Mathematical developments around Hilbert's 16th problem

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1 Overview of the Field

In his famous lecture of the 1900 International Congress of Mathematicians, David Hilbert stated a list of 23 problems with deep significance for the advance of mathematical science. There has been intensive research on these problems throughout the 20th century. Hilbert's 16th problem called "Problem of the topology of algebraic curves and surfaces" is one of the few problems which is still completely open. This problem has two parts. The first part asks for the relative positions of closed ovals of an algebraic curve given by the set of points which are solutions of a polynomial equation $P(x, y) = 0$. The maximum number was given by Harnack. As for the relative positions, even if this is a purely algebraic problem, there has been little progress on the general case, while there is progress for small values of the degree of the polynomial P (degree less or equal to 7). The workshop "Mathematical developments around Hilbert's 16th problem", held in BIRS on March 12-16 2007, focused on the second part of the problem which is a problem in ordinary differential equations, but with the components of the vector field given by polynomials, or equivalently by an algebraic Pfaff form. The second part of Hilbert's 16th problem asks for the maximal number $H(n)$ and relative positions of limit cycles of planar polynomial (real) vector fields of a given degree n. This problem, opened for more than a century, has been at the center of many developments in differential equations. The main difficulty of Hilbert's problem is that, although a polynomial vector field is an algebraic object, its trajectories are not algebraic. In the neighbourhood of singular points they may not even be analytic. The fascination of Hilbert's 16th problem comes from the fact that it sits at the confluence of analysis, algebra, geometry and even logic.

As mentioned above, Hilbert's 16th problem, second part, is completely open. It was mentioned in Hilbert's lecture that the problem "may be attacked by the same method of continuous variation of coefficients... ". Even if the problem was stated as early as 1900 it was only in 1987 that Ecalle and Ilyashenko proved independently that a polynomial vector field has a finite number of limit cycles. Both proofs are a real "tour de force" and each requires a 300 pages volume. The idea is to compactify the phase space to the Poincaré disk. Then, as limit cycles are isolated periodic solutions, if there were an infinity of them, they would need to accumulate on a graphic (also called polycycle). By blowing up the singularities it is possible to limit oneself to graphics with hyperbolic or semi-hyperbolic (one nonzero eigenvalue) singularities. For such a graphic one studies the return map in the neighborhood of the graphic and shows that it is not oscillating.

While the result of Ecalle and Ilyashenko shows that each individaul polynomial vector field has a finite number of limit cycles it is impossible to derive from it any uniform estimate on the number of limit cycles. Approaches to an accurate estimate $H(n)$ have come from two sides. On the one hand there is the construction of bounds from below. The best lower bound is of the order $H(n) \geq Cn^2 \ln n$. Such a bound was found by Christopher and Lloyd [3] through the construction of polynomial vector fields of degree n with such a number of limit cycles. This is done with a mixture of bifurcation methods and use of symmetries. So far, the only approaches from above are of the form $H(n) < \infty$ also called "Existential part of Hilbert's 16th problem" or also "Finiteness part of Hilbert's 16th problem". An important contribution is this direction is the program started in 1991 by Dumortier-Roussarie-Rousseau [7] and reducing the proof that $H(2) < \infty$ to the proof that 121 graphics have finite cyclicity. The program followed an idea of Roussarie [17] that, compactifying both the phase space and the space of coefficients, the global finiteness would follow from local finiteness. Indeed limit cycles can only accumulate in the product of phase space times parameter space on limit periodic sets. It hence suffices to show that each limit periodic set has finite cyclicity (i.e. can give rise to a finite number of limit cycles) inside the family of vector fields for which finiteness is expected. This idea of Roussarie is very powerful and has given rise to intense research on developing methods to prove the finite cyclicity of limit periodic sets.

There exists many variants of Hilbert's 16th problem. Several of them consists in addressing Hilbert's question for a simpler class of polynomial equations. In all cases the sub-questions mentioned above are also considered namely bounds from below for the number of limit cycles and also the finiteness problem. Among the subfamilies considered are Abel equations, classical Liénard equations and generalized Liénard equations.

