# Bounded diagonalization and Ramseyan results on edge-labeled ternary trees

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**Goal.** Compare the complexity of diagonalization using only the values $\{0, 1, 2\}$ with the complexity of constructing a random set, e.g. a 1-random set.

The solution takes us into Ramseyan combinatorics. Given a ternary tree with certain edges labeled 0 or 1, its paths have induced labels which are binary words. We try to find a perfect binary subtree (branching at *every* level) whose paths have few induced labels.

$f, g, h, \ldots$ are variables for total functions from $\omega$ to $\omega$.

$f \leq_T g \iff (\exists e)[f = \Phi_e^g]$

Let $A^B$ be the set of functions from $B$ to $A$.

A **mass problem** is a set of total functions from $\omega$ to $\omega$, i.e. a subset of $\omega^\omega$.

$\mathcal{A}$,$\mathcal{B}, \ldots$ are variables for mass problems.

The "solutions" to a mass problem $\mathcal{A}$ are simply the elements of $\mathcal{A}$.

## Examples of mass problems

$\mathrm{DNR} = \{f : (\forall e)[f(e) \neq \varphi_e(e)]\}$.

For $k \in \omega$,

$\mathrm{DNR}_k = \{f : (\forall e)[f(e) < k \quad \& \quad f(e) \neq \varphi_e(e)]\} = \mathrm{DNR} \cap k^\omega$

For $A \subseteq \omega$, the *problem of enumerating A* is the set of functions with range $A$.

Let PA be the set of complete extensions of Peano arithmetic. PA can be viewed as a mass problem via Gödel numbering and identifying subsets of $\omega$ with their characteristic functions.

## Comparing Mass Problems

**Definition.** Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems. Then $\mathcal{A}$ is *weakly reducible* to $\mathcal{B}$ (denoted $\mathcal{A} \leq_w \mathcal{B}$) if

$$(\forall g \in \mathcal{B})(\exists f \in \mathcal{A})[f \leq_T g]$$

This definition is due to Muchnik, and the reducibility is also known as Muchnik reducibility.

**Definition.** $\mathcal{A}$ is *strongly reducible* to $\mathcal{B}$ (denoted $\mathcal{A} \leq_s \mathcal{B}$) if

$$(\exists e)(\forall g \in \mathcal{B})[\Phi_e^g \in \mathcal{A}]$$

This definition is due to Medvedev, and the reducibility is also known as Medvedev reducibility.

**Definition**

$\mathcal{A} \equiv_s \mathcal{B}$ if $\mathcal{A} \leq_s \mathcal{B}$ and $\mathcal{B} \leq_s \mathcal{A}$

The **strong degree** of $\mathcal{A}$, denoted $[\mathcal{A}]_s$ is $\{\mathcal{B} : \mathcal{B} \equiv_s \mathcal{A}\}$.

Define $[\mathcal{A}]_s \leq [\mathcal{B}]_s$ to mean $\mathcal{A} \leq_s \mathcal{B}$. This is a partial ordering of the strong degrees. The strong degrees form a **distributive lattice** under this partial order.

Define weak degrees analogously. The analogous partial ordering is also a distributive lattice.

**Theorem.** (Jockusch-Soare). $\mathrm{DNR}_2 \equiv_s$ PA .

Let $\mathcal{B} >_s \mathcal{A}$ mean that $\mathcal{A} \leq_s \mathcal{B}$ and $\mathcal{B} \not\leq_s \mathcal{A}$.

**Theorem.** (Jockusch)

(i) For all $i, j \geq 2$, $\mathrm{DNR}_i \equiv_w \mathrm{DNR}_j$.

(ii) $\mathrm{DNR}_2 >_s DNR_3 >_s DNR_4 >_s \ldots$.

Let $\omega^{<\omega} = \cup_{n \in \omega} \omega^n$. Thus $\omega^{<\omega}$ is the set of all finite strings of natural numbers. Let $\sigma, \tau, \ldots$ be variables for such strings.

$[\sigma] = \{f \in \omega^\omega : f \supseteq \sigma\}$

$\mathcal{A}$ is *effectively open* or $\Sigma^0_1$ if $\mathcal{A} = \cup_{\sigma \in S}[\sigma]$ for some c.e. set $S \subseteq \omega^{<\omega}$.

$\mathcal{A}$ is *effectively closed* or $\Pi^0_1$ if $\omega^\omega \setminus \mathcal{A}$ is effectively open.

Alternatively $\mathcal{A}$ is $\Pi^0_1$ iff there is a computable tree $T \subseteq \omega^{<\omega}$ such that $\mathcal{A} = [T] := \{f : (\forall n)[f \upharpoonright n \in T]\}$.

