Algebra & Number Theory
http://jant.org
(both paper and electronic)

Editorial board
Georgia Benkart, Dave Benson, Richard E. Borcherds,
John H. Coates, Jean-Louis Colliot-Thélène,
Brian D. Conrad, David Eisenbud, Hélène Esnault,
Hubert Flenner, Edward Frenkel, Andrew Granville,
Joseph Gubeladze, Ehud Hrushovski, Craig Huneke,
Mikhail Kapranov, Yujiro Kawamata, János Kollár,
Hendrik W. Lenstra, Yuri Manin, Barry Mazur,
Susan Montgomery, Shigefumi Mori, Andrei Okounkov,
Raman Parimala, Bjorn Poonen, Victor Reiner, Karl Rubin,
Peter Sarnak, Michael Singer, Ronald Solomon,
Vasudevan Srinivas, J. Tobias Stafford, Bernd Sturmfels,
Richard Taylor, Ravi Vakil, Michel van den Bergh,
Marie-France Vignéras, Kei-Ichi Watanabe,
Andrei Zelevinsky, and Efim Zelmanov

x^2 + y^3 = z^7

Bjorn Poonen

Advertisement

x^p + y^q = z^r

General theorems
Known cases
Why 2,3,7?

Descent
Etale covers of a stack
Klein quartic

1. Finding twists
Reducible 7-torsion
Irreducible 7-torsion
Degree-168 map
Local test
10 curves

2. Rational points
Faltings & Vojta
Mordell-Weil rank
Chabauty's method
Brauer-Manin obstruction

x^2 + y^3 = z^p
Reducible p-torsion
Irreducible p-torsion
\[ x^2 + y^3 = z^7 \]

Bjorn Poonen

University of California at Berkeley

(joint work with Edward F. Schaefer and Michael Stoll)

June 8, 2007
**Primitive integer solutions to** $x^p + y^q = z^r$

Fix $p, q, r \in \mathbb{Z}_{>0}$. An integer solution $(x, y, z)$ to $x^p + y^q = z^r$ will be called **primitive** if $\gcd(x, y, z) = 1$.

Define

$$\chi := \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1.$$ 

Generalizations of Fermat's **descent** reduce the problem of determining the primitive integer solutions to the determination of the rational points on a finite list of curves (over number fields) whose Euler characteristic $2 - 2g$ is a positive integer multiple of $\chi$. Therefore:

**Theorem (Beukers 1998)**

*If $\chi > 0$, there are infinitely many primitive solutions, coming in finitely many parametrized families.*

**Theorem (Darmon-Granville 1995 + Faltings 1983 (and Fermat and Euler for $\chi = 0$))**

*If $\chi \leq 0$, there are at most finitely many primitive solutions.*
Known \((p, q, r)\) cases now solved

- \((1, q, r)\)
- \((2, 2, n)\)
- \((2, 3, n)\) for \(n \leq 10\)
- \((2, 4, n)\) for \(n \leq 8\) and prime \(n \geq 211\)
- \((2, 2n, 3)\) for prime \(7 < n < 10^7\) with \(n \neq 31\)
- \((2, n, n)\)
- \((3, 3, n)\) for \(n \leq 6\) and prime \(17 \leq n \leq 10000\)
- \((3, n, n)\)
- \((2n, 2n, 5)\)
- \((n, n, n)\)
- permutations of all these except \((2, 3, 10), (2, 4, 7), (2, 2n, 3),\) and \((2, 4, n)\) for prime \(n \geq 211,\)
- others that reduce immediately to these

Some of the people involved: Bennett, Beukers, Brown, Bruin, Chen, Darmon, Denes, Edwards, Ellenberg, Euler, Fermat, Ghioca, Kraus, Kummer, Lucas, Merel, Mordell, P., Schaefer, Skinner, Stoll, Zagier, based on fundamental work by Breuil, Conrad, Diamond, Frey, Mazur, Ribet, Serre, Shimura, Taylor, Wiles, etc. (this list could be made much longer)
The case \((p, q, r) = (2, 3, 7)\) is of especial difficulty because

- It achieves the negative value of \(\chi\) closest to 0, namely
  \[\frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 = -\frac{1}{42}.\]

- There exist solutions, some of which are large.