A very important variant of Hilbert's problem is the "tangential" or "infinitesimal part" of Hilbert's 16th problem. This problem is related to the birth of limit cycles by perturbation of an integrable system with an annulus of periodic solutions. Under the perturbations usually only a finite number of periodic solutions remain. When the integrable system is Hamiltonian, then the number of limit cycles appearing in a small perturbation of the system is obtained by counting the number of zeroes of Abelian integrals, at least as long as one remains far from polycycles. The subject is very active and has been covered and discussed widely during the workshop.

It is more than a century that Hilbert's problem was stated and we do not yet know if there exists a uniform upper bound for the number of limit cycles of degree n. Specialists tend to believe that such a bound exists and all efforts are in this direction. It is tempting here to mention Khovanskii's theory of fewnomials. Indeed the essence of Hilbert's problem is that, although the trajectories of a polynomial vector field are not algebraic, the algebraic nature of the system should leave its footprints and imply finiteness properties. The theory of fewnomials applied to Pfaff forms is remarkable in that aspect. It proves non-oscillating properties of separating solutions of Pfaff forms, which in itself is a spectacular demonstration that the phase curves "know" they have some kind of algebraic nature. Moreover the method of Khovanskii is extremely powerful for proving existence of upper bounds on the number of solutions of equations or systems of equations.

The developments around Hilbert's 16th problem and those on complex dynamics, in particular Fatou and Julia sets, had long developed in parallel, although they share some common problematics and methods of solutions. On purpose, researchers representing both groups had been invited together. In particular Adrien Douady had been invited to the workshop. He passed away between the time that the workshop was planned and the effective time of the workshop. His heritage is immense and his far reaching ideas have had an influence far outside his own field of complex dynamics. In particular it is his idea and the thesis of Lavaurs which made possible to identify the space of modules for unfoldings of parabolic points.

2 Theme and plenary lectures

The workshop brought together researchers making significant contributions to domains of differential equations related to Hilbert's 16th problem. The focus was put on the following subjects:

(i) singularities of differential equations and complex foliations, and related normal forms,

(ii) bifurcations of differential equations and finite cyclicity problems,

(iii) algebro-geometric techniques in differential equations.

In order to present an overview of the most significant directions the workshop started with invited lectures given by Robert Roussarie, Pavao Mardešič, Vadim Kaloshin, Jean-Christophe Yoccoz and Abdelraouf Mourtada with the purpose of describing the principal breakthroughs or main directions in the subject.

The lecture of **Robert Roussarie** enlightened the importance of the study of singular perturbed systems in the study of Hilbert's 16th problem. A brilliant demonstration of this is given in the recent paper [6] of Dumortier–Panazzolo–Roussarie where they give a counter-example to the celebrated conjecture of Lins Neto-Pugh-de Melo on the number of limit cycles of classical Liénard equations stating that a classical Liénard system of degree $2n$ or $2n+1$ has at most n limit cycles. As this conjecture was cited by Smale in his list of problems at the beginning of the twentieth century, it is also called Smale's conjecture. The study of singular perturbed systems is likely to bring new conjectures on the maximum number of limit cycles for polynomial vector fields. Indeed, it is possible by perturbation of such systems to create exponentially small regions in parameter space in which we observe more limit cycles than those which can be created by the standard bifurcation techniques.

The expository lecture of **Pavao Mardešič** focused on techniques and difficulties of what is called the infinitesimal and tangential Hilbert 16-th problem, namely the number of zeroes of Abelian integrals and the number of limit cycles that can be created by perturbing a Hamiltonian vector field. Pavao Mardešič distinguished between the "tangential Hilbert 16-th problem" which is strictly limited to the study of the number of zeroes of Abelian integrals and the "infinitesimal Hilbert 16-th problem" which is concerned with the number of limit cycles that can be created by perturbation of a polynomial Hamiltonian system. The lecture started with the celebrated finiteness result of Varchenko-Khovanskii, stating for any integer N the existence of a uniform bound for the number of zeroes of Abelian integrals of forms of a degree $\leq N$ over ovals of a Hamiltonian of degree $\leq n$. The idea of the proof is to use the Picard-Fuchs equations satisfied by Abelian integrals and to finish the proof by a special study in the neighborhood of the polycycles. Among the recent significant generalizations we find the results of Mourtada and Novikov, both highlights of this workshop. An important property of Abelian integrals is the Chebychev property which allows to give a bound on the number of zeros of the Abelian integrals based only on the dimension of the vector space of these. A geometric explanation of the Chebychev properties of elliptic envelopes was given, coming from the fact that the Picard-Fuchs is 2-dimensional because the homology group is 2-dimensional. The generalization for Hamiltonians of higher was discussed with known examples and counter-examples and open problems.

