i \in \mathcal{I}} \left(e_{i,\xi}^1 + T_{i,\xi}^1 \overline{y}_i - \overline{x}_{i,\xi}^1\right) = s_\xi^1(p^0, \{p_\xi^1\}_{\{\xi \in \Xi\}}) \geq 0$$

Here-&-Now vs. Wait-&-See

- Basic Process: decision --> observation --> decision

  $$(x_i^0, y_i) \;\rightarrow\; \xi \;\rightarrow\; x_{i,\xi}^1$$

- Here-&-now problem!

  not all contingencies available at time 0

  (x_i^0, y_i) <u>can't depend</u> on ξ!

- Wait-&-see problem

  implicitly all contingencies available at time 0

  choose $\left(x_{i,\xi}^0,\, y_{i,\xi}^0,\, x_{i,\xi}^1\right)$ after observing x

- incomplete complete market ?

Fundamental Theorem of Stochastic Optimization

A here-and-now problem can be "reduced" to a wait-and-see problem by introducing the

appropriate 'contingency' costs (price of nonanticipativity)

Contingencies prices (nonanticipativity)

Here-&-now

$$\max E\left\{ f(\xi, z^0, z^1_\xi) \right\}$$
$$z^0 \in C^0 \subset \mathbb{R}^{n_1},$$
$$z^1_\xi \in C^1_\xi(z^0), \ \forall \xi.$$

Explicit nonanti. constraints

$$\max E\left\{ f(\xi, z^0_\xi, z^1_\xi) \right\}$$
$$z^0_\xi \in C^0 \subset \mathbb{R}^{n_1},$$
$$z^1_\xi \in C^1_\xi(z^0), \ \forall \xi.$$
$$z^0_\xi = E\{z^0_\xi\} \ \forall \xi$$
$$w_\xi \perp \mathrm{c}^{\mathrm{ste}} \ \ \mathrm{fcns}$$
$$\Rightarrow E\{w_\xi\} = 0$$

Progressive Hedging

- Step 0. $w^0(\bullet)$ so that $E\left\{w^0(\xi)\right\} = 0,\ \ \nu = 0$
- Step 1. for all ξ:

$$\left(z_\xi^{0,\nu}, z_\xi^{1,\nu}\right) \in \arg\max f(\xi;\, z^0, z^1) - \left\langle w_\xi^\nu, z^0 \right\rangle$$

$$z^0 \in C^0 \subset \mathbb{R}^{n_0},\ z^1 \in C^1(\xi, x^0) \subset \mathbb{R}^{n_1}$$

- Step 2. $w_\xi^{\nu+1} = w_\xi^\nu + \rho\left[z_\xi^{0,\nu} - E\left\{z_\xi^{0,\nu}\right\}\right],\quad \rho > 0$
  - and return to Step 1, $\nu = \nu + 1$
- Convergence: add proximal term $-\frac{\rho}{2}\left\| z_\xi^{0,\nu} - E\left\{z_\xi^{0,\nu}\right\}\right\|^2$, linear rate in $\left(z^\nu, w^\nu\right)$

Disintegration: agent's problem

with $\quad p_{\blacklozenge} = \left(p^0, \left\{ p^1_\xi \right\}_{\xi \in \Xi} \right)$

$$\left(\overline{x}^0_{i,\xi},\ \overline{y}^0_{i,\xi},\ \overline{x}^1_{i,\xi} \right) \in$$

'*i*-contingency' costs

$$\underset{x_i^0, y_i, x_i^1}{\arg\max} \left\{ u_i^0(x_i^0) - \left\langle \overline{w}_{i,\xi}, (x_i^0, y_i) \right\rangle + u_i^1(\xi, x_i^1) \right\}$$

$$\left\langle p^0, x_i^0 \right\rangle \le \left\langle p^0, e_i^0 - T_i^0 y_i \right\rangle$$

$$\left\langle p^1_\xi, x_i^1 \right\rangle \le \left\langle p^1_\xi, e^1_{i,\xi} + T^1_{i,\xi} y_i \right\rangle,$$

$$x_i^0 \in X_i^0,\ \ x_i^1 \in X^1_{i,\xi}.$$

solved for each ξ <u>**separately**</u>

Incomplete to ‘Complete’ Market

$\forall \xi \in \Xi$ (separately),

agent's problem:

$$\left(\bar{x}_i^0, \bar{y}_i^0, \bar{x}_{i,\xi}^1\right) \in \arg\max \left\{ u_i^{w_{i,\xi}}\left(x_i^0, y_i^0, x_i^1\right) \text{on } \widehat{C}_{i,\xi}(p^0, p_\xi^1) \right\}$$

$$\text{for } \{w_{i,\xi}\}_{\xi\in\Xi} \text{ associated with } (p^0, p_\xi^1)$$

clear market:

$$s^0(p^0, p_\xi^1) \geq 0, \quad s_\xi^1(p^0, p_\xi^1) \geq 0$$

Arrow-Debreu ‘stochastic’ equilibrium problem

THE WALRASIAN

$$W(p_{\bullet}, q_{\bullet}) = \langle q_{\bullet}, s(p_{\bullet}) \rangle$$

$$= \left\langle (q^0, \{q^1_\xi\}_{\xi\in\Xi}), \left(s^0(p^0, \{p^1_\xi\}_{\xi\in\Xi}), \left\{ s^1_\xi(p^0, \{p^1_\xi\}_{\xi\in\Xi}) \right\}_{\xi\in\Xi} \right) \right\rangle$$

$$W : \prod_{1+|\Xi|} \Delta \times \prod_{1+|\Xi|} \Delta \to \mathbb{R}$$

linear w.r.t. $q_{\bullet}$, continuous w.r.t. $p_{\bullet}$

$W(p_{\bullet}, p_{\bullet}) \geq 0.$

provided $s(\bullet)$ continuous w.r.t. $p_{\bullet}$

like in lecture no.3

IV. Experimentation

with PATH Solver (experimental)

- Economy: (5 goods)
  - Skilled & unskilled workers
  - Businesses: Basic goods & leisure
  - Banker: bonds (riskless), 2 stocks
- 2-stages,   280 scenarios, 2776 scenarios
- utilities: CES-functions (gen. Cobb-Douglas)
  - Utility in stage 2 assigned to financial instruments
  - only used for transfer in stage 1
- on laptop: ~4 min, ~14 min, but

  extremely parallelizable algorithm

with PATH Solver (stochastic)

- objectives: $u_i^0(x_i^0) + u_i^1(x_i^1) \quad \Rightarrow$

$$u_i^0(x_i^0) - \left\langle w_{i,\xi}^v, (x_i^0, y_i) \right\rangle - \frac{\rho_i}{2} \left| (x_i^0, y_i) - (\hat{x}_i^{0,v}, \hat{y}_i^v) \right|^2 + u_i^1(x_i^1)$$

- updating: $(\hat{x}_i^{0,v}, \hat{y}_i^v) = E_i \left\{ (x_{i,\xi}^{0,v}, y_{i,\xi}^v) \right\}$

$$w_{i,\xi}^{v+1} = w_{i,\xi}^v + \rho_i((x_{i,\xi}^{0,v}, y_{i,\xi}^v) - (\hat{x}_i^{0,v}, \hat{y}_i^v))$$

$\rho_i > 0$ yields convergence

- also requires a proximal term to `support' convergence of equilibrium prices