Hele-Shaw Flows: Historical Overview

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Henri Selby Hele-Shaw

Hele-Shaw (1854–1941) one of the most prominent engineering researchers at the edge of XIX and XX centuries, a pioneer of Technical Education, great organizer, President of several engineering societies, including the Royal Institution of Mechanical Engineers, Fellow of the Royal Society, and ...

Henri Selby Hele-Shaw

... an example of undeserved forgotten great names in Science and Engineering.

1854–1871

Hele–Shaw was born on 29 July 1854 at Billericay (Essex).

A son of a successful solicitor Mr Shaw, he was a very religious person, influenced by his mother from whom he adopted her family name ‘Hele’ in his early twenties.

1871–1876

At the age of 17 he finished a private education and was apprenticed at the Mardyke Engineering Works, Messr Roach & Leaker in Bristol.

His brother Philip E. Shaw (Lecturer and then Professor in Physics, University College Nottingham) testifies:
"... Hele's life from 17 to 24 was a sustained epic: 10 hrs practical work by day followed by night classes".

1876–1885

In 1876 he entered the University College Bristol (founded in 1872) and in 1878 he was offered a position of Lecturer in Mathematics and Engineering under Professor J. F. Main.

In 1882 Main left the College and Hele-Shaw was appointed as Professor of Engineering while the Chair in Mathematics was dropped. He organized his first Department of Engineering.

1885–1904

In 1885 Hele-Shaw was invited to organize the Department of Engineering at the University College Liverpool (founded in 1881), his second department.

He served as a Profesor of Engineering until 1904 when we moved to South Africa.

1904–1906

In 1904 Hele-Shaw became the first Professor of Civil, Mechanical and Electrical Engineering of the Transvaal Technical Institute (founded in 1903) which then gave rise to the University of Johannesburg and the University of Pretoria.

It became his third department. In 1905 he was appointed Principal of the Institute and an organizer of Technical Education in the Transvaal.

1906–1941

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- In 1922 Hele-Shaw became the President of the Institution of Mechanical Engineers.

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He possessed a great sense of humor, was a good conversationalist (testimonies of his brother Philip, colleagues), loved companies. He was a great teacher, his free-hand drawing attracted special interest to his lectures.

1906–1941

Hele-Shaw was a man of great mental and physical alertness, of great energy and of great courage. He was a self-made person and was successful and recognized during his professional life.

- He married Miss Ella Rathbone, a member of a prominent Liverpool family;

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- He retired at the age 85 from his office in London and died 1.5 year later on 30 January 1941.

Hele-Shaw’s inventions

Two greatest inventions: Stream-line Flow Methods (1896-1900) and Automatic Variable-Pitch Propeller (1924), jointly with T. Beacham. Apart form these two:

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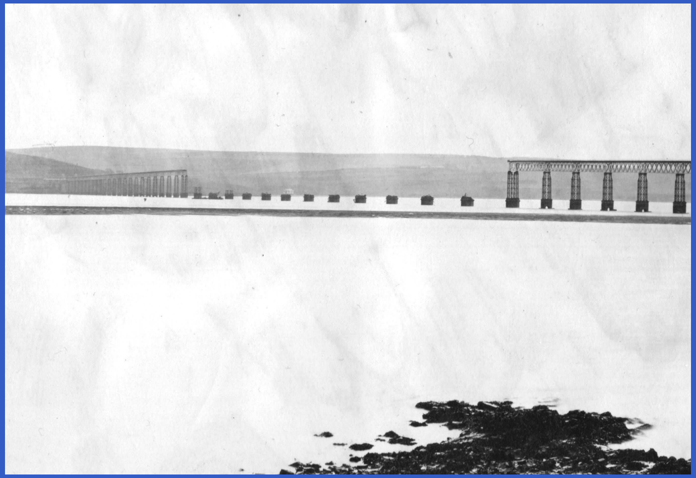
- Earliest original work (1881): the measurement of wind velocity (Tay Bridge disaster, 28 December 1879);



Invention of a new integrating anemometer.

Hele-Shaw’s inventions

- Special stream-line filter to purify water from oil pollution.
- Hele-Shaw (the first) Friction Clutch (1905) for cars, patent #GB795974. At a notable Paris Motor Show (1907) about 80% exhibited cars had the Hele-Shaw clutch.

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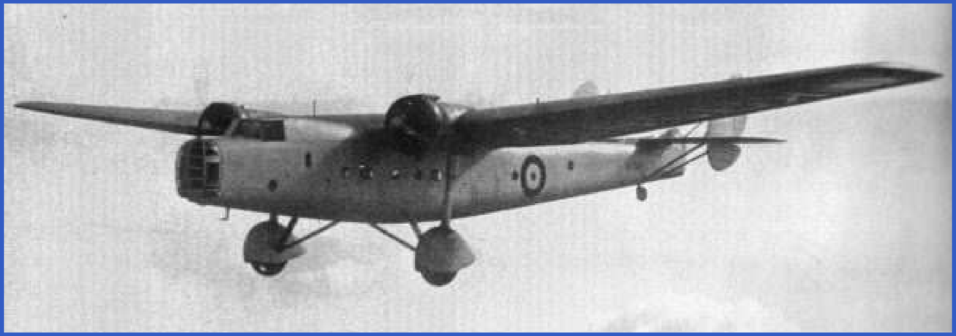
- Hele-Shaw hydraulic transmission gear (1912).
- Hele-Shaw pump (1923), ... etc. 82 patents.

Hele-Shaw's inventions

H. S. Hele-Shaw and T. E. Beacham patented the first constant speed, variable pitch propeller in 1924, patent #GB250292

Hele-Shaw’s inventions

- $\approx$ 1929 Fairey and Reed in UK, Curtiss in the USA;
- 1932 Variable pitch propellers were introduced into air force service;



- 1933 Boeing 247, passenger aircraft;
- 1935 Bristol Aeroplane Company/Rolls-Royce: Bristol Type 130 Bombay, medium bomber.

Hele-Shaw’s inventions

Further developments of variable pitch propeller by Hele-Shaw:

- 1929 Adjustable pitch propeller drive, patent #GB1723617;
- 1931 Control system for propeller with controllable pitch, patent #GB1829930.
- 1932 Hele-Shaw and Beacham invented ‘Exactor Control’, a remote mechanism to reproduce the control movements in aircrafts. Hele-Shaw was already 78!

Hele-Shaw Prizes

- Hele-Shaw Prize (University of Bristol) to the students in their Final Year in any Department with a good academic or social record not otherwise covered;
- Hele-Shaw Prize (University of Liverpool) for a candidate who has specially distinguished himself in the Year 2 examination for the degree of Bachelor or master of Engineering.

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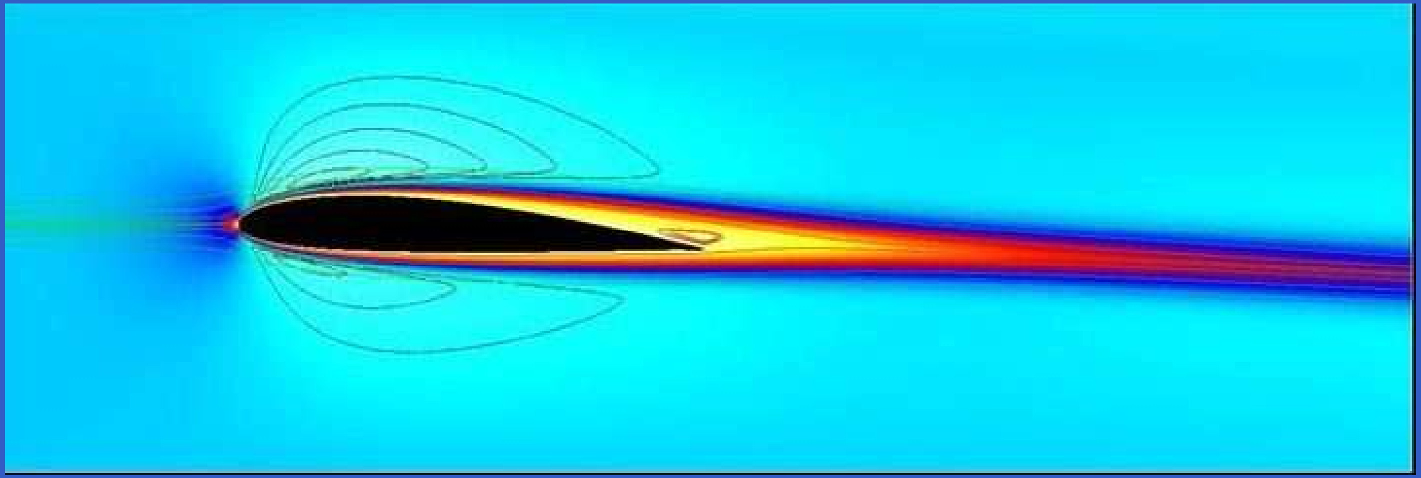
The sum is small: $\pounds 50$ and $\pounds 30$ each

Stream-line Flow Methods

The most notable Hele-Shaw's scientific research came from his desire to exhibit on a large screen the character of the flow past an object contained in a lantern slide for students in Liverpool:



Hele-Shaw wanted to visualize stream lines. He tried colouring liquid (unsuitable, immediately mixed), sand (formed eddies, modified the flow)...

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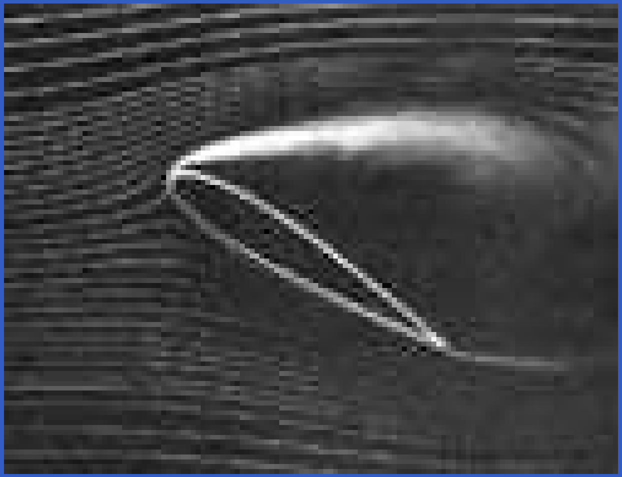

Apparently the glass got a small accidental leak providing small air bubbles acting as continuous tracers (1897).

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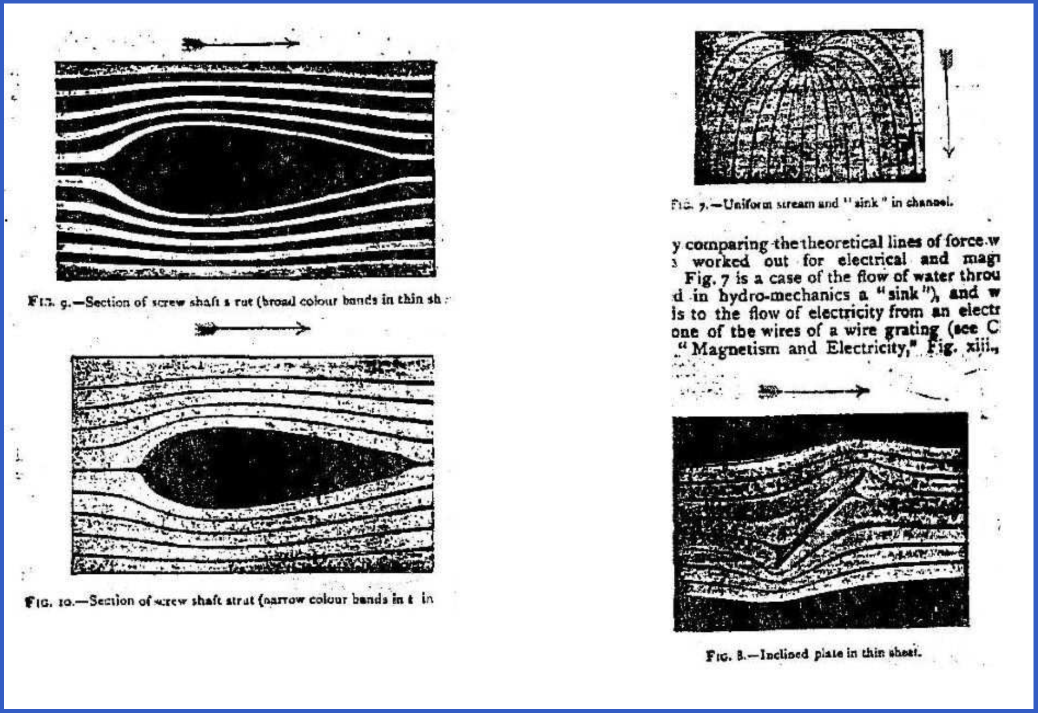

Hele-Shaw's photos taken from his 1898 paper.