To make the link with algebraic dynamical systems **Jean-Christophe Yoccoz** lectured on the recent results of Buff and Cheritat on the geometry and size of Siegel disks of quadratic polynomials which allow them in particular to find parameters for which the corresponding Julia set has positive Lebesgue measure. He concentrated on the case where the multiplier λ of the quadratic polynomial $P(z) = \lambda z + z^2$ is of the form $\lambda = \exp(2\pi i\alpha)$ where α is a Liouvillian irrational number and he showed the limiting process which leads to the existence of a Julia set with positive Lebesgue measure.

The lecture of **Aldelraouf Mourtada** summarized his immense work spread over several years to prove the finite cyclicity of hyperbolic polycycles in compact families of analytic vector fields on the sphere S^2, including the introduction of several algebras of quasi-analytic functions and the use of the theory of fewnomials of Khovanskii. The work was started many years ago with the generic case of an attracting or repelling polycycle. The proof sketched at the workshop included the case where the polycycle is an accumulation of cycles (it is the boundary of an annulus of periodic solutions). The ideas of this proof are then used to extend the result of Khovanski-Varchenko about Abelian integrals, to the neighbourhood of hyperbolic polycycles. And this gives rise to the following general result: let H be a Morse polynomial of degree $d+1$ which is generic at infinity (but maybe with multiple critical values). Then there exist a number $N(d)$ (depending only on d), such that every perturbation of dH (of degree d and with non vanishing Abelian integrals) has at most $N(d)$ limit cycles on the real plane.

The lecture of **Vadim Kaloshin** reported on recent breakthroughs in the restricted or planar 3-body problem. Among the problems discussed where the Hausdorff dimension of oscillatory motions, and the Arnold diffusion. In private conversations, Vadim Kaloshin also discussed recent ideas on embedding planar polynomial vector fields in Hamiltonian vector fields of dimension 4 with the hope of using Floer homology to obtain new lower bounds for $H(n)$.

3 Presentation highlights

3.1 The role of singular perturbations in Hilbert's 16th problem

After more than a century of work on Hilbert's 16th problem and long lasting conjectures on the maximum number of limit cycles for some special classes of polynomial systems it became very clear only recently that the study of slow-fast systems is going to be one of the keys in the estimation of uniform upper bounds for the number of limit cycles. Robert Roussarie explained the recent example of Daniel Pannazolo of a classical Liénard system providing a negative answer to Smale's system. What is particularly remarkable in that case is that no "classical method" allows to track this polynomial system with 5 limit cycles where only 4 were expected by the conjecture. It gives a new light on the long standing conjecture that $H(2) = 4$. And it makes the link with the remark of Joan Carlos Artés, Jaume Llibre and Dana Schlomiuk [1] that a corner of the bifurcation diagram of quadratic systems is a very degenerate slow-fast system.

3.2 The role of the study of singularities in Hilbert's 16th problem

A group of lectures dealt with the study of singularities of analytic vector fields. This study is one of the most basic and fundamental part of the subject as the singularities are the organizing centers of the foliations. Among the study of singularities lies the problem of the center, which is fundamental in the context of Hilbert's 16th problem. The lecture of Emmanuel Paul sat in that context, where he discussed the Galoisian reducibility for a germ of quasi-homogeneous foliation.

Singularities of Fuchsian systems. The lecture of Caroline Lambert illustrated the link between the unfoldings of the confluent hypergeometric equation and that of the Riccati equation unfolding a saddle-node. In this lecture she showed how to cover all parameter values in the confluence from the hypergeometric equation to the confluent hypergeometric equation. In particular she was the first to identify the parametric resurgence phenomenon in this context and explain it. She could completely calculate the unfolding of the Martinet-Ramis modulus for a Riccati equation unfolding a saddle-node of codimension 1. Rodica Costin discussed the linearization of nonlinear perturbations of Fuchsian systems.