- The exponents are prime, so the equation cannot be immediately related to one with smaller exponents. This also prevents solution via elementary factorization arguments, i.e., descent via (geometrically) abelian covers. The descent for \((2, 3, 7)\) will involve the simple group of order 168.

**Theorem (P.-Schaefer-Stoll)**

There are exactly 16 primitive integer solutions to \(x^2 + y^3 = z^7\):

- \((\pm 1, -1, 0),\) \((\pm 1, 0, 1),\) \((\pm 0, 1, 1),\) \((\pm 3, -2, 1),\)
- \((\pm 71, -17, 2),\) \((\pm 2213459, 1414, 65),\)
- \((\pm 15312283, 9262, 113),\) \((\pm 21063928, -76271, 17).\)
The highbrow explanation of the \((2, 3, 7)\) descent

(We paraphrase Darmon’s explanation of the descent.)

primitive integer solutions to \(x^2 + y^3 = z^7\)

\[=\]

integer points on the scheme
\[S: \{x^2 + y^3 = z^7\} - \{(0, 0, 0)\}\] in \(\mathbb{A}^3_{\mathbb{Z}}\).

Let’s work over \(\mathbb{C}\) temporarily:

- \(\mathbb{G}_m\) acts on \(S\) by \((x, y, z) \mapsto (\lambda^{21}x, \lambda^{14}y, \lambda^6z)\).
- Stack quotient:
  \([S/\mathbb{G}_m] = \mathbb{P}^1\) with \(0, 1, \infty\) replaced by \(\frac{1}{2}\)-pt, \(\frac{1}{3}\)-pt, \(\frac{1}{7}\)-pt.
- \(\chi = -1/42 = \) Euler characteristic of this stack.
- Étale covers of \([S/\mathbb{G}_m]\) and hence \(S\) can be constructed by finding Galois covers of \(\mathbb{P}^1\) with ramification of order \(2, 3, 7\) above \(0, 1, \infty\).
- The Riemann Existence Theorem implies that the Galois group \(G\) should be generated by \(a, b, c\) satisfying \(a^2 = b^3 = c^7 = abc = 1\) (a Hurwitz group).
(Highbrow explanation, continued)

- The smallest nontrivial Hurwitz group is \( G = \text{PSL}_2(\mathbb{F}_7) \) (the simple group of order 168).
- The corresponding étale cover of the stacky \( \mathbb{P}^1 \) is the Klein quartic

\[
X: \quad x^3y + y^3z + z^3x = 0 \quad \text{in} \quad \mathbb{P}^2.
\]

In fact, this defines an étale cover over \( \mathbb{Z}[1/42] \).
- Descent reduces the original problem to finding the \( \mathbb{Q} \)-points on twists of \( X \) by cocycles unramified outside 2, 3, 7. By Hermite, there are \emph{finitely many} such twists.

Thus the remainder of the proof consists of the following:

1. Find the relevant twists.
2. Find the rational points on these twists.
Step 1: Finding the relevant twists

We use modularity: \( X \rightarrow \mathbb{P}^1 \) is the same as \( X(7) \rightarrow X(1) \).

- Each twist of \( X(7) \) parametrizes elliptic curves with a nonstandard level-7 structure.
- Each solution \((a, b, c)\) to the original equation gives rise to a “Frey curve” \( E_{(a,b,c)} \) with rather special (but not impossible) 7-torsion, and hence a rational point on a special twist as above.

**Case 1a: Suppose that \( E_{(a,b,c)}[7] \) is reducible.**

- Then the element of \( H^1(G_{\mathbb{Q}}, \text{PSL}_2(\mathbb{F}_7)) \) classifying the twist comes from \( H^1(G_{\mathbb{Q}}, B) \) for the Borel subgroup \( B = \Gamma_0(7)/\Gamma(7) \) (nonabelian of order 21).
- Since \( B \) is a semidirect product, we can construct each such twist in two stages, twisting by a cyclic group each time.
- Since the action on \( B \) on the Klein quartic \( X \) is known explicitly, these twists may be constructed explicitly by Galois descent.
Case 1b: Suppose that $E_{(a,b,c)}[7]$ is irreducible.