Stream-line Flow Methods

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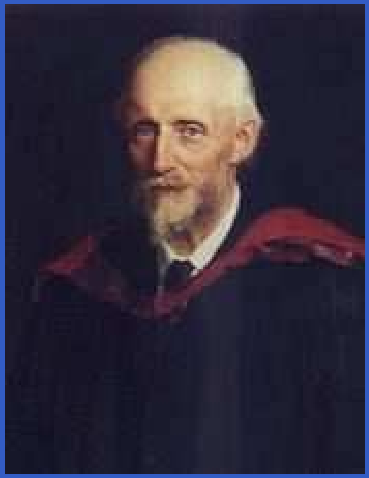
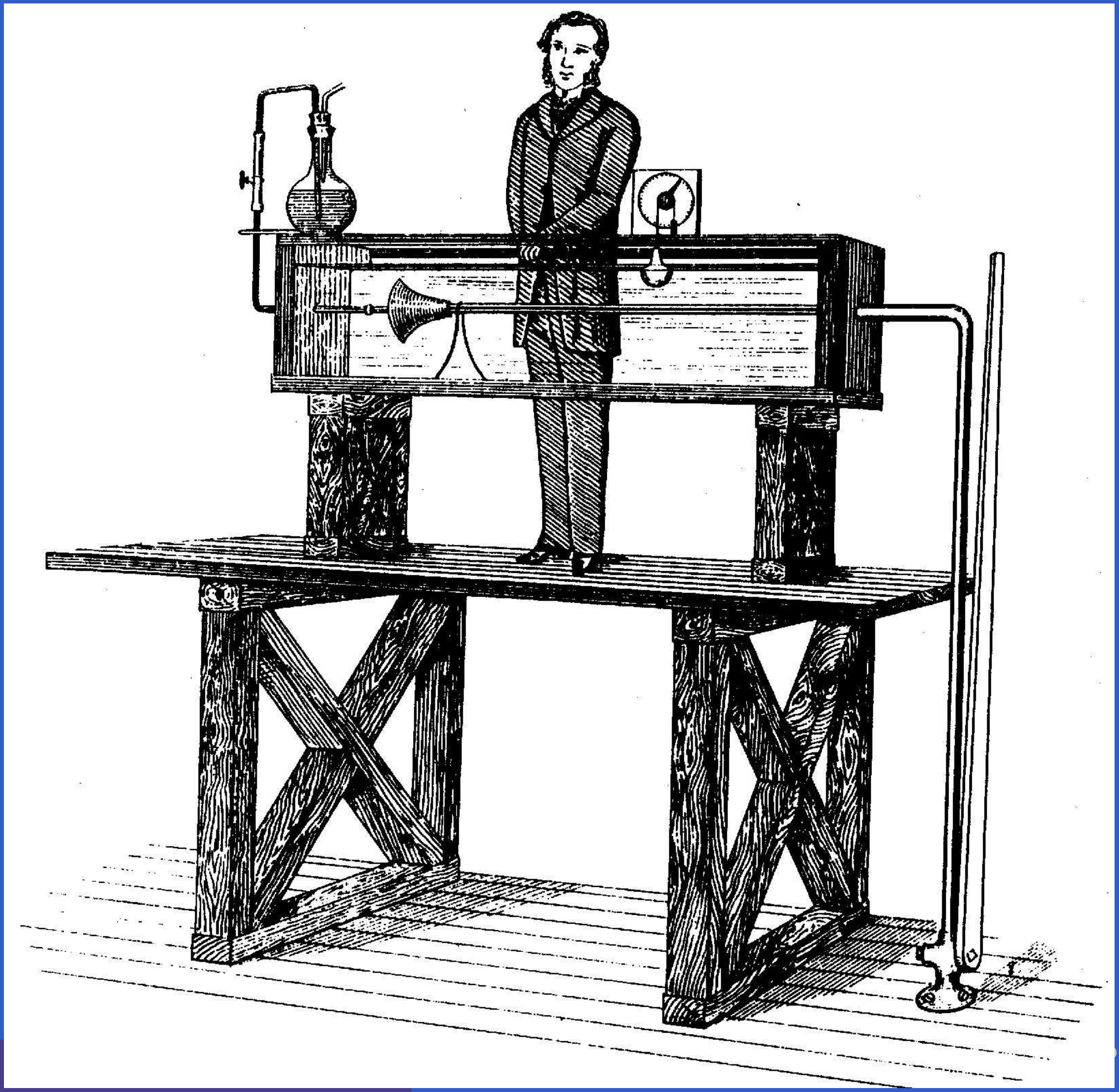


Later in 1898, Osborne Reynolds (1842–1912) criticized experiments by Hele-Shaw expecting turbulence at higher velocities.

Experiment by Reynolds

O. Reynolds (1873) revealed the turbulence phenomenon:



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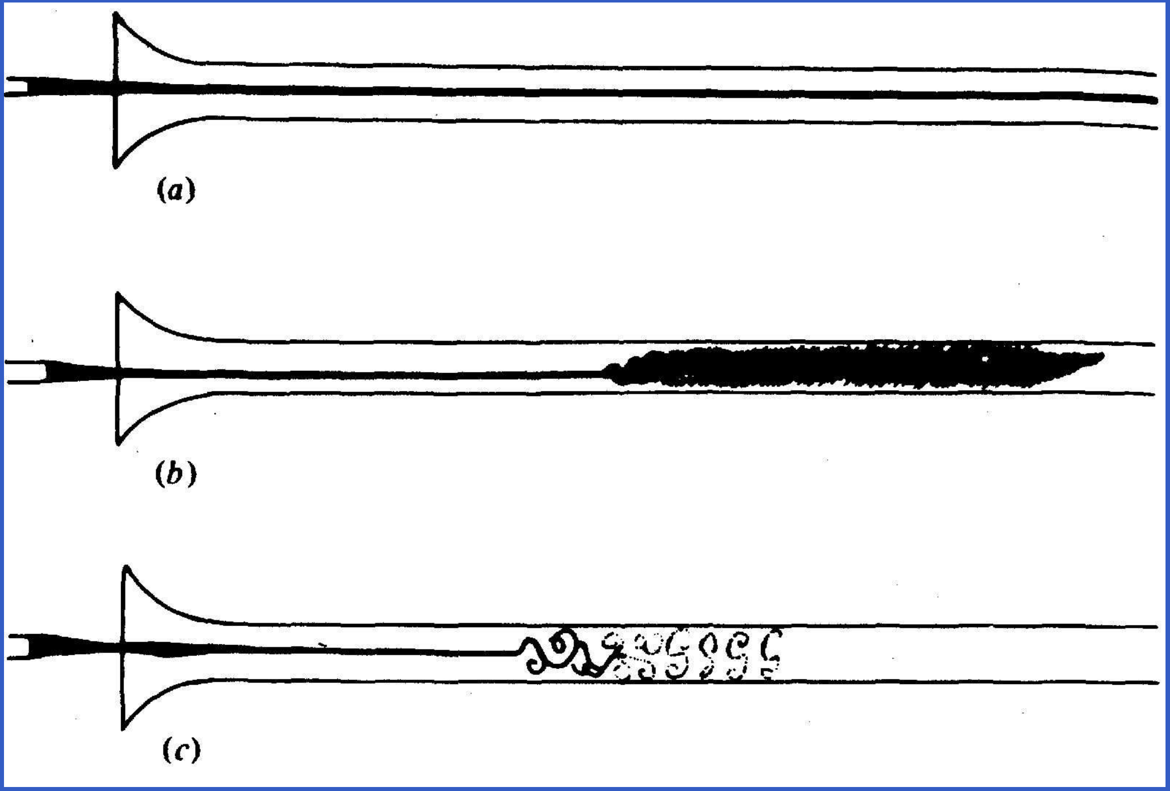

Sketches of Reynold’s dye experiment are taken from his 1883 paper.

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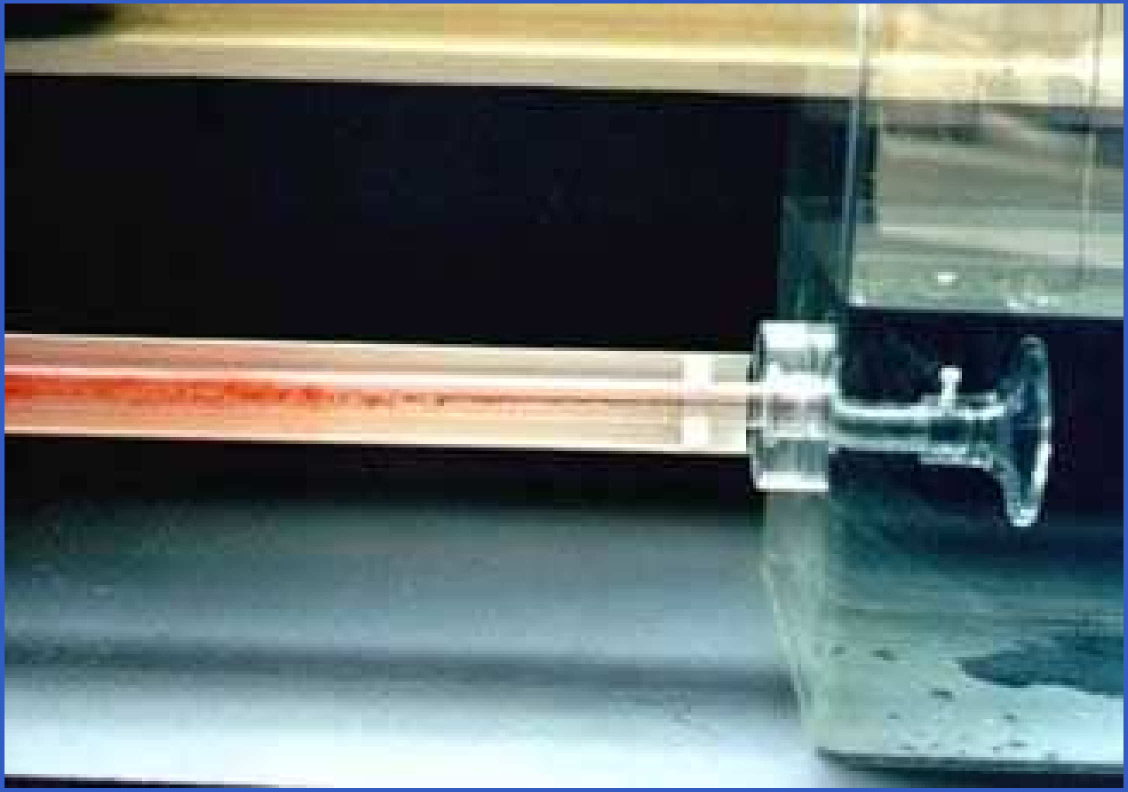
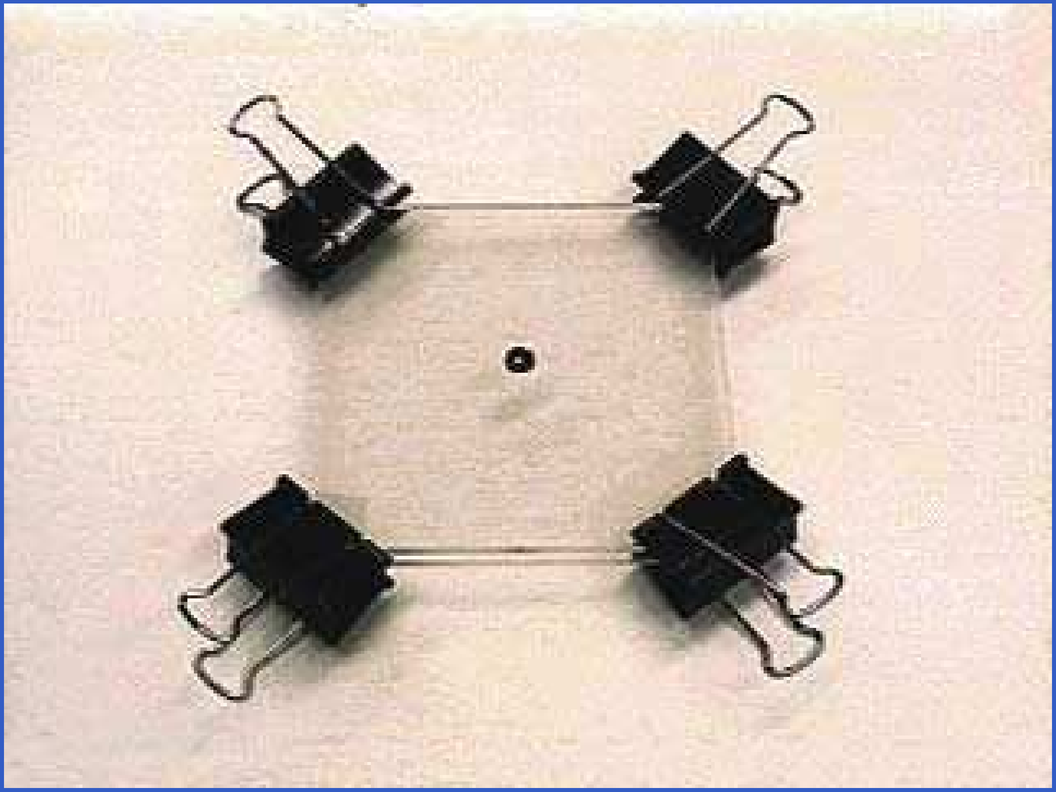


Hele-Shaw Cell

Hele-Shaw's greatest discovery: If the glass plates are mounted sufficiently close (0.02 inch) of each other, then the flow is laminar at all velocities!



Hele-Shaw Cell

Hele-Shaw got the Gold Medal from the Royal Institution of Naval Architects in 1898

NATURE [MAY 12, 1898

JAMES WIMSHURST.

Member of the Consultative Staff, Board of Trade. Qualifications: (1) Improvements in Electrical Influence Machines, which are now universally approved and adopted by Physicists; (2) an Influence Machine which gives charges of electricity, alternating from positive to negative with each rotation of the disc (in this type the glass discs, without any metal upon them, are freely self-exciting); (3) has delivered a lecture upon Influence Machines at the Royal Institution, April 27, 1888, and read papers at the Physical Society, April 17, 1891, and June 22, 1893.

THE FLOW OF WATER.

MORE than one hundred years ago, the French philosopher Coulomb caused a disc suspended by a torsion wire to oscillate in a vessel of liquid, and he thus ascertained that the resistance to various bodies under such circumstances, when the movement is a slow one, varies directly as the velocity of the motion. This law of resistance, it should be noted, is quite contrary to that of the friction between solid bodies as investigated by General Morin. Colonel Beaufoy, Froude, and others, however, found that, at higher velocities, the resistance varied more nearly as the square of the velocity. The difference of the two conditions in which the variation was directly, or, as the higher power, undoubtedly represented on the one hand the condition of water in which the mere viscosity came into play, resisting the shearing stress of the layers in passing over each other, and on the other hand the condition when the breaking up of the water into eddying motion caused the resistance to become much greater.

