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Lieb-Robinson Bounds and Applications to Quantum Information Theory

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Quantum Spin Systems

A quantum spin system consists of a finite or infinite number of spins, labeled by $x \in V$. For finite V, the Hilbert space of states is

$$\mathcal{H}_V = \bigotimes_{x \in V} \mathbb{C}^{n_x}.$$

For the purpose of this talk you can just think of a collection of qubits.

(But it could also be a physical system of interest for other reasons: atomic magnetic moments in a material, or of specific atoms embedded in a large molecule or deposited by STM on a substrate, or an array of quantum dots.)

For each spin, the observables are the complex $n \times n$ matrices: M_n. For the whole system the algebra of observables is

$$\mathcal{A}_V = \bigotimes_{x \in V} M_{n_x} = \mathcal{B}(\mathcal{H}_V).$$

If $X \subset V$, we have $\mathcal{A}_X \subset \mathcal{A}_V$, by identifying $A \in \mathcal{A}_X$ with $A \otimes \mathbb{1} \in \mathcal{A}_V$. For such $A \in \mathcal{A}_V$, we say that the support of A is in X.

For infinite V, the algebra of observables is the completion of the algebra of local observables given by

$$\mathcal{A}_V = \bigcup_{X \subset V} \mathcal{A}_X$$

where the union is over finite subsets X.

Interactions

An interaction for a quantum spin system is a map Φ from the set of finite subsets of V to $\mathcal{A}_V$ such that $\Phi(X) \in \mathcal{A}_X$, and $\Phi(X) = \Phi(X)^*$, for all $X \subset V$, and, for finite V, the Hamiltonian is

$$H = \sum_{X \subset V} \Phi(X).$$

The Heisenberg dynamics, $\{\tau_t\}_{t \in \mathbb{R}}$, defined by

$$\tau_t(A) = e^{itH} A e^{-itH}, \quad A \in \mathcal{A}_V.$$

For infinite V, one considers finite systems indexed by finite subsets of V and then takes appropriate limits (assuming suitable conditions on Φ).

The Heisenberg Model

The first and most important example of a quantum spin system.

Let V be the set of vertices of a graph with edges $(xy) \in E$.

$$H = -\sum_{(xy)\in E} J_{xy} \mathbf{S}_x \cdot \mathbf{S}_y$$

Here $S^j_x = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes S^j \otimes \cdots \mathbb{1}$, the jth spin matrix in the n_x-dimensional irrep of $su(2)$, acting on the xth factor in the tensor product. $J_{xy} \in \mathbb{R}$ are the coupling constants.

Heisenberg introduced this model in 1926 to describe magnetism in solids. Since then, the Heisenberg model and variants of it have come up in many different contexts.

Quasi-Locality of Extended Systems

V is often called "the lattice" due to its origins in the crystal lattice of magnetic atoms. V usually has a graph structure (a set of edges $(xy), x, y \in V$), but it doesn't have to be a lattice. Let d be a metric on V. E.g., if V is a graph, one can take $d(x, y)$ to be given by the graph distance, i.e., the length of the shortest path of edges connecting x and y.

For $X, Y \subset V$, we define

$$d(X, Y) = \inf\{d(x, y) \mid x \in X, y \in Y\}.$$

We will assume that there is a non-increasing function $F : [0, \infty) \to (0, \infty)$, with the following properties:
(i) $\|F\| := \sup_{x \in V} \sum_{y \in V} F(d(x, y)) < \infty$,
(ii) there is a constant $C > 0$ such that for pairs $x, y \in V$,

$$\sum_{z \in V} F(d(x, z)) F(d(z, y)) \leq C F(d(x, y))$$

Note that if F satisfies (i) and (ii) then so thus $F_a(r) = e^{-ar} F(r)$, for all $a \geq 0$, with $\|F_a\| \leq \|F\|$, and $C_a \leq C$.
If $V = \mathbb{Z}^\nu$, we can take $F(r) = \frac{1}{(1+r)^{\nu+\epsilon}}$, for any $\epsilon > 0$.