- By modularity, there is a newform $f$ associated to $E_{(a,b,c)}$.

- Ribet’s level lowering shows that if $E_{(a,b,c)}[7]$ is irreducible, then “$f \equiv f' \pmod{7}$” for some weight-2 newform $f'$ on $\Gamma_0(N)$ with $N \mid 2^63^3$ (up to quadratic twist).

- Stein’s tables show that each $f'$ is a quadratic twist of one of 14 newforms $f''$, of which 13 have coefficients in $\mathbb{Z}$.

- The 14th has coefficients in $\mathbb{Z}[\sqrt{13}]$, in which 7 is inert, and cannot be congruent mod 7 to a newform with coefficients in $\mathbb{Z}$.

- Thus $E_{(a,b,c)}[7] \simeq E[7]$ where $E$ is one of the 13 curves $24A1, \ldots, 864C1$ (up to quadratic twist).
Recall: $X(7)$ is the smooth projective model of the $\mathbb{Q}$-variety $Y(7)$ representing the functor

$$S \mapsto \{(E', \phi) : E'/S \text{ elliptic, } \phi : \mu_7 \times \mathbb{Z}/7\mathbb{Z} \cong E'[7]\}$$

where the $\cong$ indicates an isomorphism such that $\wedge^2 \phi : \mu_7 \to \mu_7$ (using the Weil pairing on the right) is the identity.

Given $E/\mathbb{Q}$, define the twist $X_E(7)$ as the smooth projective model of $Y_E(7)$ representing

$$S \mapsto \{(E', \phi) : E'/S \text{ elliptic, } \phi : E[7] \cong E'[7]\}.$$

For each $a \in (\mathbb{Z}/7\mathbb{Z})^\times$, there is another twist $X^a_E(7)$ defined as for $X_E(7)$, but for which $\phi$ transforms the Weil pairing on $E$ to the $a^{th}$ power of the Weil pairing on $E'$.

The isomorphism type of $X^a_E(7)$ is unchanged if $a$ is multiplied by a square, so as $a$ varies we get only two curves, which we call $X_E(7)$ and $X^{-}_E(7)$.
Each twist of $X(7)$ is a non-hyperelliptic genus-3 curve over $\mathbb{Q}$, and hence is given as $F(x, y, z) = 0$ for some degree-4 form $F$.

For $E: y^2 = x^3 + ax + b$, an equation for $X_E(7)$ (a form $F(x, y, z)$ with coefficients in $\mathbb{Z}[a, b]$) was given by Halberstadt and Kraus.

Then we noticed that Salmon’s 1879 *Treatise on the higher plane curves* gives an order 4 contravariant $\Psi_{-4}$ of ternary quartic forms; we conjectured and proved that when it is evaluated at the equation of $X_E(7)$, it gives $X_E^-(7)$.

Thus we can write down $X_E(7)$ and $X_E^-(7)$ for each of the 13 elliptic curves over $\mathbb{Q}$. 
Step 1, continued: maps to $\mathbb{P}^1$

We need explicit equations not only for the twists of $X(7)$, but also for their degree-168 maps to $\mathbb{P}^1$ given by the $j$-invariant, so that given points on these twists, we can compute the associated $j$-invariants and hence the associated primitive solutions to $x^2 + y^3 = z^7$.

- To find the maps, we exploit the fact that they are $\text{PSL}_2(\mathbb{F}_7)$-invariant.
- Specifically, we construct them as ratios of covariants of ternary quartic forms.
- If $F = 0$ is the equation of a twist $X(7)'$ in $\mathbb{P}^2$, then the map is

\[
X(7)' \longrightarrow \mathbb{P}^1 \\
(x : y : z) \longmapsto \frac{\psi_{14}(F)^3}{\psi_0(F) \psi_6(F)^7},
\]

where the $\psi_i$ are covariants.
Step 1, continued: the local test

- For each of the finitely many twists constructed, we check whether for every prime \( p \) it has \( \mathbb{Q}_p \)-points that give rise to \( \mathbb{Z}_p \)-points on \( S \); if not, it gives no primitive integer solutions to \( x^2 + y^3 = z^7 \) so we discard it.