Prof. Osborne Reynolds, about 1883, investigated the critical velocity at which this change of state occurs,

Hele-Shaw Cell

Sir George Gabriel Stokes, 1st Baronet (1819–1903)

wrote: "Hele-Shaw's experiments afford a complete graphical solution, experimentally obtained, of a problem which from its complexity baffles mathematicians except in a few simple cases".

Hele-Shaw Papers

- Experiments on the flow of water. *Trans. Liverpool Engn. Soc.*, 1897;
- Investigation of the nature of surface resistance of water and of stream line motion under certain experimental conditions, *Trans. Inst. Nav. Archit.*, 1898 [Gold Medal];
- Experimental investigation of the motion of a thin film of viscous fluid, *Rep. Brit. Assoc.*, 1898 [Appendix by G. Stokes]
- Experiments on the character of fluid motion, *Trans. Liverpool Engn. Soc.*, 1898;

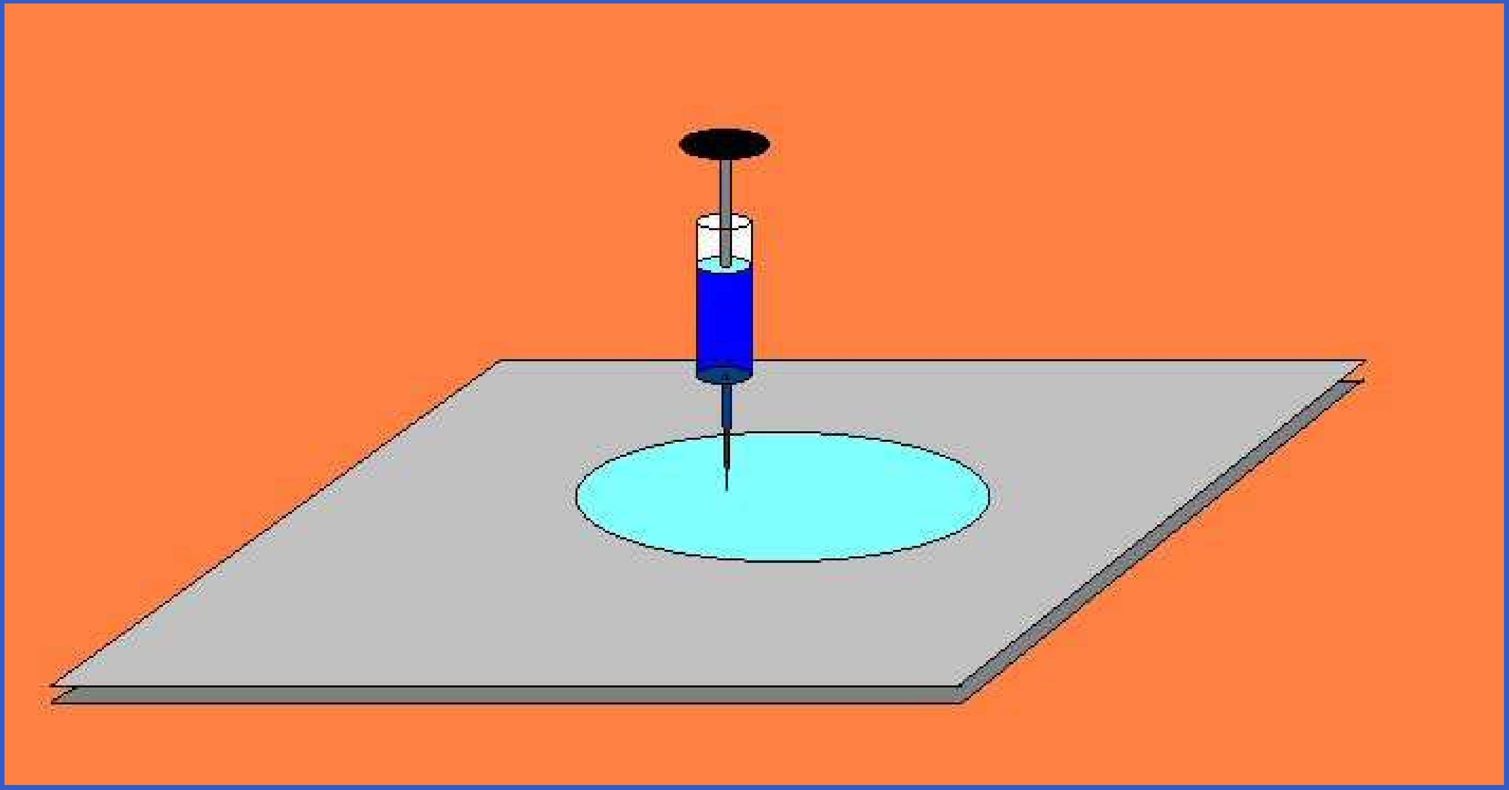
Hele-Shaw Papers

- The flow of water, *Nature*, 1898.
- The motion of a perfect fluid, *Not. Proc. Roy. Inst.*, 1899.

USSR CONTRIBUTION

Stokes-Leibenzon Model

A model of the Hele-Shaw cell with a finite source/sink:



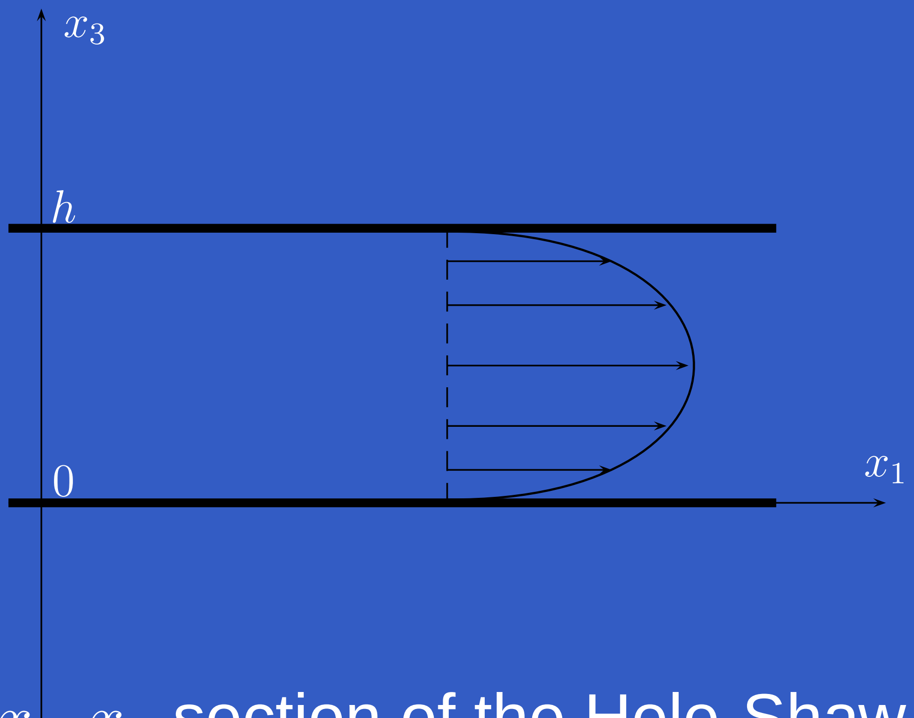
Stokes-Leibenzon Model

Leonid Samuilovich (Leib Shmulevich) Leibenzon (1879–1951)

born in Kharkov (Ukraine), Russian/Soviet engineer and mathematician, member of Soviet Academy of Sciences (1943).

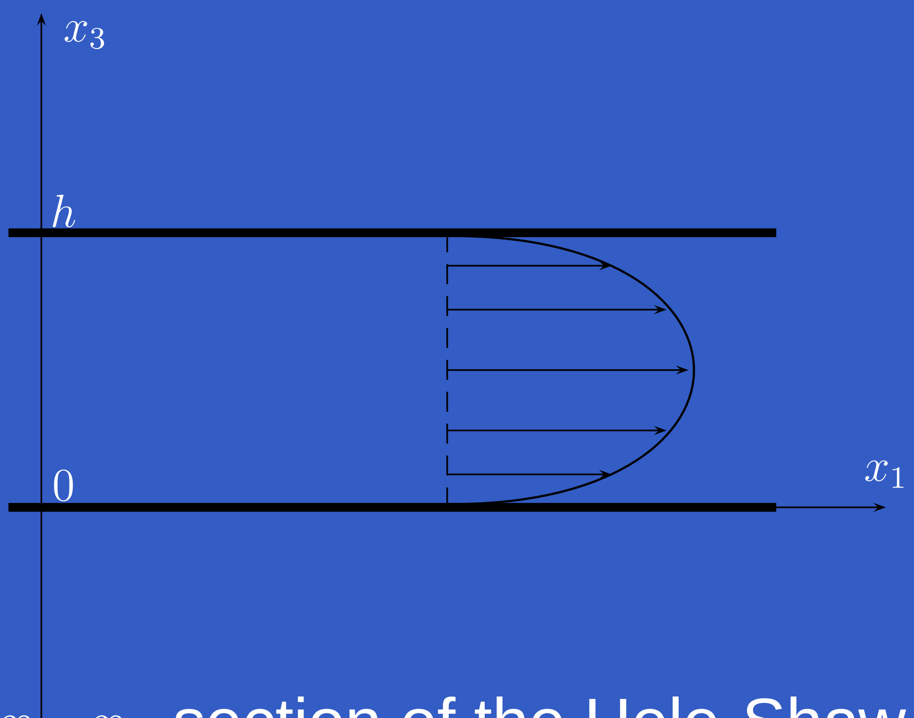
L.S.Leibenzon: The motion of natural fluids and gases in porous media, 1947.

Stokes-Leibenzon Model



x_1, x_3-section of the Hele-Shaw cell.

Stokes-Leibenzon Model

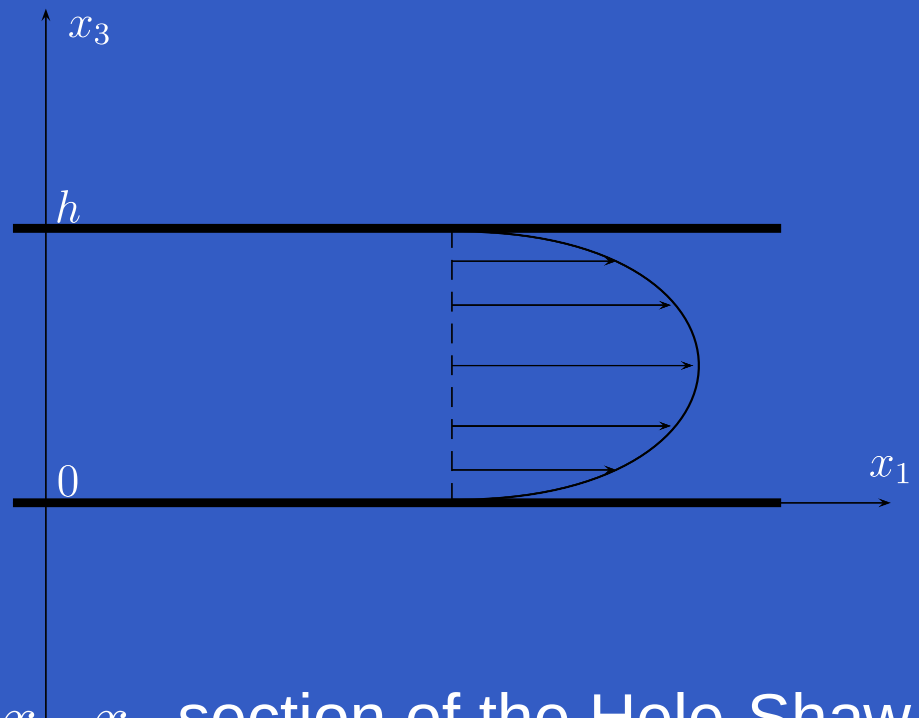


x_1, x_3-section of the Hele-Shaw cell.

Suppose that the flow is parallel and slow:

$$\frac{\partial \mathbf{V}}{\partial t} = 0, \quad V_3 = 0.$$

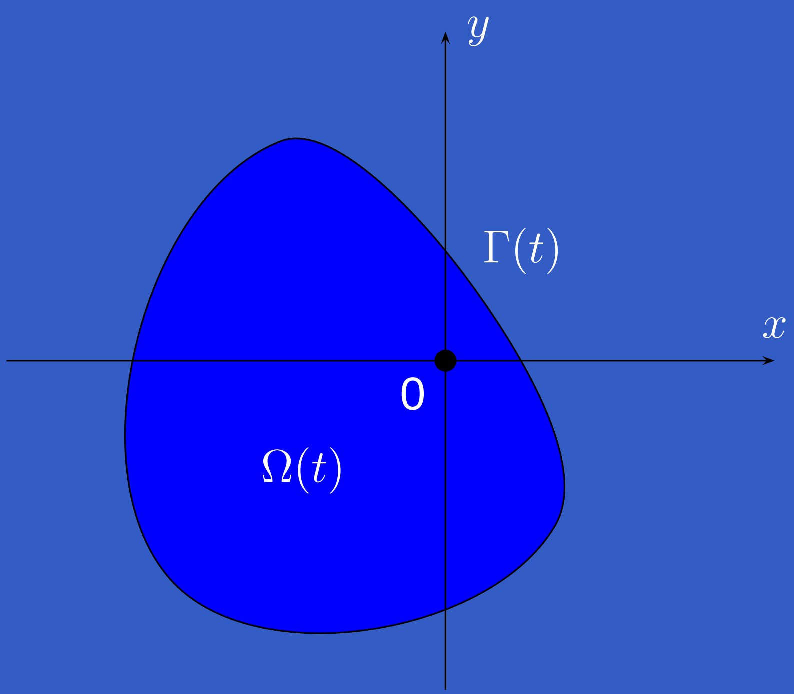
Stokes-Leibenzon Model



x_1, x_3-section of the Hele-Shaw cell.