Moduli of analytic classification of families unfolding resonant singularities. This was discussed by the group of lectures of Loïc Teyssier (modulus space for germs of families unfolding saddle-nodes of codimension k), Javier Ribon (analytic classification of unfoldings of resonant diffeomorphisms) and Christiane Rousseau (analytic classification of unfoldings of resonant diffeomorphisms and moduli spaces in the codimension 1 case). The moduli of analytic classification of resonant singularities have been a tool in Ecalle's proof that limit cycles of an analytic vector field cannot accumulate on a polycycle. Later. in the paper [5]. the Martinet-Ramis modulus of a saddle-node was used as a tool to prove the finite cyclicity of some graphics of the DRR program [7]. While these graphics were generic, it became clear that to tackle the same kind of questions for graphics which can produce an annulus of periodic solutions, then it was necessary to control the behaviour in the parameter to be able to control the number of limit cycles which can appear in a perturbation. Christiane Rousseau presented her joint work with Colin Christopher where they determine the space of moduli of germs of generic 1-parameter families unfolding a diffeomorphism with a parabolic fixed point. It is the first time that a space of moduli can be determined for a **family** of dynamical systems. In the same spirit the lecture of Loïc Teyssier explored the higher codimension case for unfolding of saddle-node vector fields. The description was based on the decomposition of a neighborhood of the singularities in sectors over which the space of leaves is given by $\mathbb{C}$. Now a complete modulus of analytic equivalence or conjugacy has been given for a family unfolding the codimension k saddle-node and all the machinery is ready for attacking the description of the space of moduli. Javier Ribon discussed a complete system of analytic invariants for a germ of one-parameter family of diffeomorphisms unfolding a diffeomorphism with a parabolic point. His results complete and extend those of [13]. In particular they are valid for any codimension and the family need not be generic. Better criteria for deciding the analytic conjugacy of two such families are given.

Topology of leaves of analytic foliations on Stein manifolds. This was the topic of the lecture of Tanya Firsova. Her theorem stated that, for a generic singular foliation of a Stein manifold, all leaves except possibly a countable number are topological disks and the rest are topological cylinders. This result is likely to open a new domain in analytic foliations.

3.3 Algebraic dynamical systems

The trend on algebraic dynamical systems and iteration of rational maps, started with the lecture of Jean-Christophe Yoccoz and was followed by the lecture of Alexey Glutsyuk presenting that the horospheric lamination of the orbit space of a rational function is topologically transitive, provided that the rational function under consideration does not belong to an explicit list of exceptions.

3.4 o-minimal structures and Hilbert's 16th problem

o-minimal structures have been studied for several years in conjunction with Hilbert's 16th problem. This comes from the fact that the properties of quasi-analytic solutions of analytic ordinary differential are well captured by the language of algebra and logic, in the same way as they can be studied by the theory of fewnomials of Khovanskii. For instance it is well known that the non-spiraling leaves of real analytic foliations of codimension 1 all belong to the same o-minimal structure. Some specialists of o-minimal sructures had to cancel their coming to BIRS. Nevertheless the subject was represented at the workshop. As a follow-up to the lecture of Abdelraouf Mourtada, the lecture of Reinhard Schäfke, discussed how non-oscillating trajectories of real analytic vector fields sit inside o-minimal structures and explained that, under certain assumptions, such a trajectory generates an o-minimal and model complete structure together with the analytic functions. The proof uses the asymptotic theory of irregular singular ordinary differential equations in order to establish a quasi-analyticity result from which the main theorem follows. An application was given of an infinite family of o-minimal structures such that any two of them do admit a common extension, and also an example of non-oscillating trajectory of a real analytic vector field in dimension 5 that is not definable in any o-minimal extension of the reals.

3.5 Extension of the Varchenko-Khovanskii theorem to the Darboux integrable case

Dmitry Novikov presented a beautiful extension of the Varchenko-Khovanskii theorem to the Darboux integrable case, yielding a bound on the number of zeroes of Abelian integrals in the latter case. The tools introduced for the proof of this result introduced new perspectives in tangential Hilbert's 16th problem as the classical tools for studying Abelian integrals do not work when we switch from a Hamiltonian system to a Darboux integrable one. In particular the Pontrjagin-Melnikov integrals in that case have no natural extension to the complex domain.