- We are left with 10 genus-3 curves whose rational points we must find.
The 10 genus-3 curves

\[ C_1: 6x^3 y + y^3 z + z^3 x = 0 \]
\[ C_2: 3x^3 y + y^3 z + 2z^3 x = 0 \]
\[ C_3: 3x^3 y + 2y^3 z + z^3 x = 0 \]
\[ C_4: 7x^3 z + 3x^2 y^2 - 3xyz^2 + y^3 z - z^4 = 0 \]
\[ C_5: -2x^3 y - 2x^3 z + 6x^2 yz + 3xy^3 - 9xy^2 z + 3xyz^2 - xz^3 + 3y^3 z - yz^3 = 0 \]
\[ C_6: x^4 + 2x^3 y + 3x^2 y^2 + 2xy^3 + 18xyz^2 + 9y^2 z^2 - 9z^4 = 0 \]
\[ C_7: -3x^4 - 6x^3 z + 6x^2 y^2 - 6x^2 yz + 15x^2 z^2 - 4xy^3 - 6xyz^2 - 4xz^3 + 6y^2 z^2 - 6yz^3 = 0 \]
\[ C_8: 2x^4 - x^3 y - 12x^2 y^2 + 3x^2 z^2 - 5xy^3 - 6xy^2 z + 2xz^3 - 2y^4 + 6y^3 z + 3y^2 z^2 + 2yz^3 = 0 \]
\[ C_9: 2x^4 + 4x^3 y - 4x^3 z - 3x^2 y^2 - 6x^2 yz + 6x^2 z^2 - xy^3 - 6xyz^2 - 2y^4 + 2y^3 z \]
\[ - 3y^2 z^2 + 6yz^3 = 0 \]
\[ C_{10}: x^3 y - x^3 z + 3x^2 z^2 + 3xy^2 z + 3xyz^2 + 3xz^3 - y^4 + y^3 z + 3y^2 z^2 - 12yz^3 + 3z^4 = 0 \]

Example

The rational point \((0, 1, 1)\) on \(C_7\) gives rise to

\[ 21063928^2 + (-76271)^3 = 17^7. \]
Step 2: Determining $C_i(\mathbb{Q})$

Theorem (Faltings 1983, reproved by Vojta 1991)

If $X$ is a curve of genus $\geq 2$ over a number field $k$, then $X(k)$ is finite.

- With work, the proofs of Faltings and Vojta give an upper bound on $\#X(k)$, but this does not let one compute $X(k)$, even in principle.
- In fact, no current algorithm is known to determine $X(k)$ in general, even for genus-2 curves over $\mathbb{Q}$.
- Nevertheless, there are methods, independent of the proofs of Faltings and Vojta, that sometimes succeed for individual curves.
Let $J_i$ be the Jacobian of $C_i$.

**Step 2a: Determine the rank of $J_i(\mathbb{Q})$.**

- The rank is determined by 2-descent, a 2-Selmer group computation.

- It is not yet known how in practice to compute 2-Selmer groups of general genus-3 Jacobians: the most obvious methods require the class group of a number field obtained by adjoining the coordinates of at least one point of $J[2]$, but such a number field is generically of degree 63. (There is, however, work in progress by Bruin, Flynn, P., and Stoll, showing that one can get by with degree-28 class groups.)

- So we developed a method especially for twists of $X(7)$: the geometry of $X(7)$ shows that the Galois action on $J_i[2]$ looks like the Galois action on the 2-torsion of a hyperelliptic genus-3 Jacobian. Then only degree-8 class groups are required.
Step 2b: Use Chabauty’s method to determine $C_i(\mathbb{Q})$ for $i \neq 5$

By adapting Skolem’s $p$-adic method for solving $S$-unit equations, Chabauty proved

**Theorem (Chabauty 1941)**

*Let $X$ be a curve of genus $g$ over a number field $k$. Let $J = \text{Jac} X$. If rank $J(k) < g$, then $X(k)$ is finite.*