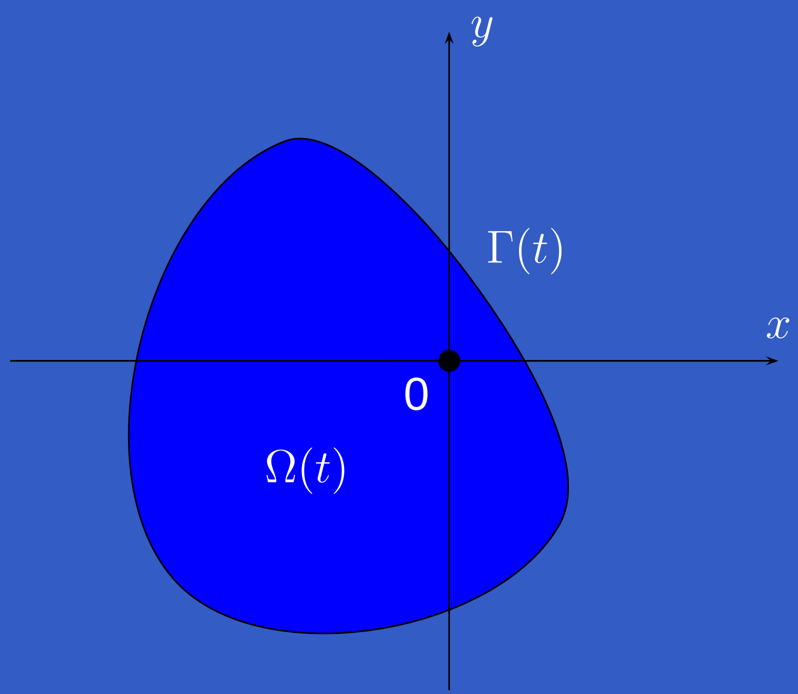
H. Lamb, Hydrodynamics, *Dover Publ., New York*, 1932.

Stokes-Leibenzon Model



- p– pressure;
- $\mathbf{v}$ – velocity field;
- z– phase variable;
- μ– viscosity;
- h– the gap between plates.

Stokes-Leibenzon Model



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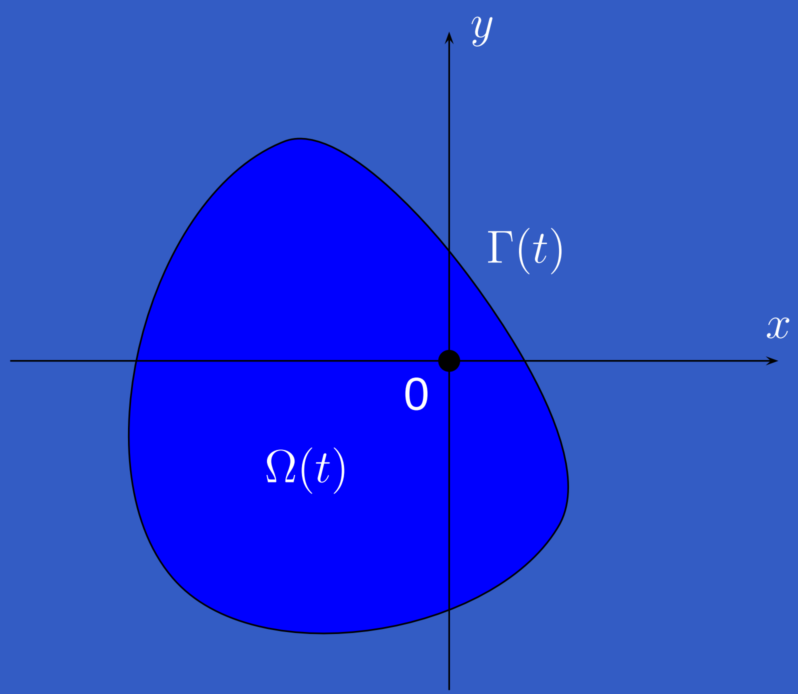
Averaging across the vertical direction, the Navier-Stokes equations reduce to $\mathbf{v} = -\frac{h^2}{12\mu}\nabla p$, or...

Stokes-Leibenzon Model



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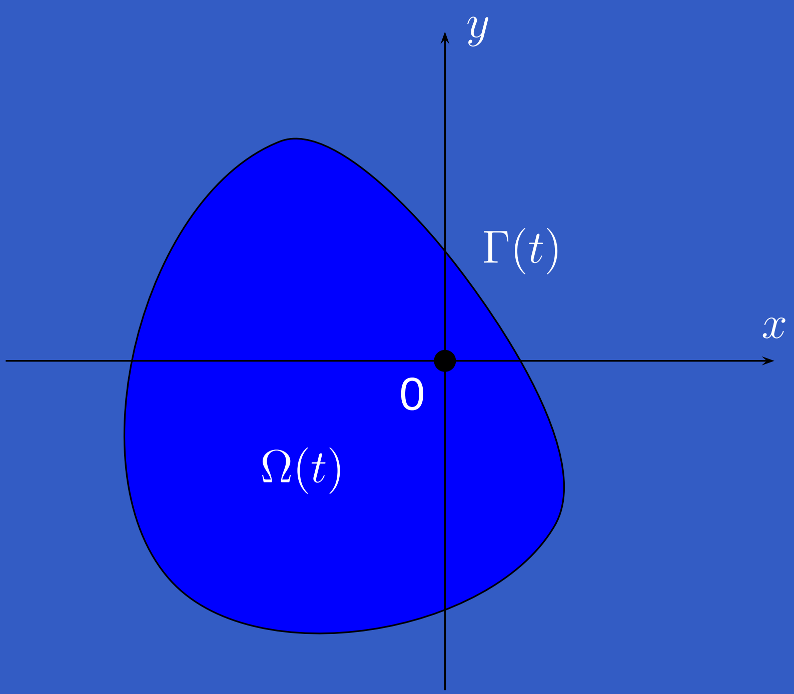
- the Laplace equation $\Delta p = \gamma(z,t)$, where $\gamma(z,t)$ is a measure.

Stokes-Leibenzon Model



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- in the case of a pointwise source/sink we have $\Delta p = Q\delta_0(z)$, where Q is the strength and $\delta_0(z)$ is the Dirac measure.

Free Boundary Problem

- $\Delta p = Q\delta_0(z)$, for $z \in \Omega(t)$;
- $p\Big|_{z\in\Gamma(t)} = 0$, where $\Gamma(t) = \partial\Omega(t)$;
- $\dfrac{\partial p}{\partial \mathbf{n}}\Big|_{z\in\Gamma(t)} = -\mathbf{v}_{\mathbf{n}}$.

$Q < 0$ for injection, $Q > 0$ for sucction.

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In the case of surface tension replace 0 by $\beta\varkappa(z,t)$ where β is surface tension, $\varkappa$ is the mean curvature.

P. Ya. Polubarinova-Kochina

Pelageya Yakovlevna Polubarinova-Kochina (13 May 1899–3 July 1999).

One of the most important women in mathematics in the Soviet Union and one of its leading scientists.

P. Ya. Polubarinova-Kochina

Pelageya Polubarinova was born in Astrakhan, a city is situated in the delta of the Volga River,100 km from the Caspian Sea. Her father Yakov Stepanovich Polubarinov, an accountant, discovered Pelageya's particular interest in science and decided to go to St Petersburg where se graduated from Pokrovskii Women's Gymnasium.

In 1918, after father's death, Pelageia Polubarinova took a job at the Main Geophysical Laboratory to bring in enough money to allow her to continue her education. She worked under supervision of Aleksandr Aleksandrovich Friedmann (1888–1925).

P. Ya. Polubarinova-Kochina

In 1921 she got a degree in pure mathematics. In 1921–23 she met Nikolai Yevgrafovich Kochin (1901–1944) who graduated from the Leningrad State University.

They married in 1925 and had two daughters Ira and Nina.

In 1934 she returned to a full time post being appointed as professor at Leningrad University. In the following year her N.Ye. Kochin was appointed to Moscow University and the family moved to Moscow.

P. Ya. Polubarinova-Kochina

In 1939 Kochin became Head of the Mechanics Institute of the USSR Academy of Sciences, and memeber of the Ac.Sci. USSR, Pelageya worked at the same institute.

Kochina and her two daughters were evacuated to Kazan in 1941 when Germans approached Moscow. However, N.Kochin remained in Moscow carrying out military research.

In 1943 she returned to Moscow but Kochin became ill and died. He had been in the middle of lecture courses and Kochina took over the courses and completed delivering them. His research was on meteorology, gas dynamics and shock waves in compressible fluids.

P. Ya. Polubarinova-Kochina

In 1958 P.Ya. Polubarinova-Kochina was elected a member USSR Academy of Sciences, and moved to Novosibirsk to building the Siberian Branch of the Academy of Sciences.

For the next 12 years she worked in Novosibirsk where she was Director at the Hydrodynamics Institute and also Head of the Department of Theoretical Mechanics at the University of Novosibirsk.

In 1970 she returned to Moscow and became the Director in the Mathematical Methods of Mechanics Section of the USSR Academy of Sciences.

P. Ya. Polubarinova-Kochina

One of her major contributions is the complete solution of the problem of water filtration from one reservoir to another through a rectangular dam. There she established connections with the Riemann P-function, Hilbert problems and Fuchsian equations.

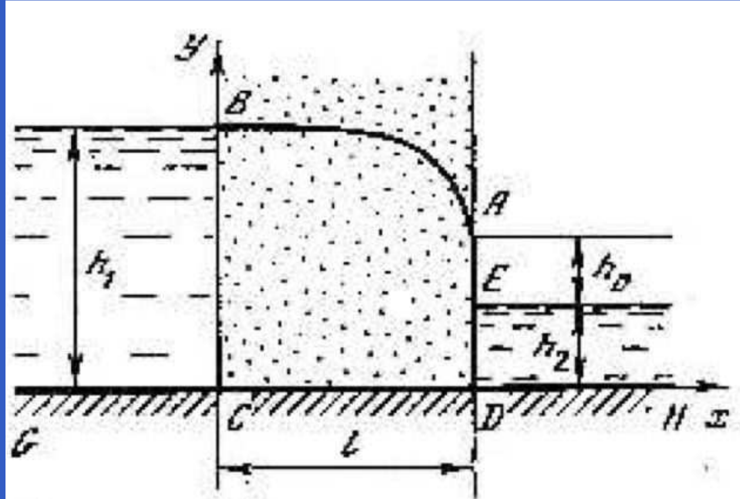

Рис. 3

лением p таким образо

$$\varphi = -k\left(\frac{p}{\rho g} + y\right),$$

где k — коэффициент ф
ции земляного слоя, p
ление, ρ — постоянная
ность, g — ускорение
тяжести, y — высота, от
ваемая от непроницаемо
нования.

Рассмотрим условия

типа

L. A. Galin

Lev Alexandrovich Galin (28 September 1912– 16 December 1981) was born in Bogorodsk (Gor'kii region), graduated from the Technology Institute of Light Industry in 1939 and started to work at the Mechanics Institute led by N.Ye.Kochin.

Professor at the Moscow State University from 1956.
Correspondent Member of the Soviet Academy of Sciences from 1953.

Hele-Shaw problem

P. Ya. Polubarinova-Kochina, L. A. Galin (1945) gave a conformal formulation of the Hele-Shaw problem.

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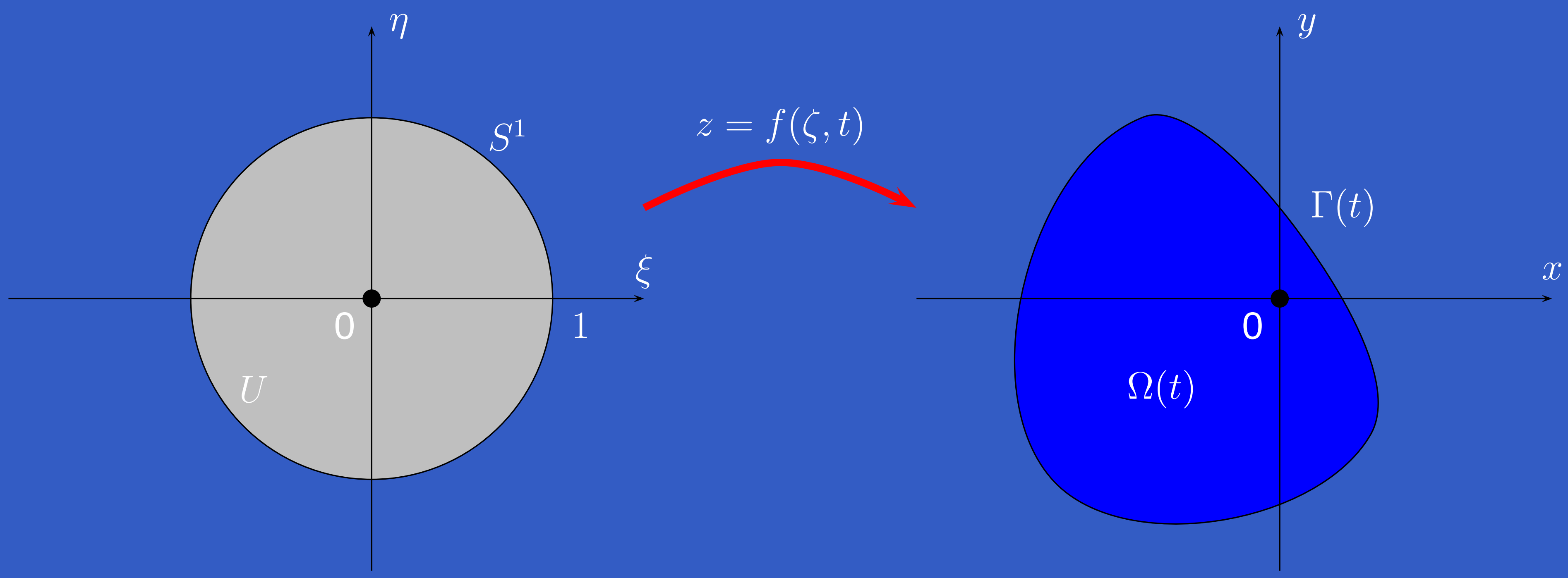
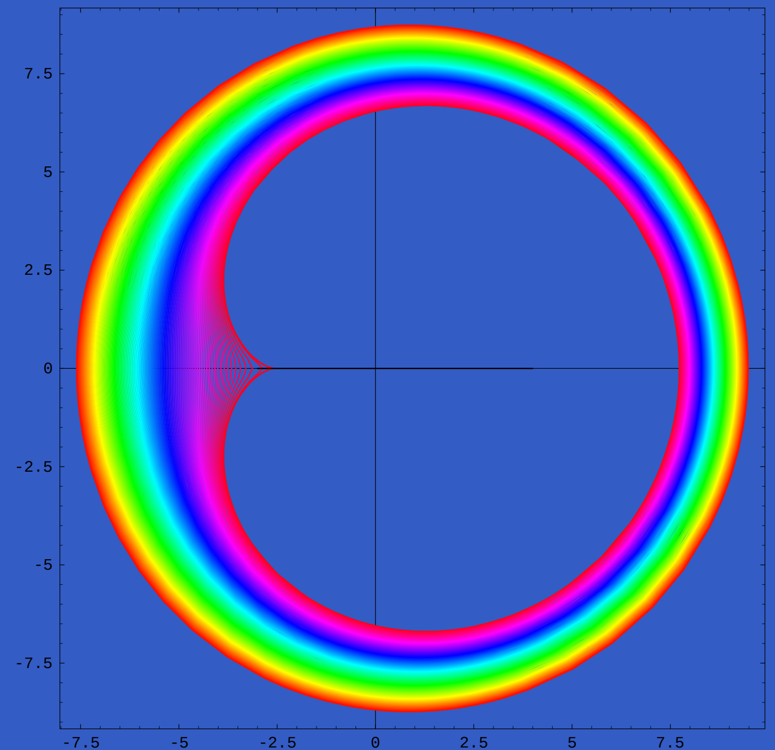