3.6 Results for particular classes of polynomial vector fields in the spirit of Hilbert's 16th problem.

A special emphasis was put on special subclasses of polynomial vector fields for which Hilbert's 16th problem is much studied.

Quadratic vector fields. They have been studied very systematically by Joan Carlos Artés, Jaume Llibre and Dana Schlomiuk for several years. They now have a complete bifurcation diagrams of quadratic systems with a weak focus of order greater or equal to 2 [1] and they start attacking the case of a weak focus of order 1. Their study of the quadratic systems with weak order of order 2 has enlightened some corners of the bifurcation diagram which they conjecture will produce the larger number of limit cycles in the family of quadratic vector fields. For their work they mix algebro-geometric techniques with numerical simulations and Joan Carlos Artés lectured on the numerous phase portraits expected for a system with a weak focus of order 1.

Classical Liénard systems. The recent work of Freddy Dumortier and Magdalena Caubergh shows the existence of a uniform explicit bound for the number of limit cycles of a classical Liénard system of degree n

as long as one stays in a compact subsetof the parameter space, far from the singular perturbed system. The finiteness part of Hilbert's 16th problem for this subfamily is thus reduced to the proof of the finite cyclicity of the graphics in the singular perturbed systems.

The question of the number of critical periods of a polynomial vector field with a center is often studied in parallel and in the same spirit as the questions on the number of limit cycles of a polynomial vector field. Freddy Dumortier presented recent results with Peter De Maesschalck on the period function of the classical Liénard systems. Here again the problem of the existence of a uniform bound is reduced to the study of slow-fast systems.

Unicity of a limit cycle of a vector field and unicity of the critical period in an annulus of periodic solutions surrounding a center. A number of results in this direction were presented. Jordi Villadelprat discussed the period fonction of quadratic centers. Liénard systems were discussed Jaume Llibre. An application of results in Liénard equations to a predator-prey system was discussed in the lecture of Huaiping Zhu. Abel equations were discussed by Armengol Gasull and Rafel Prohens Sastre.

4 Recent Developments and Open Problems

The most important developments in the last years around Hilbert's 16th problem are the following:

4.1 The role of singular perturbations for tracking additional limit cycles.

The techniques of geometric singular perturbation theory become more and more sophisticated and permit now to treat problems where the slow manifold has a large number of local extrema and there are several breaking parameters.

In this context an open problem stated by Roussarie is the following: what can be said of the number and type of fixed points of a composition of applications of the form

$$R_i(x) = \alpha_i + x^{r_i}$$

with $x > 0$ and $\alpha_1 \in \mathbb{R}$ and $r_i > 0$? Such compositions have been studied locally by Mourtada in the case where $\alpha_i \sim 0$ and for x close to 0. The new global problem appears naturally in singular perturbations problem. Although very simple to state, very little is known on this problem. It seems easier to expect lower bounds, but any result on bounds from below or bounds from above would be welcome.

Several finiteness problems for subfamilies of polynomial vector fields in the spirit of Hilbert's 16th problem have been reduced to conjectures on slow-fast systems.

While the methods are becoming more sophisticated it remains the case that precise results on the number of limit cycles are much harder to get when there is more than one limit cycle. This comes from the fact that the analysis must be pushed to a much finer level in order to be able to conclude