- Coleman and others showed how to refine this into an effective method for determining $X(k)$, when $J(k)$ is known.
- For $i \neq 5$, we have rank $J_i(\mathbb{Q}) < 3$ and Chabauty’s method determines $C_i(\mathbb{Q})$.
- For $i = 5$, we have rank $J_5(\mathbb{Q}) = 3$ and Chabauty’s method gives no information.
Step 2b’: Use the Brauer-Manin obstruction (sieving out residue classes) to attempt to determine $C_5(\mathbb{Q})$

- Let $C = C_5$ and $J = J_5$.
- Embed $C$ in $J$.
- It is hard to determine which points of $J(\mathbb{Q})$ lie on $C$.
- But for a prime $p$ of good reduction, we can determine the subset of points of $J(\mathbb{Q})$ whose image in $J(\mathbb{F}_p)$ lies in $C(\mathbb{F}_p)$. (It will be a union of cosets of a finite-index subgroup of $J(\mathbb{Q})$.)
- If the intersection of these subsets over several $p$ is empty, then we know that $C(\mathbb{Q})$ is empty. (This turns out to be a special case of the Brauer-Manin obstruction, modulo finiteness of $\text{III}(J)$.)

\[ C(\mathbb{Q}) \longrightarrow \prod_{p \in S} C(\mathbb{F}_p) \]
\[ J(\mathbb{Q}) \longrightarrow \prod_{p \in S} J(\mathbb{F}_p). \]

- This doesn’t work, since $C(\mathbb{Q})$ is nonempty.
In fact, even today we still don’t know $C(\mathbb{Q})$. We got around this problem as follows:

- Points in $C(\mathbb{Q})$ give rise to solutions that are primitive away from 2 and 3, but there are 2-adic and 3-adic conditions that must be satisfied to obtain truly primitive solutions.

- Thus we need only determine the points in $C(\mathbb{Q})$ satisfying these conditions.

- We show that there are none, by incorporating these conditions into the sieve on the previous slide.

- Since $p = 2$ and $p = 3$ are bad for $C$, in the sieve we must replace $C(\mathbb{F}_p) \hookrightarrow J(\mathbb{F}_p)$ by $C^\text{smooth}(\mathbb{F}_p) \hookrightarrow J(\mathbb{F}_p)$, where $C^\text{smooth}$ is the smooth locus of the minimal proper regular model of $C$ at $p$, and $J$ is the Néron model of $J$. 

\[ x^2 + y^3 = z^7 \]
Example

For $p = 2$, after iteratively blowing up the initial model eight times, one finds that the special fiber at 2 of the minimal proper regular model of $C_5$ is

Combining the sieve information from the bad primes 2 and 3 with the sieve information from the good primes 13, 23, and 97, one rules out rational points in the relevant 2-adic and 3-adic regions.

This completes the proof. □
\[ x^2 + y^3 = z^p \text{ for } p > 7? \]

Our approach generalizes to reduce the study of \( x^2 + y^3 = z^p \text{ for } p > 7 \) to the determination of rational points on twists of \( X(p) \).

Some steps become easier, but others become harder.

Each solution gives rise to a Frey curve \( E \) as before.

**Case 1: Reducible \( E[p] \).**

- The reducible \( E[p] \) case becomes almost trivial for \( p > 7 \) with \( p \neq 13 \), since there are only finitely many \( j \)-invariants of elliptic curves over \( \mathbb{Q} \) with reducible \( E[p] \) (and none at all for \( p > 163 \)).

- The reducible \( E[13] \) case should also be easy: one can reduce to studying rational points on a finite list of twists of the genus-2 curve \( X_1(13) \).

- Modularity and level lowering apply as before.
- In fact, the 14 newforms are the same as before.
- The 14\textsuperscript{th} newform can be excluded for all $p \neq 13$ using a method I learned from a paper by Calegari: a given newform with non-integral coefficients can be congruent mod $p$ to a newform with integral coefficients only for a finite, effectively determinable list of $p$.
- Hence one reduces to determining $X_E(p)$ and $X_E^-(p)$ for the same 13 elliptic curves $E$ as before (plus a problem with the 14\textsuperscript{th} newform if $p = 13$).
- This may be difficult, however, since the genus is much larger (already $g = 26$ for $p = 11$), and again some of these curves have relevant points.

Example

For any $p$, we have the primitive solution $3^2 + (-2)^3 = 1^p$, associated to $E = 864B1$. 