- Re $\left[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}\right] = \frac{-Q}{2\pi}$, $f(\zeta,t) = a_1(t)\zeta + \dots$ on S^1;
- $f(\zeta, 0) = f_0(\zeta)$.

First exact solution

P. Ya. Polubarinova-Kochina, L. A. Galin (1945)

A polynomial solution $f(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2$ under suction:



Papers

- P. Ya. Polubarinova-Kochina: Concerning unsteady motions in the theory of filtration. *Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.]* **9**, (1945), 79–90.
- P. Ya. Polubarinova-Kochina: On a problem of the motion of the contour of a petroleum shell. *Dokl. Akad. Nauk SSSR* **47**, (1945), no. 4, 254–257.
- L. A. Galin: Unsteady filtration with a free surface. *Dokl. Akad. Nauk SSSR* **47**, (1945), no. 4, 246–249.

Papers

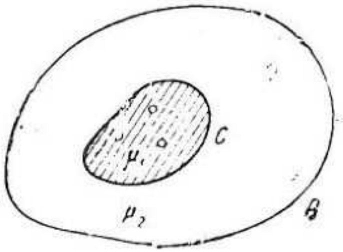
Институт механики Академии Наук Союза ССР
Прикладная математика и механика, том IX, 1945

О НЕУСТАНОВИВШИХСЯ ДВИЖЕНИЯХ В ТЕОРИИ ФИЛЬТРАЦИИ
1. О ПЕРЕМЕЩЕНИИ КОНТУРА НЕФТЕНОСНОСТИ

П. Я. Полубаринова-Кочина

(Москва)

В теории фильтрации нефти большой интерес представляет такая задача. L начальный момент времени в пористой среде дана плоская область, ограниченная контуром C, занятая нефтью, и содержащая несколько скважин (фиг. 1). Эта область окружена водой, причем на внешнем, неподвижном контуре этой области поддерживается постоянное давление (такой контур называют контуром питания [3]). На контурах скважин давление также имеет постоянные значения: $p_1, p_2, \ldots$ Нефть и вода обладают соответственно вязкостями μ_1 и μ_2. Спрашивается, как будет перемещаться контур C с течением времени. В частности, когда наступит момент обводнения скважин, т. е. когда вода вступит в первую из скважин.



Фиг. 1

В общей постановке эта задача чрезвычайно трудна для точного решения (хотя возможно указать приближенные методы ее решения). Muskat [2] рас-

P. P. Kufarev

Pavel Parfenievich Kufarev (1909–1968) was born in Tomsk on 18 March, 1909. His life was always linked with the Tomsk State University where he studied (1927–1932), was appointed as docent (1935), professor (1944), State Honor in Sciences (1968). His main achievements are in the theory of Univalent Functions where he generalized in several ways the famous Löwner parametric method. But the first works were in Elasticity Theory and Mechanics.

P. P. Kufarev

Kufarev was greatly influenced by Fritz Noether (Erlangen 1884– Orel 1941), the brother of Emmy Noether, and Stefan Bergman (1895–1977), who immigrated from nazi Germany (under anti-Jewish repressions) to Tomsk (1934). Bergman moved to Paris in 1937. Noether's life turned to be more tragic. He was arrested during the Great Purge, and sentenced to a 25-year imprisonment for being a 'German spy'. While in prison, he was accused of 'anti-Soviet propaganda', sentenced to death, and shot in the city of Orel in 1941.

Kufarev's exact solutions 1947–1952

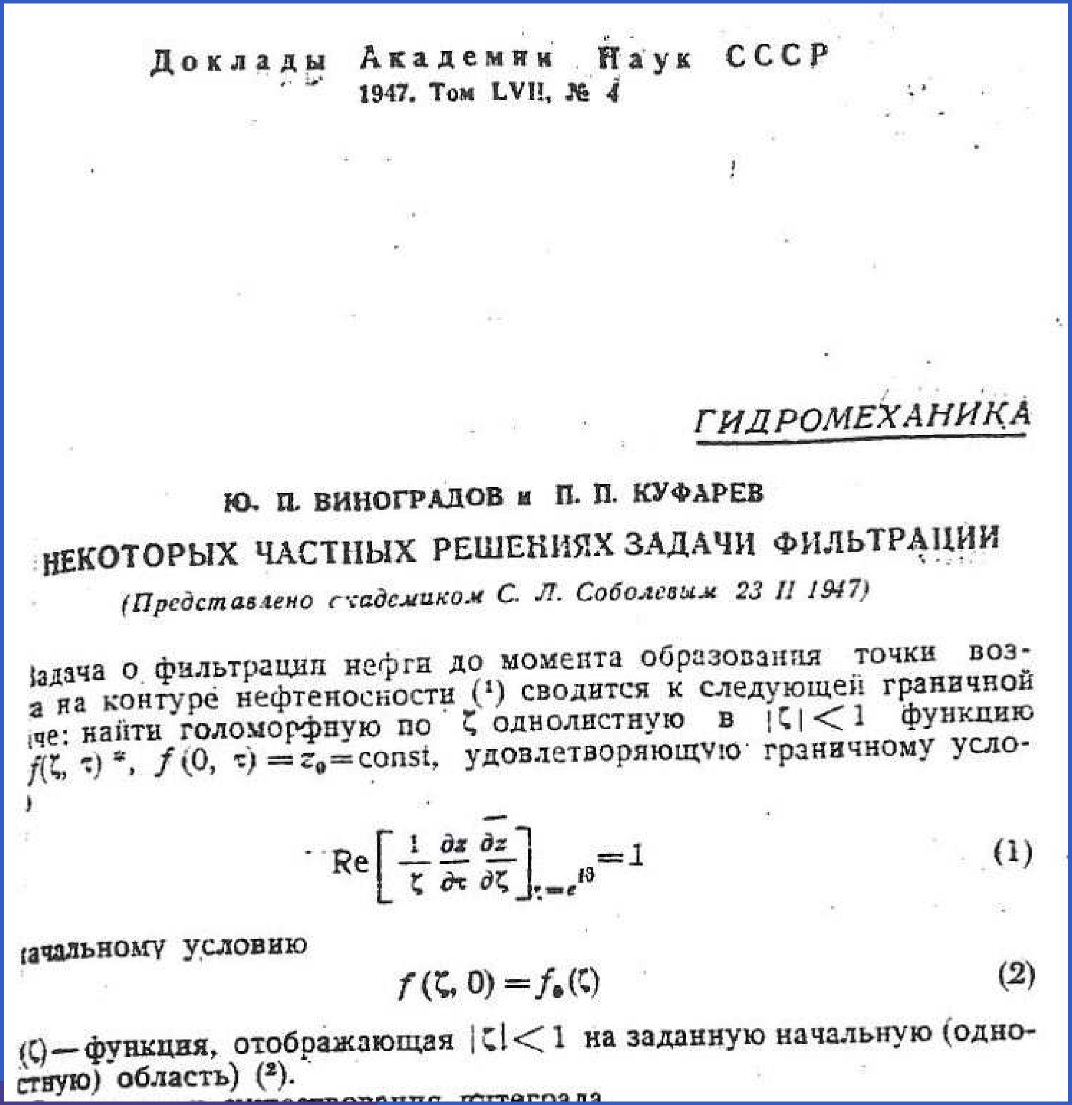
- Yu. P. Vinogradov, P. P. Kufarev: On some particular solutions to the filtration problem. *Dokl. Akad. Nauk SSSR* **57**, (1947), no. 4, 335–338.
- P. P. Kufarev: A solution of the boundary problem for an oil well in a circle. *Dokl. Akad. Nauk SSSR* **60**, (1948), no. 8, 1333–1334.
- P. P. Kufarev: Solution of a problem on the contour of the oil-bearing region for lodes with a chain of gaps. *Dokl. Akad. Nauk SSSR* **75**, (1950), no. 4, 353–355.

Kufarev's exact solutions 1947–1952

- P. P. Kufarev: The problem of the contour of the oil-bearing region for a circle with an arbitrary number of gaps. *Dokl. Akad. Nauk SSSR* **75**, (1950), no. 4, 507–510.
- P. P. Kufarev: On free-streamline flow about an arc of a circle. *Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.]* **16**, (1952), 589–598.

In these papers Kufarev gave many exact solutions: when the initial domain is a strip or a half-plane; when the initial domain is a disk with a non-centered sink; rational exact solutions; the case of several sinks/sources, etc.

Kufarev’s exact solutions 1947–1952



Доклады Академии Наук СССР
1947. Том LVII, № 4

ГИДРОМЕХАНИКА

Ю. П. ВИНОГРАДОВ и П. П. КУФАРЕВ

НЕКОТОРЫХ ЧАСТНЫХ РЕШЕНИЯХ ЗАДАЧИ ФИЛЬТРАЦИИ

(Представлено академиком С. Л. Соболевым 23 II 1947)

Задача о фильтрации нефти до момента образования точки воз-
а на контуре нефтеносности (1) сводится к следующей граничной
іче: найти голоморфную по ζ однолистную в $|\zeta|<1$ функцию
$f(\zeta, \tau)$ *, $f(0, \tau)=z_0=\text{const}$, удовлетворяющую граничному усло-
)

$$\operatorname{Re}\left[\frac{1}{\zeta}\frac{\partial z}{\partial \tau}\frac{\overline{\partial z}}{\partial \zeta}\right]_{\zeta=e^{i\theta}}=1 \qquad (1)$$

ачальному условию

$$f(\zeta, 0)=f_0(\zeta) \qquad (2)$$

(ζ) — функция, отображающая $|\zeta|<1$ на заданную начальную (одно-
стную) область) (2).

The proof of local existence

The most important Kufarev's contribution was the first proof of the existence and uniqueness of the Polubarinova-Galin equation (joint work with Kufarev's student Vinogradov):

- Yu. P. Vinogradov, P. P. Kufarev: On a problem of filtration. *Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.]* **12**, (1948), 181–198.

The modern proof was given only in 1993 by M. Reissig and L. von Wolfersdorf

The proof of local existence

Институт механики Академии Наук Союза ССР
Прикладная математика и механика. Том XII, 1948

ОБ ОДНОЙ ЗАДАЧЕ ФИЛЬТРАЦИИ

Ю. П. Виноградов, П. П. Куфарев

(Томск)

В работе рассматривается следующая задача из теории плоских неустановившихся течений жидкости.

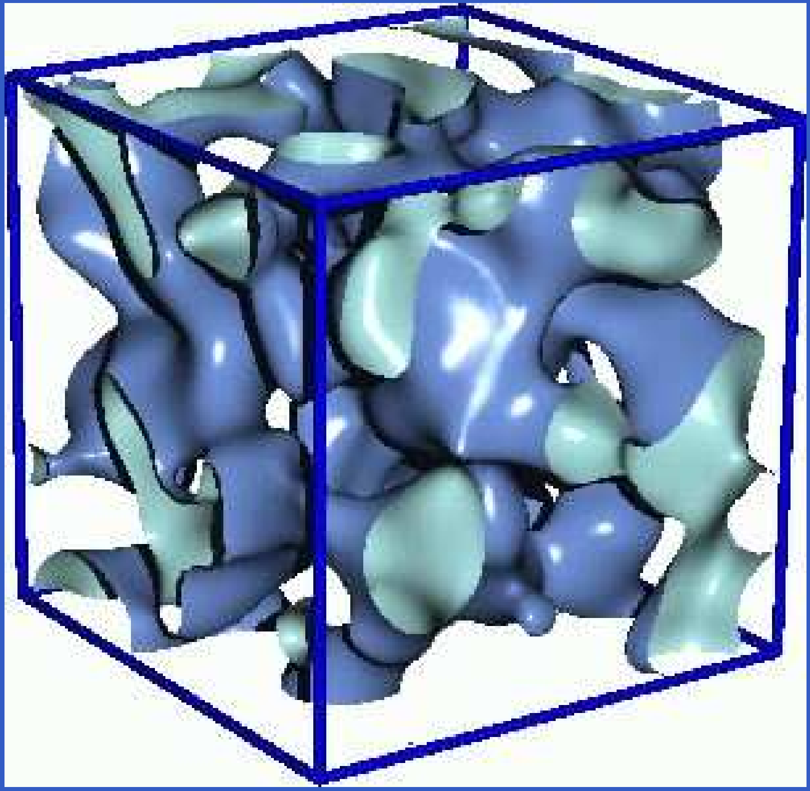
В несжимаемой жидкости, заполняющей некоторую меняющуюся во времени τ однолистную область G_τ, в точке $z = z_0 = 0$ имеется источник (сток), мощность которого $2\pi q(\tau)$ в общем случае также переменна. Известно, что в течение некоторого промежутка времени потенциал скорости имеет постоянное значение на границе C_τ области G_τ. Требуется определить характер движения и вид области G_τ, если в начальный момент $\tau = 0$ область G_0, занимаемая жидкостью, задана.