We will assume the following quasi-locality condition on the interaction Φ:

Suppose we have V with a function F as above, and let $a \geq 0$.

$$\|\Phi\|_a := \sup_{x,y \in V} \frac{1}{F_a(d(x,y))} \sum_{X \ni x,y} \|\Phi(X)\| < \infty$$

Under these conditions, the dynamics τ_t has a quasi-locality property, expressed by an estimate for commutators of the form

$$[\, \tau_t(A)\, ,\, B\,]\, ,$$

where $t \in \mathbb{R}$, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, for finite $X, Y \subset V$. The first such estimates were proved by Lieb & Robinson (1972).

Lieb-Robinson Estimates

Theorem (N-Sims 06, Hastings-Koma 06, N-S-Ogata 06)

Let V, $a \geq 0$, F_a, C_a, $\|\Phi\|_a$, *and* τ_t *be as above, and* $t \in \mathbb{R}$, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, *for some* $X, Y \subset V$. *Then, if* $d(X, Y) > 0$, *we have the bound*

$$\| [\tau_t(A) , B] \| \leq \frac{2\|A\| \, \|B\|}{C_a} \left(e^{2C_a\|\Phi\|_a|t|} - 1\right) \sum_{x \in X} \sum_{y \in Y} F_a(d(x, y))$$

In the important cases of finite-range or exponentially decaying interactions, one has $\|\Phi\|_a < \infty$ for some $a > 0$. In that case the double sum can be estimated by

$$\sum_{x \in X} \sum_{y \in Y} F_a(d(x,y)) \leq \|F\| \min(|X|,|Y|) e^{-ad(X,Y)}$$

and one gets the familar Lieb-Robinson bound of the form

$$\| [\tau_t(A), B] \| \leq C e^{-a(d(X,Y) - v|t|)}$$

for some C which depends only on A and B, and constants $a, v > 0$, which only depend on the model.

The two main improvements over the original result by Lieb and Robinson are:

1) The new proof does not require the Fourier transform and therefore extends to non-lattice (V, d).

2) The constant C is better. In particular ours does not depend on the dimension(s) of the single-spin Hilbert spaces, and grows only linearly, rather than exponentially, in the size of supports of the observables A and B, one of which may even be infinite.

These improvements are essential for some applications.

Sketch of the Proof

Suppose V is finite. Consider the function $f : \mathbb{R} \to \mathcal{A}$ defined by

$$f(t) := [\tau_t(A), B]. \tag{1}$$

Differentiate to see that f satisfies the following differential equation

$$f'(t) = -i\,[f(t), \tau_t\,(H_X)] - i\,[\tau_t(A), [\tau_t(H_X), B]]\,, \tag{2}$$

with the notation

$$H_Y = \sum_{\substack{Z \subset V: \\ Z \cap Y \neq \emptyset}} \Phi(Z), \tag{3}$$

for any subset $Y \subset V$. The first term in (2) above is norm-preserving, and therefore we have

$$\|\,[\tau_t(A), B]\,\| \;\leq\; \|[A, B]\| + 2\|A\| \int_0^{|t|} \|\,[\tau_s(H_X), B]\,\|\,ds \tag{4}$$

Define the quantity

$$C_B(X,t) := \sup_{A\in\mathcal{A}_X} \frac{\|[\tau_t(A),B]\|}{\|A\|}, \tag{5}$$

then (4) implies that

$$C_B(X,t) \leq C_B(X,0) + 2 \sum_{\substack{Z\subset V:\\ Z\cap X\neq\emptyset}} \|\Phi(Z)\| \int_0^{|t|} C_B(Z,s)ds. \tag{6}$$

Clearly, one has that

$$C_B(Z,0) \;\leq\; 2\,\|B\|\,\delta_Y(Z), \tag{7}$$

where $\delta_Y(Z) = 0$ if $Z \cap Y = \emptyset$ and $\delta_Y(Z) = 1$ otherwise. Using this fact, iterate (6) and find that

$$C_B(X,t) \;\leq\; 2\|B\| \sum_{n=0}^{\infty} \frac{(2|t|)^n}{n!} a_n, \tag{8}$$

where the coefficients are given by

$$a_n = \sum_{\substack{Z_1 \subset V: \\ Z_1 \cap X \neq \emptyset}} \sum_{\substack{Z_2 \subset V: \\ Z_2 \cap Z_1 \neq \emptyset}} \cdots \sum_{\substack{Z_n \subset V: \\ Z_n \cap Z_{n-1} \neq \emptyset}} \prod_{i=1}^{n} \|\Phi(Z_i)\| \, \delta_Y(Z_n). \qquad (9)$$

Using the properties of the function F_a and the norm $\|\Phi\|_a$ one can estimate a_n:

$$a_n \leq \|\Phi\|_a^n C_a^{n-1} \sum_{x \in X} \sum_{y \in Y} F_a(d(x,y)),$$

and thus complete the proof for finite V.