4.2 The recent progress in the tangential (infinitesimal) Hilbert's 16th problem.

The generalization by D. Novikov of the finiteness result of Varchenko-Khovanskii for zeroes of Abelian integrals to the case of a perturbation of an integrable foliation of Darboux type is a real breakthrough. Varchenko-Khovanskii's theorem states for any integer N the existence of a uniform bound for the number of zeroes of Abelian integrals of forms of a degree $\leq N$ over ovals of a Hamiltonian of degree $\leq n$. The generalized theorem by Novikov states the existence of such a bound when one replaces a Hamiltonian foliation by an integrable foliation of Darboux type. Such a foliation has a first integral of the form $\prod P_i^{\alpha_i}$, where the P_i are bivariate polynomials such that the algebraic curves $P_i = 0$ are invariant for the foliation. The existence of the uniform bound depends only on the degree of the rational form defining the foliation and the degree of the pertubation. The difficulty of the proof comes from the fact that the Abelian integrals do not satisfy any more a Picard-Fuchs equation. Also the level curves of the first integral are no more nice Riemann surfaces on which it is possible to extend the Abelian integrals. The method used for the proof is a clever application of Khovanskii theory. As in the case of the Khovanskii-Varchenko theorem it uses a special study of the asymptotic expansion of the Abelian integrals in the neighborhood of the polycycles. One difficulty is that these extensions can have small denominators. However a solution can be found using the fact that the singular points in the corners of the polycycles are integrable.

This result opens the new field of the study of Abelian integrals appearing in the perturbations of Darboux integrable systems in several directions. The bound obtained by Novikov is not explicit. Also it is only obtained under generic conditions on the first integral. Two natural generalizations are in the direction of obtaining explicit bounds under more precise conditions and to generalize to the limit cases of generalized Darboux integrable systems: these occur when two or more algebraic invariant curves coallesce.

4.3 The first space of modulus for a germ of analytic family of vector fields unfolding a resonant singularity is identified.

While it is already known for some years that the Ecalle-Voronin modulus of a germ of diffeomorphism with a parabolic fixed point or the Martinet-Ramis modulus of a saddle-node can be unfolded to yield a modulus for the unfolded system, the dependence of the unfolded modulus on the parameter was completely open. This came from the fact that no construction of this modulus could make a full turn in the parameter. All constructions yield to multiple descriptions of the modulus for some values of the parameter. The key for understanding the dependence on the parameter was to express that these two descriptions yielded the same dynamics. This “compatibility condition” ensures then that the unfolded modulus was $1/2$-summable in the parameter.

This result opens many questions. Several of them are concerned with the generalization to higher codimension. Others concern the applications. For instance what can we say of the indifferent fixed points which are born by perturbation of a parabolic fixed point. It is certainly interesting to also consider applications to problems of finite cyclicity of graphics not satisfying a genericity condition.

The fruitful interactions of the studies of polynomial vector fields and algebraic dynamical systems. The common thread between these two domains is the fact that the singularities organize the dynamics. Moreover the study of singularities of 2-dimensional vector fields can sometimes be reduced to the study of singularities of 1-dimensional dynamical systems. Although the workshop produced no specific output on this particular item, all participants appreciated the fruitful discussions between the two fields.

The reduction of the finiteness part of Hilbert’s 16th problem for classical Liénard equations to the study of singular perturbations in this family. Classical Liénard equations are ones for which the tools of singular perturbations work quite efficiently. The reduction performed by Freddy Dumortier and Magdalena Caubergh opens the hope that a proof of uniform finte cyclicity be given soon, at least for degrees not too high.

The better understanding of the leaves of the foliation of a saddle-node. The lecture of Loïc Teyssier was particularly impressive with his programming of the leaves in the neighborhood of a saddle-node or an unfolding of a saddle-node. In particular he illustrated in a brilliant way how the leaves near a hyperbolic point could have both a node or saddle behaviour depending in which direction we approach the singular point. Examples of his drawings appear in Figures 1 and 2.

5 Outcome of the Meeting

The meeting was a real success. This was the opinion of al participants. Most of them really appreciated the broad sense given to the theme and the wide spectrum of expertises. The mixing of specialists from different areas permitted animated discussions and mixing of ideas during the workshop. Several talks have been absolutely exceptional in quality, in particular the plenary talks which played the role of presenting a mature view of the state of the subject. Moreover several participants were hearing for the first time the details of some of the new significant results of the subject. Some students and young researchers attended the workshop and could discuss and exchange with the senior researchers. Two students: Tanya Firsova and Caroline Lambert gave lectures. Moreover several subgroups used the opportunity to start new work.