По существу к этому вопросу сводится в простейшем случае одной скважины задача о фильтрации нефти в постановке Л. С. Лейбензона[1]. Задача изучалась ранее П. Я. Кочиной[2,3] (в более общей постановке) и Л. А. Галиным[4].

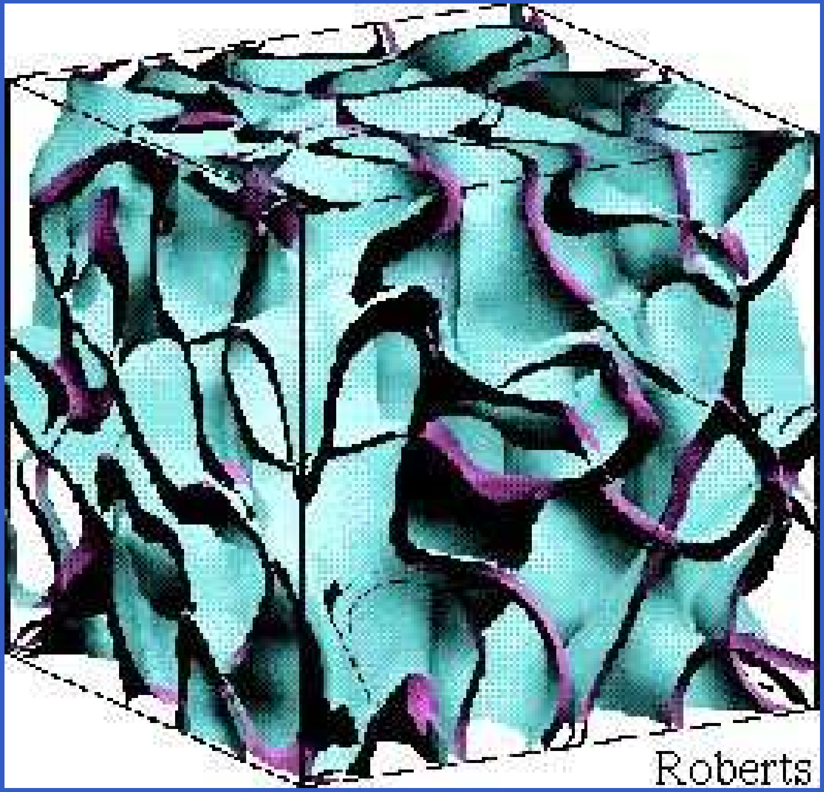
POROUS MEDIA FLOW

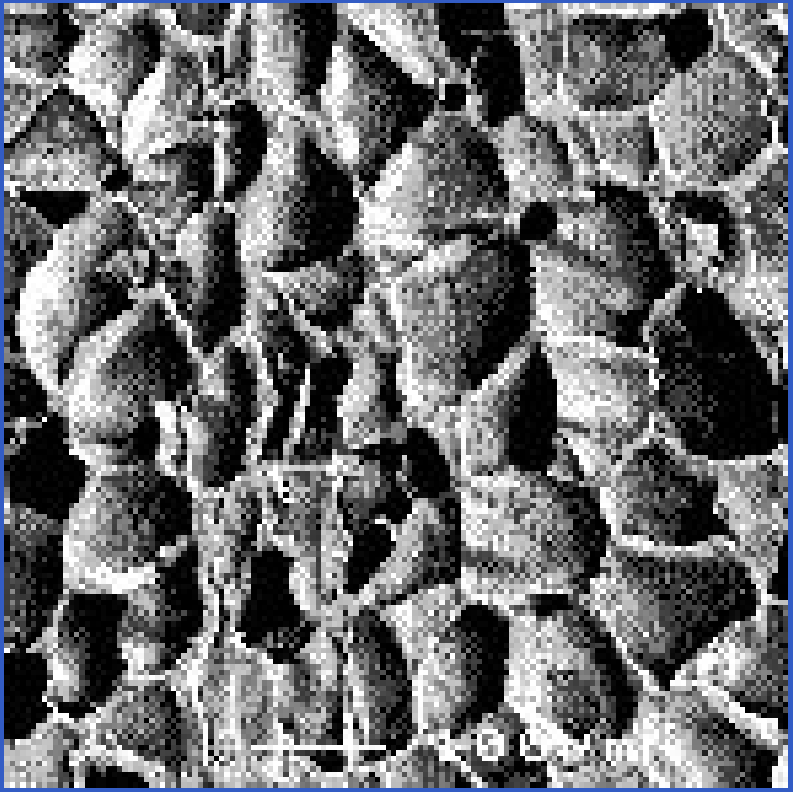
Flows in Porous Media

Random structures



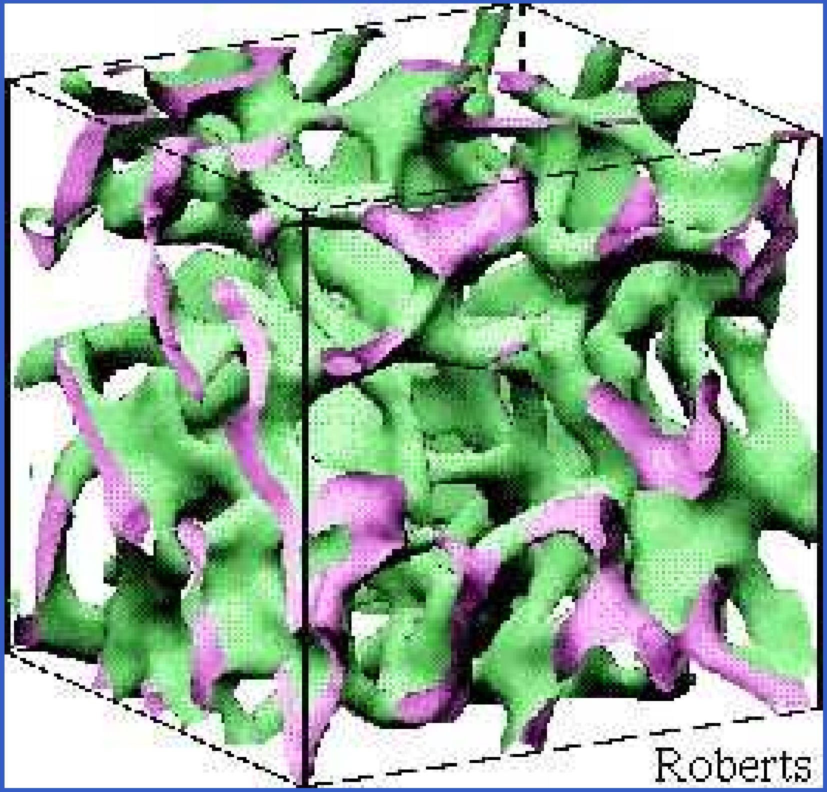
Examples of Porous Media





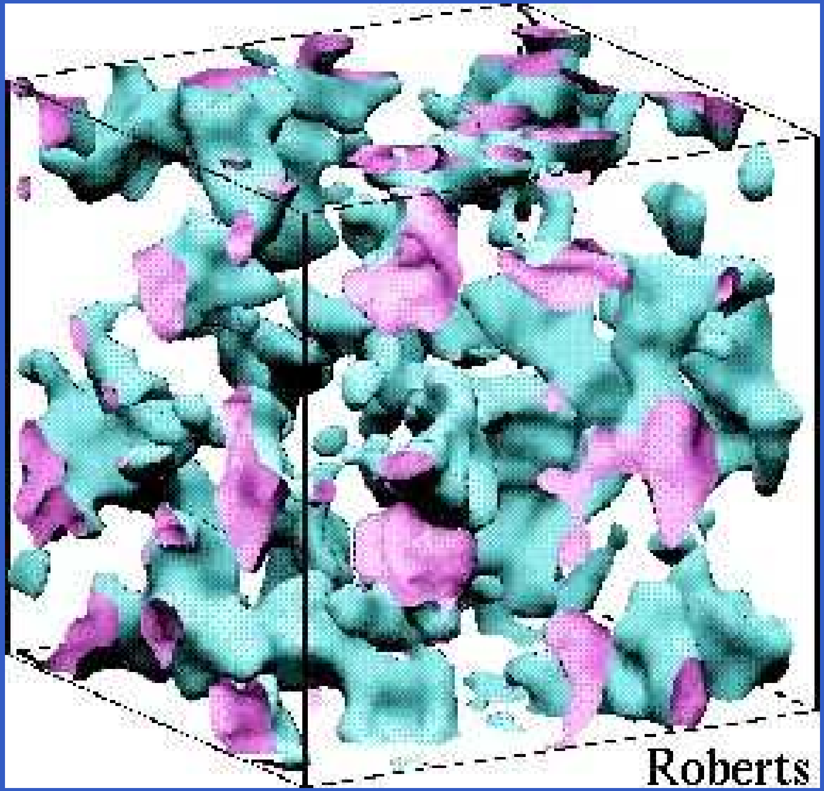
Solidificated foam,

Examples of Porous Media



Rock,

Examples of Porous Media



Silver-Wolfram composit.

Darcy’s Low

Henry Philibert Gaspard Darcy (1803–1858)

Darcy’s low- 1855.

Darcy's Low

- 1855 experimental works by Darcy.
- Mathematically proved in 1940 (M. King Hubbert), 1972 (J. Bear), 1978 (Ernan McMullin).
- Averaging across random structures we get

$$\mathbf{V} = -\frac{k}{\mu}\nabla p,$$

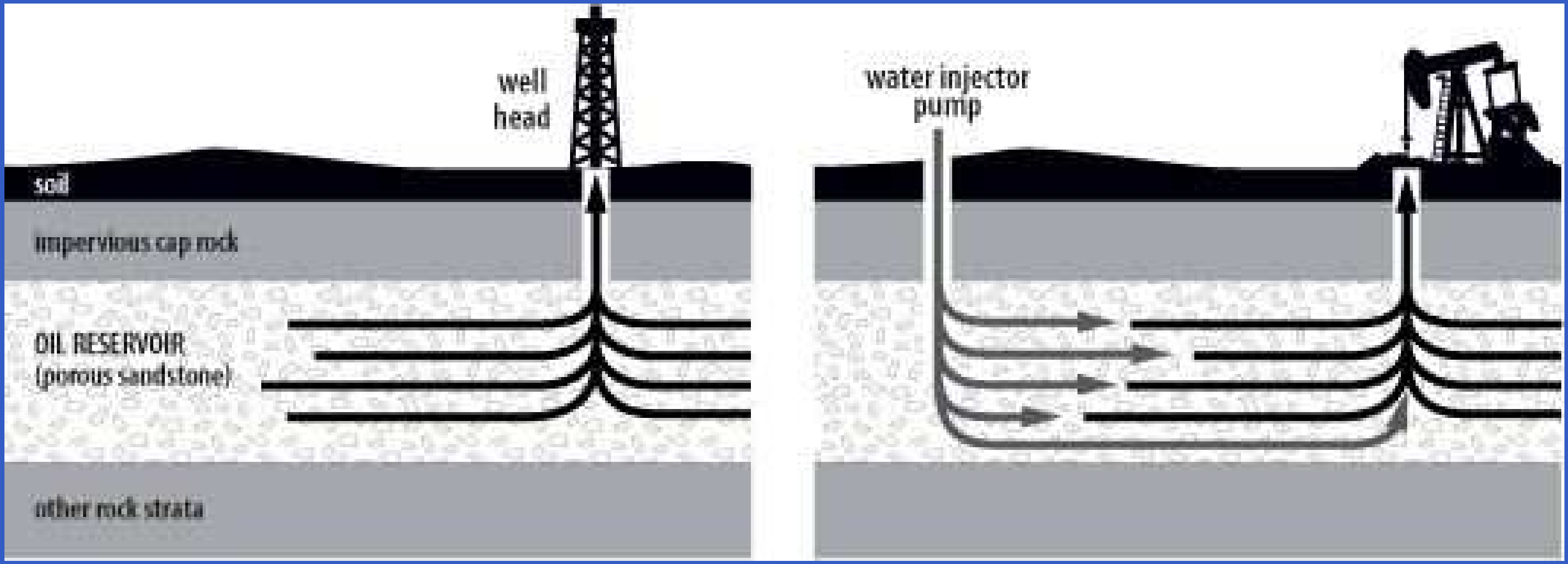
where k is permeability.