The uniformity of the estimate in V, lets one prove that the dynamics converges as V increases. And the same bounds then hold for the infinite V dynamics.

Application 1: Existence of the Dynamics

It is well-known that one can use the Lieb-Robinson bound to establish the existence of the dynamics for infinite lattice systems.

Let Λ_n be an increasing exhausting sequence of finite subsets of an infinite system, V, with interaction Φ. The essential observation is the following bound: for $n > m$

$$\|\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A)\| \leq \sum_{x \in \Lambda_n \setminus \Lambda_m} \sum_{X \ni x} \int_0^{|t|} \|[\Phi(X), \tau_t^{\Lambda_m}(A)]\| \, ds.$$

Theorem (N-Sims-Ogata 06)

Let $a \geq 0$, and Φ such that $\|\Phi\|_a < \infty$. Then, the dynamics $\{\tau_t\}_{t\in\mathbb{R}}$ corresponding to Φ exists as a strongly continuous, one-parameter group of automorphisms on $\mathcal{A}$. In particular,

$$\lim_{n\to\infty} \|\tau_t^{\Lambda_n}(A) - \tau_t(A)\| = 0$$

for all $A \in \mathcal{A} = \overline{\bigcup_n \mathcal{A}_{\Lambda_n}}$. The convergence is uniform for t in compact sets and independent of the choice of exhausting sequence $\{\Lambda_n\}$.

Application 2: Propagation of Correlations

Let $\langle \cdot \rangle$ denote the expectation in a product state, e.g., one determined by a product vector $\Omega = \bigotimes_{x \in V} \Omega_x$, with $\Omega_x \in \mathcal{H}_{\{x\}}$, for all $x \in V$. Then, for $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, with $X, Y \subset V$, $X \cap Y = \emptyset$, we have

$$\langle AB \rangle = \langle A \rangle \langle B \rangle$$

i.e., no correlations.

Question: how quickly can a dynamics with finite-range (or other) interactions build up correlations between A and B?

Similar question: how quickly can the state in the regions X and Y become substantially entangled?

More precisely, we want to bound

$$|\langle \tau_t(AB) \rangle - \langle \tau_t(A) \rangle \langle \tau_t(B) \rangle|$$

Theorem (N-Sims-Ogata 06; similar results by Bravyi-Hastings-Verstraete, and Eisert-Osborne)

Let V, $a \geq 0$, F_a, C_a, $\|\Phi\|_a$, τ_t, *and* $\langle \cdot \rangle$ *be as above, and* $t \in \mathbb{R}$, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, *for some* $X, Y \subset V$. *Then, if* $d(X, Y) = d > 0$, *we have the bound*

$$
\begin{aligned}
&|\langle \tau_t(AB) \rangle - \langle \tau_t(A) \rangle \langle \tau_t(B) \rangle| \\
&\leq \widetilde{C}_a \|A\| \, \|B\| \left(e^{2C_a \|\Phi\|_a |t|} - 1 \right) \\
&\times \left[\sum_{x \in X} \sum_{y : d(y,X) \geq d/2} + \sum_{y \in Y} \sum_{x : d(x,Y) \geq d/2} \right] F_a(d(x, y))
\end{aligned}
$$

If $V = \mathbb{Z}^\nu$, and Φ is a pair interaction that decays exponentially as e^{-ar}, with $a > 0$, and X and Y are finite, our theorem gives

$$
\begin{aligned}
&|\langle \tau_t(AB) \rangle - \langle \tau_t(A) \rangle \langle \tau_t(B) \rangle| \\
&\leq c\|A\| \, \|B\| \left(e^{2C_a \|\Phi\|_a |t|} - 1 \right) (|X| + |Y|) e^{-ad(X,Y)/2}
\end{aligned}
$$

which means that it takes a time

$$
t \sim \frac{a \, d(X, Y)}{2C_a \|\Phi\|_a}
$$

before significant correlations appear between the regions X and Y.