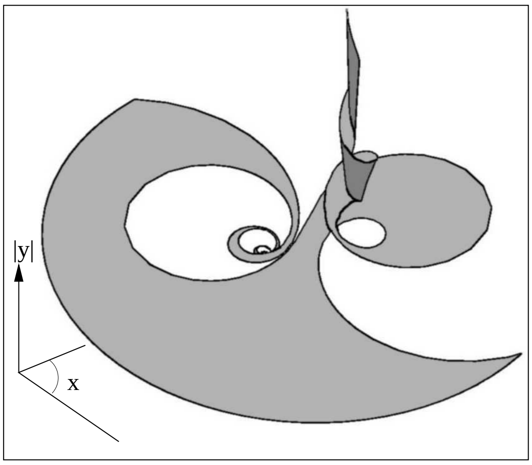

Figure 1: Modulus of a leaf a vector field unfolding a saddle node. The drawing justifies the terms “node type” (on the left of the figure) and “saddle type” (on the right) qualifying the singular points.

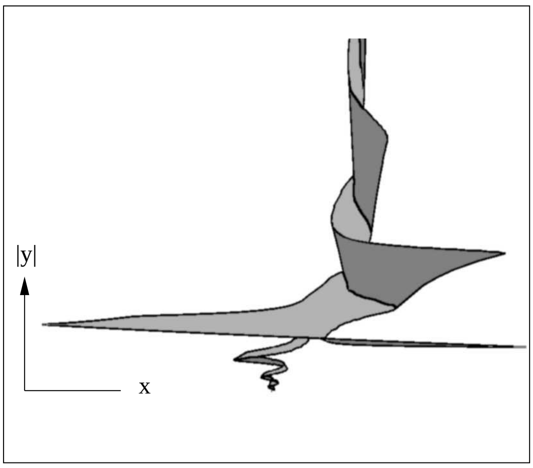

Figure 2: Another view of the same leaf.

While there has been important developments around Hilbert's 16th problem in the last years, it is clear that no complete solution is expected in the near future and there is a consensus that new ideas are still needed in order to make a breakthrough towards a complete solution.

6 Appendix: list of participants and titles of lectures

- Waldo Arriagada-Silva (Montreal)
- Joan C. Artés (UAB, Barcelona): *Quadratic vectors fields of codimension 1*
- Patrick Bonckaert (Hasselt University): *Invariant manifolds close to linear non-hyperbolic singularities*
- Magdalena Caubergh (Hasselt University): *Large Amplitude Limit Cycles for Liénard systems*
- Rodica Costin (Ohio State): *Nonlinear perturbations of Fuchsian systems: linearization criteria and classification*
- Freddy Dumortier (Hasselt University): *The period function of classical Liénard equations*
- Remy Etoua (Montrea)
- Tanya Firsova (Toronto): *Topology of leaves of analytic foliations on Stein manifolds*
- Armengol Gasull (UAB Barcelona): *Some results on periodic orbits for Abel-type equations*
- Alexey Glutsyuk (ENS, Lyon): *On density of horospheres in dynamical laminations*
- Vadim Kaloshin (Penn. State): *Oscillatory motions and instabilities for the planar 3-body problem*
- Caroline Lambert (Montreal): *Confluence of the hypergeometric equation and Riccati equation*
- Jaume Llibre (UAB, Barcelona): *On the limit cycles of the Liénard differential systems*
- Pavao Mardesic (Dijon): *Infinitesimal and tangential 16-th Hilbert problem*
- Abdelraouf Mourtada (Dijon): *Hilbert's 16th problem for hyperbolic polycycles and extension of Khovanskii-Varchenko theorem to algebraic polycycles*
- Dmitry Novikov (Weizmann): *Extension of the Varchenko-Khovanskii theorem to the integrable case*
- Emmanuel Paul (Toulouse): *Galoisian reducibility for a germ of quasi-homogeneous foliation*
- Rafel Prohens Sastre(Illes Balears, Spain): *On the number of limit cycles of some systems on the cylinder*
- Javier Ribón (IMPA): *Analytic classification of unfoldings of resonant diffeomorphisms*
- Robert Roussarie (Dijon): *Slow-fast systems and Sixteenth Hilbert's Problem*
- Christiane Rousseau (Montréal): *The space of modules of unfoldings of germs of generic diffeomorphisms with a parabolic point*
- Reinhard Schfke (Strasbourg): *Quasi-analytic solutions of analytic ordinary differential equations and o-minimal Structures*
- Loc Teyssier (Strasbourg): *Confluence of singular points in a family of holomorphic vector fields*
- Jordi Villadelprat (Universitat Rovira i Virgili,Spain): *A new result on the period function of quadratic reversible centers*
- Jean-Christophe Yoccoz (ENS, Paris): *Siegel disks and Julia sets of quadratic polynomials, according to X.Buff and A.Chritat*
- Huaiping Zhu (York): *Bifurcation of limit cycles from a Nilpotent Center in a Near-Hamiltonian System*