- Compare with the Hele-Shaw equation:

$$\mathbf{V} = -\frac{h^2}{12\mu}\nabla p.$$

Oil Recovery

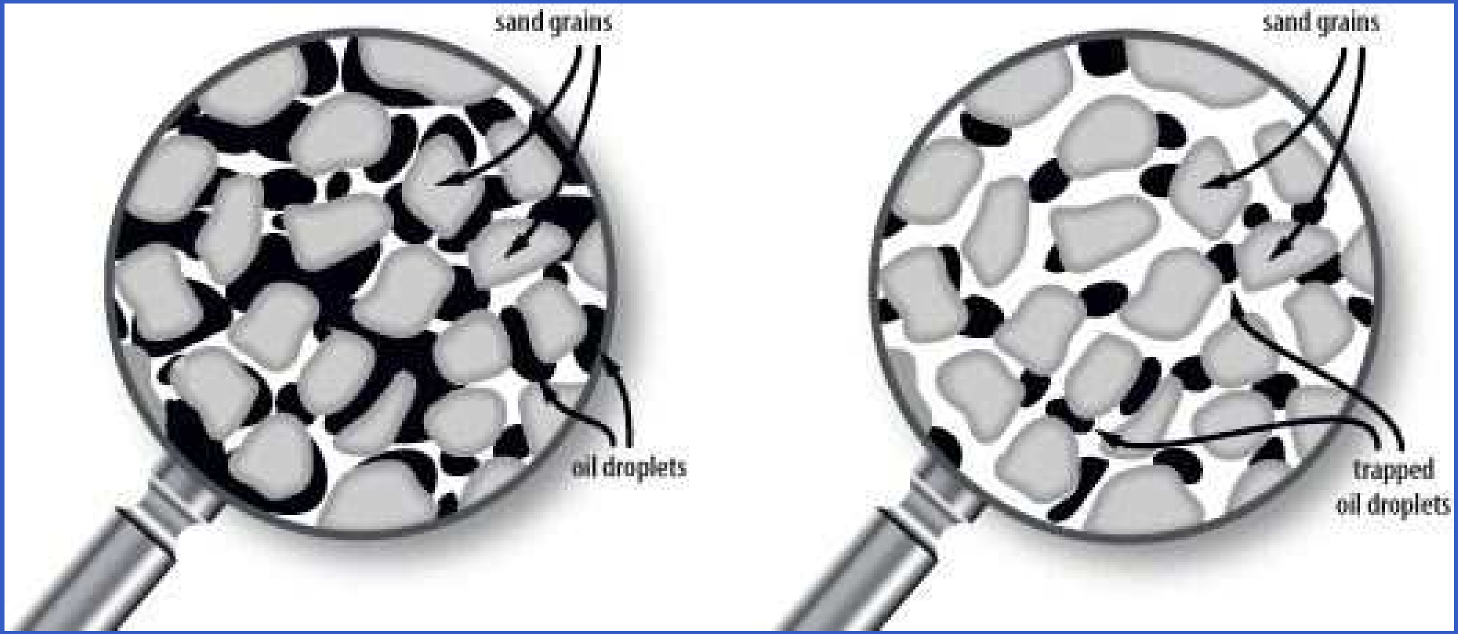
Oil Recovery



Beginning of recovery

Some years later

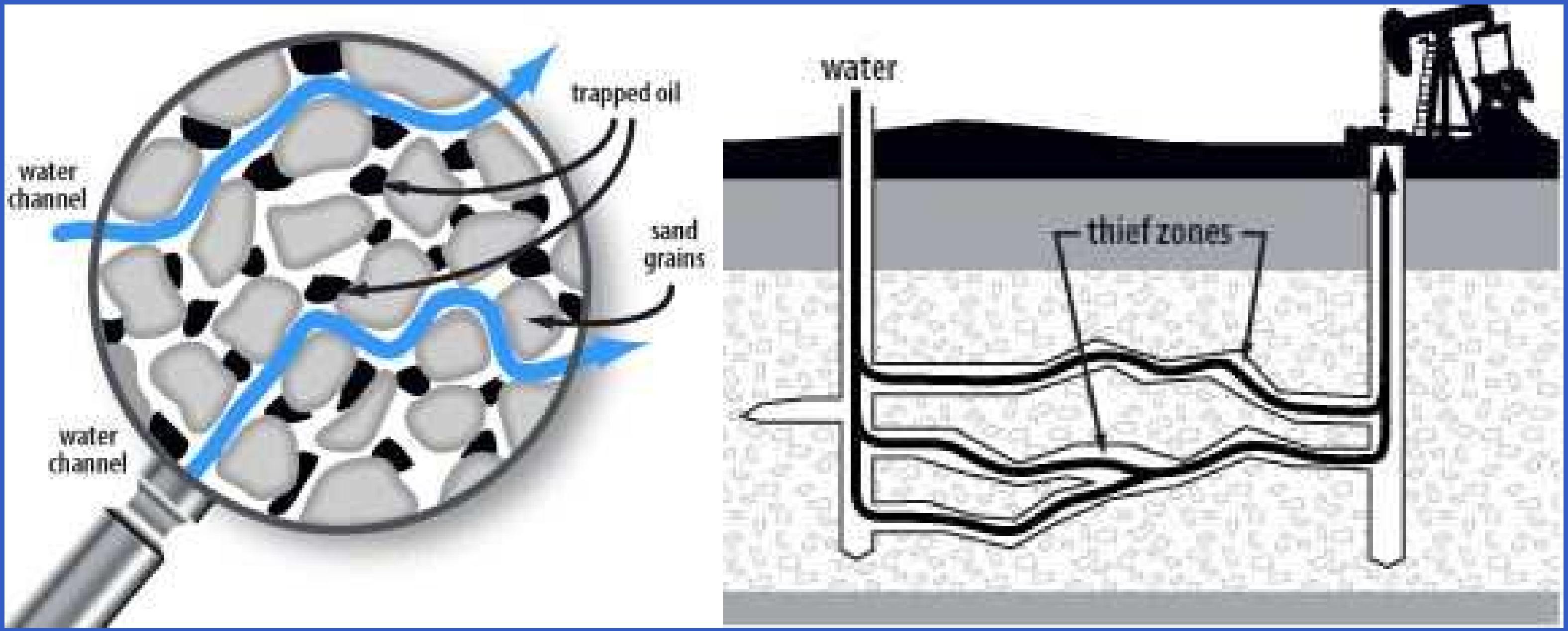
Microscopic Image



Beginning of recovery

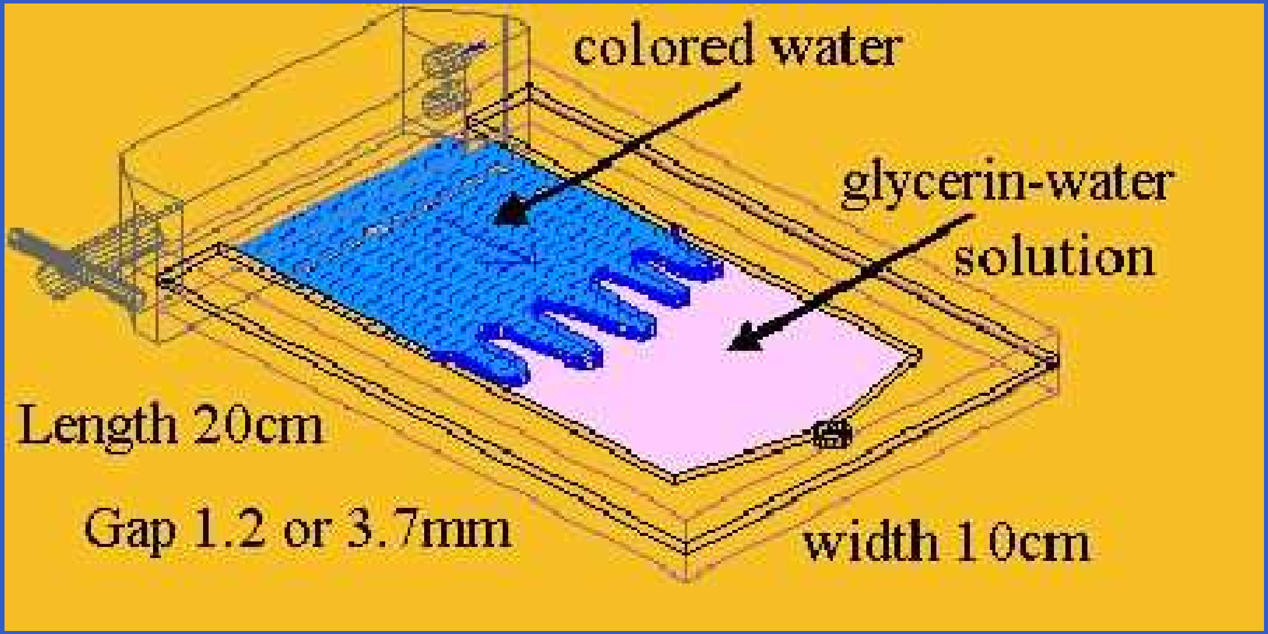
Some years later

Fingering Phenomenon



Microscopic image Water entering

Fingering Phenomenon



Modelling by the Hele-Shaw cell.

Fingering and ill-posed problems

- Receding viscous fluid performs an ill-posed problem.
- Kinetic undercooling regularization (Reissig, Hohlov, Rogozin, Entov from 1995)

$$\beta \frac{\partial p}{\partial n} + p = 0, \quad \text{on} \quad \Gamma(t), \quad \beta > 0.$$

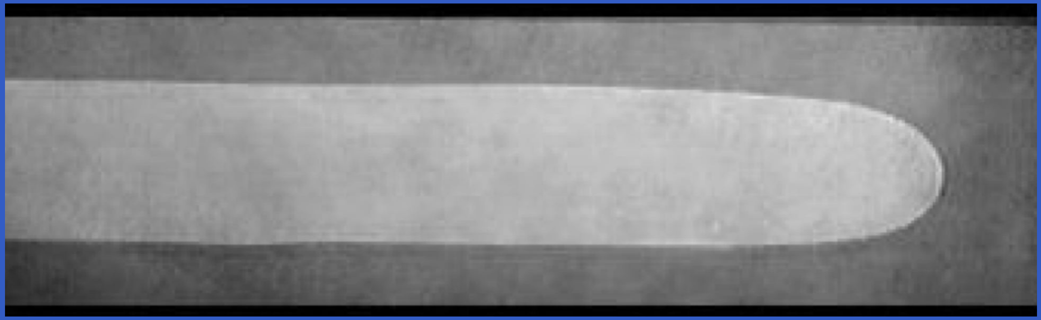
- Surface tension regularization

$$p\Big|_{z\in\Gamma(t)} = \beta\varkappa(z,t).$$

UK CONTRIBUTION

Saffman-Taylor Finger

P. G. Saffman, G. I. Taylor, The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid, *Proc. Royal Soc. London*, Ser. A, **245** (1958), no. 281, 312–329.



The first stable exact solution of the ill-posed problem.

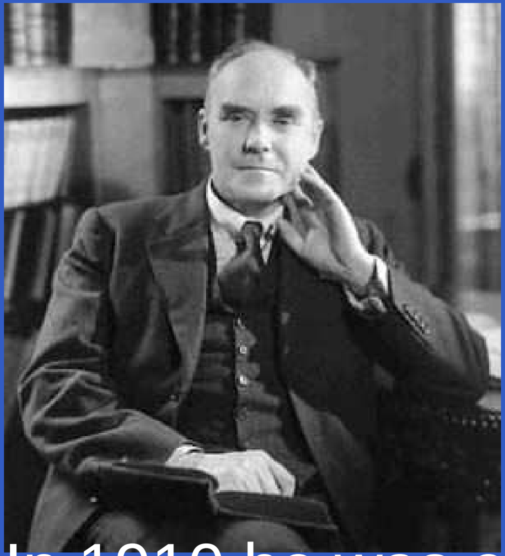
Saffman-Taylor Finger

Mathematical Review comments: "...the authors' analysis does not seem to be completely rigorous, mathematically. Many details are lacking. Besides, the authors do not seem to be aware of the fact that there exists a vast amount of literature concerning viscous fluid flow into porous (homogeneous and non-homogeneous) media in Russian and Romanian. A number of these contributions are reviewed in Mathematical Reviews." (1958)

Google search: 24 000 references.

Geoffrey I. Taylor

Sir Geoffrey Ingram Taylor (7 March 1886, London - 27 June 1975, Cambridge)



His mother– Margaret Boole, his grandfather– George Boole.

FRS-1919

Knighthood- 1944

In 1910 he was elected to a Fellowship at Trinity College, Cambridge. During World War II Taylor worked on applications of his expertise to military problems and became a member of the British delegation for the Manhattan project in Los Alamos between 1944 and 1945

Philip G. Saffman

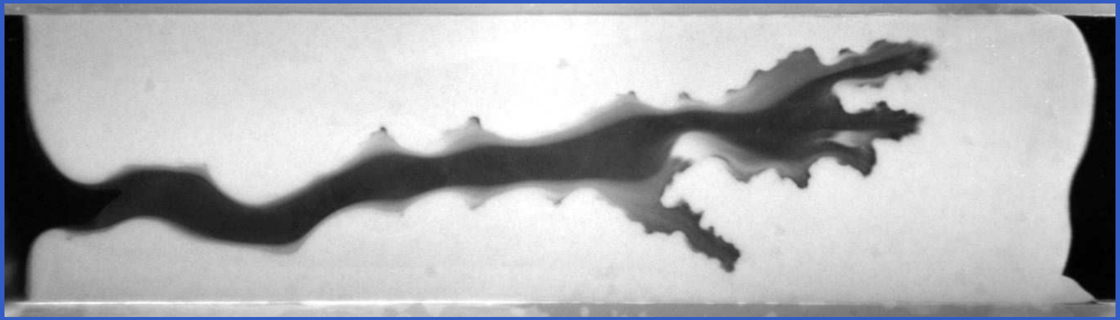
Philip Geoffrey Saffman (born 1931)

George Keith Batchelor (1920-2000), FRS–1957, an Australian scientist, student of Taylor, founder of the Journal of Fluid Mechnics (1956).