Application 3: The Exponential Clustering Theorem

In the physics literature the term massive ground state implies two properties: a spectral gap above the ground state and exponential decay of spatial correlations. It has long been (correctly) believed that the first implies the second and it is often also (incorrectly) believed that the second implies the first.

The setup is as before. For simplicity of the presentation, we limit ourselves to the case of a unique ground state.

We assume that $H \geq 0$ and that $\Omega \in \mathcal{H}$ is the unique normalized vector state such that $H\Omega = 0$.

We say that the system has a spectral gap if there exists $\delta > 0$ such that $\mathrm{spec}(H) \cap (0, \delta) = \emptyset$ and in that case the spectral gap, γ, is defined by

$$\gamma = \sup\{\delta > 0 \mid \mathrm{spec}(H) \cap (0, \delta) = \emptyset\}.$$

Note: these conditions may be applied in a new representation of the system, e.g., to cover cases of infinite systems with discrete symmetry breaking.

In the following theorem the interactions are assumed to be exponentially decaying in the sense that there is $a > 0$ such that $\|\Phi\|_a < \infty$.

Theorem (N-Sims 2006, Hastings-Koma 2006)

Let V, $a > 0$, F_a, C_a, $\|\Phi\|_a$, $\Omega \in \mathcal{H}$ and $\gamma > 0$ be as above. Then, there exists $\mu > 0$ and a constant c such that for $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, $X, Y \subset V$,

$$|\langle \Omega, AB\Omega \rangle - \langle \Omega, A\Omega \rangle \langle \Omega, B\Omega \rangle| \leq c\|A\| \, \|B\|(|X|+|Y|)e^{-\mu d(X,Y)}.$$

One can take

$$\mu = \frac{a\gamma}{\gamma + 4\|\Phi\|_a}.$$

Application 4: A D-Dimensional Lieb-Schultz-Mattis Theorem

The Lieb-Schultz-Mattis (LSM) theorem is about specific models. The classic result (LSM 1961) is about the spin-1/2 AF chain. It was generalized to other 1D models by Affleck and Lieb (1985). Heisenberg chain: $V = [1, L]$, and $\mathcal{H}_x \cong \mathbb{C}^2$,

$$H_L = \sum_{x=1}^{L-1} \mathbf{S}_x \cdot \mathbf{S}_{x+1}$$

The LSM Theorem states that if the ground state of H_L is unique, then the gap to the first exited state is bounded by C/L. A result of Lieb and Mattis (1966), shows that for the particular model H_L with L even, this is indeed the case.

One expects that when the conditions of the LSM Theorem are fullfilled, the excitation spectrum in the thermodynamic limit has no gap above the ground state.
Recently, Hastings derived a higher-dimensional version (2004, not obviously rigorous). We now have a rigorous proof of such a result.
Let $V = [1, L] \times V_L^\perp \subset \mathbb{Z}^\nu$, with $V_L^\perp \subset \mathbb{Z}^{\nu-1}$, such that $|V_L^\perp| \leq CL^{\nu-1}$. We will assume periodic boundary conditions in the first coordinate. The spin systems on each copy of $V^\perp$ are the same and have a total half-integer spin: $\sum_{x\in V_L^\perp} s_x = k + 1/2$, for some nonnegative integer k.

The simplest model to which our theorem applies is the antiferromagnetic Heisenberg Hamiltonian on V. Only translation invariance in the first coordinate is used.

Theorem (N-Sims, arXiv:math-ph/0608046)

If the ground state of H_V is non-degenerate, then the gap above it is bounded by

$$C\frac{\log L}{L}$$

The proof of this theorem is by construction of a suitable variational state for the low-lying excitation. But ...

The proof of this theorem is complicated by the fact that the ground state is unknown. Conceptually, the variational state is the ground state of a modified Hamiltonian, H_θ, in which the interactions in one hyperplane have been twisted by an angle θ. The idea of Hastings' (2004) was to describe this state as the solution of a differential equation in the variable θ with the original ground state as initial condition.

Both the new Lieb-Robinson bounds and the Exponential Clustering Theorem play a crucial role in estimating the energy of the variational state and proving its orthogonality to the ground state.

Conclusion

We continue to improve our understanding of the dynamics of quantum spin systems, in particular its locality properties.

New knowledge about the dynamics can be used to better understand correlations, excitations etc., and may lead to further applications, in particular to perturbation theory and to study computability of quantum states.