References

[1] J.C. Artes, J. Llibre and D. Schlomiuk, The geometry of quadratic differential systems with a weak focus of second order, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **16** (2006), 3127–3194.

[2] X. Buff and A. Cheritat, Ensembles de Julia quadratiques de mesure de Lebesgue strictement positive, *C. R. Math. Acad. Sci. Paris* **341** (2005), 669–674.

[3] C. Christopher and N. Lloyd, Polynomial systems: a lower bound for the Hilbert numbers, *Proc. Roy. Soc. London Ser. A* **450** (1995), 219–224.

[4] C. Christopher and C. Rousseau, The moduli space of unfoldings of germs of generic diffeomorphisms with a paraolic point, Preprint 2007.

[5] F. Dumortier, Y. Ilyashenko, and C. Rousseau, Normal forms near a saddle-node and applications to the finite cyclicity of graphics, *Ergod. Theor. Dyn. Systems* **22** (2002), 783–818.

[6] F. Dumortier. D. Panazzolo and R. Roussarie, More limit cycles than expected in Liénard equations, preprint.

[7] F. Dumortier, R. Roussarie, and C. Rousseau, Hilbert's 16-th problem for quadratic vector fields, *J. Differential Equations* **110** (1994), 86–133.

[8] A. A. Glutsyuk, "Confluence of singular points and nonlinear Stokes phenomenon", *Trans. Moscow Math. Soc.*, **62** (2001), 49–95.

[9] D. Hilbert, *Mathematische Probleme* (lecture), The Second International Congress of Mathematicians, Paris 1900, Nachr. Ges.Wiss. Göttingen Math.-Phys Kl. **1900**, 253–297.

[10] Yu. Ilyashenko, Dulac's memoir "On limit cycles" and related topics of the theory of differential equations, *Uspekhi Matematicheskikh Nauk* **40** (1985), 41–70.

[11] Yu Ilyashenko, Nonlinear Stokes phenomena *Advances in Soviet Mathematics*, **14**, American Mathematical Society, (1993), 1-55.

[12] Yu Ilyashenko and S. Yakovenko, Finite cyclicity of elementary polycycles, *American Mathematical Society Translations*, series 2 **165**, 1995, 21-95.

[13] P. Mardešić, R. Roussarie and C. Rousseau, "Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms", *Moscow Mathematical Journal*, **4** (2004), 455–502.

[14] J. Martinet and J.-P. Ramis, Problèmes de modules pour des équations différentielles non linéaires du premier ordre, *Publ. Math., Inst. Hautes Etud. Sci.* **55** (1982), 63–164.

[15] D. Novikov, On limit cycles appearing by polynomial perturbation of Darbouxian integrable systems, preprint.

[16] J.-P. Rolin, F. Sanz and R. Schäfke, Quasianalytic solutions of differential equations and o-minimal structures, preprint.

[17] R. Roussarie, A note on finite cyclicity and Hilbert's 16th problem, in *Dynamical Systems, Valparaiso 1986* (R Bamón et al. eds.), Lecture Notes in Math. 1331, Springer-Verlag, Berlin-Heidelberg-New York 1998, 161–168.

[18] C. Rousseau and L. Teyssier, Analytical moduli for unfoldings of saddle-node vector-fields, preprint, 2007.

[19] L. Teyssier, "Équation homologique et cycles asymptotiques d'une singularité nœud-col", *Bulletin des Sciences Mathématiques*, vol. 128, **3** (2004), 167–187.

[20] L. Teyssier, "Analytical classification of singular saddle-node vector fields", *Journal of Dynamical and Control Systems*, vol. 10, **4** (2004), 577–605.