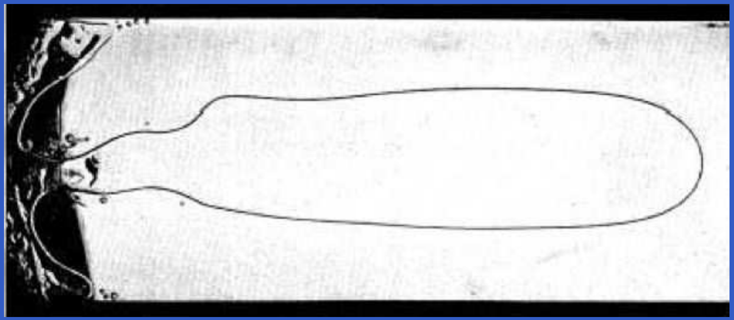
Saffman was a student of Batchelor. He is a Theodore von Kármán Professor at the California Institute of Technology. FRS– 1988.

Saffman-Taylor experiments

Experiments showed:



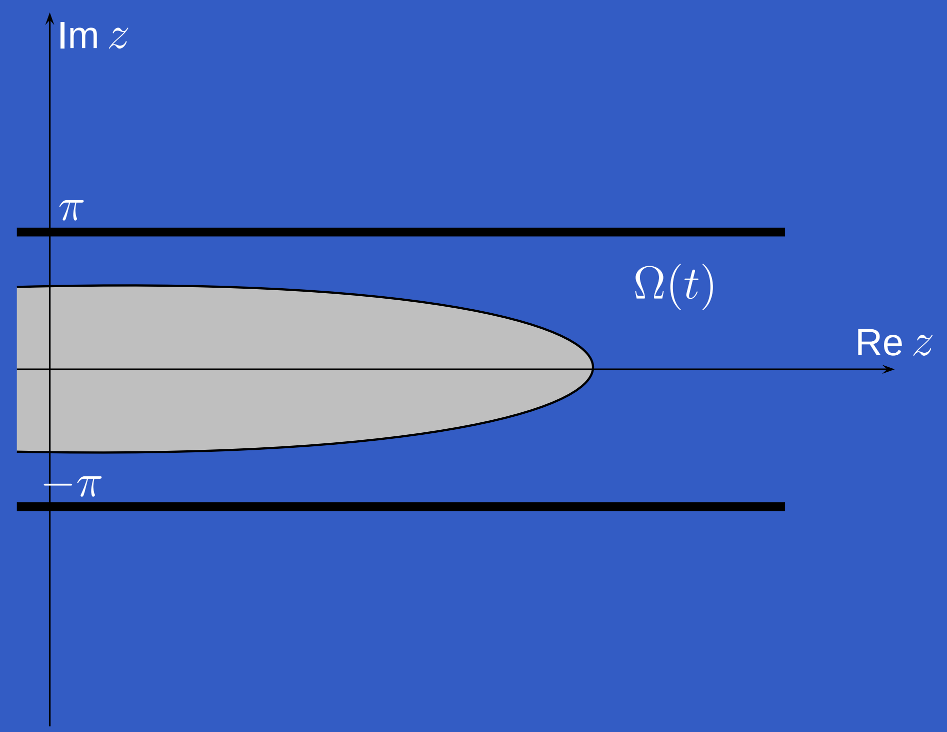
- the instability of an interface moving towards a more viscous fluid;



- Growth of a single long finger.

Conformal formulation

Saffman-Taylor exact solution:



The function $f(\zeta,t)=\frac{Q}{2\pi\lambda}t-\log\zeta+2(1-\lambda)\log(1+\zeta)$. maps the unit disk U minus $(-1,0]$ onto the phase domain $\Omega(t)$.

Selection Problem

The parameter λ, the relative width of the finger, is freely defined in $0 < \lambda \leq 1$. But in experiments λ was found to be close to 1/2 except some very special cases (very slow flow, Saffman's unsteady solution). Why $\lambda = 1/2$ selected? (Saffman-Taylor, 1958)

They also proposed to use small surface tension β as a selection mechanism as $\beta \to 0$.

Selection Problem

This proposal was realized in:

- D. A. Kessler, J. Koplik, H. Levine: Pattern selection in fingered growth phenomena, *Adv. Phys.* **37** (1988), no. 3, 255–339.

- X. Xie, S. Tanveer: Rigorous results in steady finger selection in viscous fingering, *Arch. Ration. Mech. Anal.* **166** (2003), no. 3, 219–286.

Without use of surface tension:

- M. Mineev-Weinstein: Selection of the Saffman-Taylor finger width in the absence of surface tension: an exact result, *Phys. Rev. Lett.* **80** (1998), no. 10, 2113–2116.

Richardson: Modern Period

Stanley Richardson, received his Ph.D. from the University of Cambridge in 1968 and has been at Edinburgh since 1971.

- S. Richardson: Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel, *J. Fluid. Mech.* **56** (1972), no. 4, 609–618.

He introduced 'Harmonic Moments':

Richardson: Modern Period

Printed in Great Britain

Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel

By S. RICHARDSON

Applied Mathematics, University of Edinburgh

(Received 22 August 1972)

A blob of Newtonian fluid is sandwiched in the narrow gap between two plane parallel surfaces so that, at some initial instant, its plan-view occupies a simply connected domain D_0. Further fluid, with the same material properties, is injected into the gap at some fixed point within D_0, so that the blob begins to grow in size. The domain D occupied by the fluid at some subsequent time is to be determined.

It is shown that the growth is controlled by the existence of an infinite number of invariants of the motion, which are of a purely geometric character. For sufficiently simple initial domains D_0 these allow the problem to be reduced to the solution of a finite system of algebraic equations. For more complex initial domains an approximation scheme leads to a similar system of equations to be solved.

1. Introduction

One of the basic manufacturing processes used in the plastics industry is that of injection moulding. Molten polymer is forced into a mould of an appropriate shape through a strategically placed hole, and subsequently allowed to solidify. As a simple example, one might consider the production of a plane lamina: the hollow of the mould would consist of the narrow gap between two parallel planes, with side-wall boundaries enclosing a void whose plan-form coincided with that of the required lamina. In order to reduce the high pressures needed to force

Richardson's Moments

$\Omega(t) \subset \Omega(s)$ for $0 < t < s < t_0$, and

$$M_n(t) = \iint\limits_{\Omega(t)} z^n dxdy = \iint\limits_{U} f^n(\zeta, t)|f'(\zeta, t)|^2 d\xi d\eta,$$

He proved that

$$\begin{aligned} M_0(t) &= M_0(0) - Qt, \\ M_n(t) &= M_n(0), \quad \text{for } n \geq 1. \end{aligned}$$

Connections with the inverse problem of Potential Theory.

Future connections with integrable systems.

MODERN PERIOD

1981–PRESENT

- Nowadays, the Hele-Shaw cell is widely used as a powerful tool in several fields of natural sciences and engineering, in particular, matter physics, material science, crystal growth and, of course, fluid mechanics.
- 145 000 Google references.
- Impossible to review all developments.

Classical Solutions

Sam Howison, John Ockendon, Linda Cummings, John King *et al*:

- Several classical solutions in different geometries;
- Linear stability analysis;
- Singularities, cusp formation, and blow-up;

Evolution Geometry

Björn Gustafsson, Dmitri Prokhorov, Makoto Sakai, A.V. *et al*:

- Inheriting geometry (starlikenes, convexity, etc.);
- Distance from the boundary;
- Asymptotic behaviour;

Other Models

Darren Crowdy, Linda Cummings, Sam Howison, John King, Saleh Tanveer, Kornev *et al*:

- Presence of surface tension;
- 2D Stokes flow;
- Squeeze films;
- Muskat (2-phase) problem;
- Melting/solidification in potential flow;

Other Models

Witten, Sander, Hastings, Levitov, Carleson, Makarov, Hedenmalm, Smirnov, Werner, A.V. *et al*:

- Diffusion-Limited Aggregation;
- General Löwner theory;
- Stochastic Löwner Equation;
- Modelling on general parametric spaces (Teichmüller, Kirillov)

Weak Solutions

Elliott, Gustafsson, Duchon, Robert, Prokert, Sakai, Karp *et al*:

- Existence and uniqueness;
- Branching backward in time;
- Regularity of the boundary;
- Balayage and other connections with Potential Theory;
- Quadrature domains;

Multi-dimensional flows, PDE

Caffarelli, Di Benedetto, Friedman, Tian, Escher, Simonett *et al*:

- Existence and uniqueness for general free boundary problems;
- Viscous solution;
- Scales of Banach spaces and abstract Cauchy-Kovalevskaya theory;

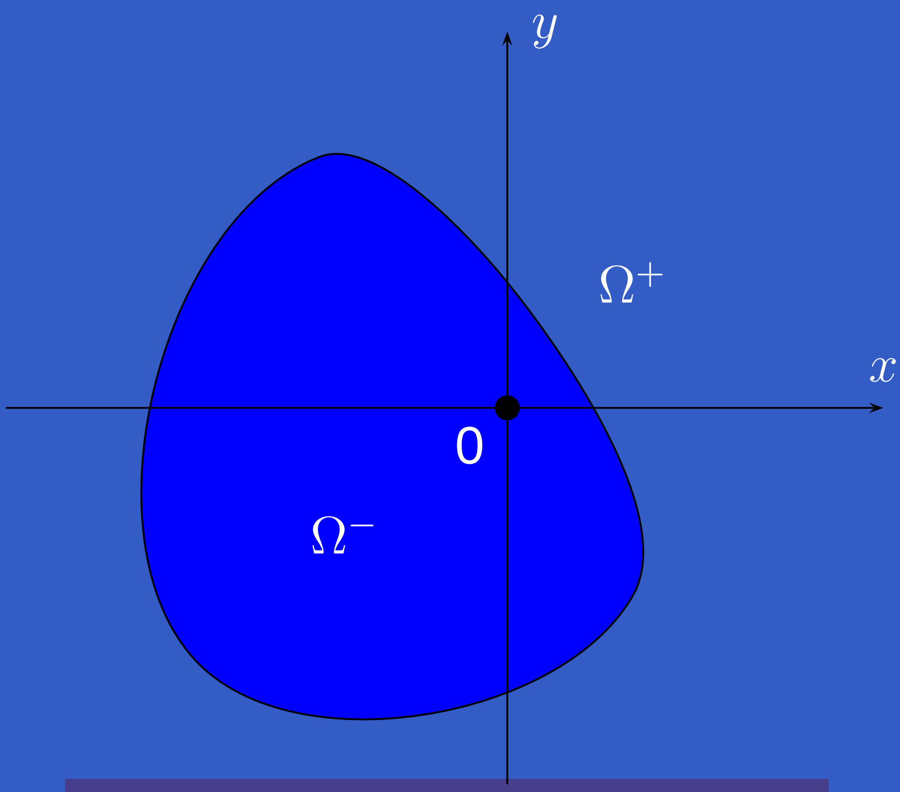
Numerical treatment

DeGregoria, Schwarz, Bensimon, Dai, Shelley, Hou, Ceniceros *et al*:

- Finite element/boundary integral methods;
- Small-scale decomposition;
- Quasi-contour methods;

Integrable Systems (2000–2007)

P.Wiegmann, M.Mineev-Weinstein, A.Zabrodin, I.Krichever, I.Kostov, A.Marshakov, T.Takebe, L.-P.Teo *et al*: Following definition of Richardson's moments define



- $M_k = -\int_{\Omega^+} z^{-k} dxdy$;
- $M_0 = |\Omega^-|$;
- $M_{-k} = \int_{\Omega^-} z^{k} dxdy$;
- $k \geq 1$,
- $t = M_0/\pi$, $t_k = M_k/\pi k$ generalized times.

Integrable Systems (2000–2007)

Moments satisfy the 2-D Toda dispersionless lattice hierarchy

$$\frac{\partial M_{-k}}{\partial t_j} = \frac{\partial M_{-j}}{\partial t_k}, \qquad \frac{\partial M_{-k}}{\partial \bar{t}_j} = \frac{\partial \bar{M}_{-j}}{\partial t_k}.$$

Real-valued τ- function, the solution of the Hirota equation

$$S_{f^{-1}}(z) = \frac{6}{z^2} \sum_{k,n=1}^{\infty} \frac{1}{z^{n+k}} \frac{\partial^2 \log \tau}{\partial t_k \partial t_n},$$

where $z = f(\zeta)$ is the parametric map of the unit disk onto the exterior phase domain.

$$\frac{C_{-k}}{\pi} = \frac{\partial \log \tau}{\partial t_k}, \qquad \frac{\bar{C}_{-k}}{\pi} = \frac{\partial \log \tau}{\partial \bar{t}_k}, \qquad k \geq 1.$$

Integrable Systems (2000–2007)

The Polubarinova-Galin equation written from the Poisson bracket viewpoint as

$$\mathrm{Im}\left(\frac{\partial f}{\partial t}\overline{\frac{\partial f}{\partial \theta}}\right) = \frac{\partial(u,v)}{\partial(\theta,t)} = \frac{\partial u}{\partial \theta}\frac{\partial v}{\partial t} - \frac{\partial v}{\partial \theta}\frac{\partial u}{\partial t} = \frac{Q}{2\pi},$$

becomes the string constrain.
Solutions lead to a reconstruction of the domain by its moments.

- Wiegmann, Zabrodin: Random matrices.
- A.V.: Hele-Shaw worldsheet.

Recommended Reading

- Sam Howison's Web-Page where one finds a survey on Hele-Shaw flows (BAMC plenary talk) and a 1898–1998 bibliography list http://www.maths.ox.ac.uk/ howison/ collected with K.Gillow;
- Survey: S. D Howison: *Complex variable method in Hele-Shaw moving boundary problem*.- Euro J. Appl. Math. **3** (1992), 209–224;
- Survey: J. R. Ockendon, S. Howison: *Kochina and Hele-Shaw in modern Mathematics, Natural Science and Industry*.- J. Appl. Maths. Mechs. **66** (2002), no. 3, 505–512;

Recommended Reading

Two monographs:

- A. N. Varchenko, P. Etingof: *Why the boundary of a round drop becomes a curve of order four?*- University Lecture Series, vol. 3, AMS, 1992.
- B.Gustafsson, A.Vasil'ev: *Conformal and potential analysis in Hele-Shaw cells*. - ISBN 3-7643-7703-8, Birkhäuser Verlag, 2006.

END