The following are all $\Pi^0_1$ sets:

DNR, $\mathrm{DNR}_k$, PA, the set of all ideals of a computable ring, the set of $k$-colorings of a computable graph

**Definition.** A $\Pi^0_1$ set $P \subseteq 2^\omega$ is *strongly universal* if $P \neq \emptyset$ and every nonempty $\Pi^0_1$ set $Q \subseteq 2^\omega$ is strongly reducible to $P$.

**Theorem**. (D. Scott) PA is strongly universal.

**Corollary.** $\mathrm{DNR}_2$ is strongly universal. For $k > 2$, $\mathrm{DNR}_k$ is weakly universal but not strongly universal.

(We can pretend that $DNR_k \subseteq 2^\omega$ via binary coding.)

Let $\mu$ be the usual coin toss measure on $2^\omega$.

**Definition.** A set $A \subseteq \omega$ is *weakly 1-random*, or *Kurtz-random*, if there is no $\Pi^0_1$ set $P \subseteq 2^\omega$ such that $\mu(P) = 0$ and $A \in P$.

**Note.** Every 1-generic set is weakly 1-random.

Let $\mathcal{K} = \{A : A \text{ is weakly 1-random}\}$.

**Definition.** An *effectively null* $\Pi^0_2$ set is a set $\mathcal{S} \subseteq 2^\omega$ of the form $\mathcal{S} = \cap_e \mathcal{S}_e$, where $\{\mathcal{S}_e\}$ is a computable sequence of $\Sigma^0_1$ subsets of $2^\omega$ with $\mu(\mathcal{S}_e) \leq 2^{-e}$ for all $e$.

Note: Every $\Pi^0_1$ set $P \subseteq 2^\omega$ of measure 0 is an effectively null $\Pi^0_2$ set.

**Definition.** A set $A \subseteq \omega$ is 1-*random* if there is no effectively null $\Pi^0_2$ set $\mathcal{S}$ such that $A \in \mathcal{S}$.

Let $\mathcal{R}_1 = \{A : A \text{ is 1-random}\}$.

$\mathcal{R}_1 \subset \mathcal{K}$

## Weak and strong degrees of nonempty $\Pi^0_1$ subsets of $2^\omega$

Stephen Simpson initiated the study of this area and has led its development. If **d** is a weak degree and there is a nonempty $\Pi^0_1$ subset of $2^\omega$ of weak degree **d**, call **d** a $\Pi^0_1$ weak degree. Let **1** be the weak degree of $\mathrm{DNR}_2$. Then **1** is the greatest $\Pi^0_1$ weak degree.

**Theorem.** (Simpson) Let $S$ be a $\Sigma^0_3$ subset of $\omega^\omega$, and let $P$ be a nonempty $\Pi^0_1$ subset of $2^\omega$. Then the weak degree of $S \cup P$ is $\Pi^0_1$.

**Corollary.** (Simpson) The weak degree $\mathbf{d}$ of DNR and the weak degree $\mathbf{r_1}$ of $\mathcal{R}_1$ (the family of 1-random sets) are each $\Pi^0_1$.

**Theorem.** $\mathbf{d} < \mathbf{r_1} < \mathbf{1}$

Here $\mathbf{r_1} < \mathbf{1}$ since almost every set is 1-random yet almost no set computes a $\text{DNR}_2$ function. Simpson and Giusto proved that $\mathbf{d} \leq \mathbf{r_1}$. This inequality is strict because Kumabe proved that there is a DNR function of minimal degree, while it is known that no 1-random set is of minimal degree.

**Theorem.** (Simpson) $\mathbf{r_1}$ is the greatest $\Pi^0_1$ weak degree containing a $\Pi^0_1$ set $P \subseteq 2^\omega$ of positive measure.

In contrast, Terwijn and Simpson-Slaman showed that there is no greatest $\Pi^0_1$ strong degree containing a $\Pi^0_1$ set $P \subseteq 2^\omega$ of positive measure.

Let $\mathbf{d}_k^*$ be the strong degree of $\mathrm{DNR}_k$. Recall that $\mathbf{d}_2^* > \mathbf{d}_3^* > \ldots$.

**Theorem.** (Simpson) Let $P$ and $Q$ be $\Pi^0_1$ subsets of $2^\omega$ with $P$ of positive measure. Let $\mathbf{p}$, $\mathbf{q}$ be the strong degrees of $P, Q$, respectively. If $\mathbf{d}_k^* \leq \sup(\mathbf{p}, \mathbf{q})$, then $\mathbf{d}_k^* \leq \mathbf{q}$.

Thus, $\Pi^0_1$ sets of positive measure are not helpful in "computing" $\mathrm{DNR}_k$.

**Theorem.** (Simpson) Let $\mathbf{p}$ be the strong degree of a $\Pi^0_1$ set $P \subseteq 2^\omega$ of positive measure, and let $\mathbf{d}^*_k$ be the strong degree of $\mathrm{DNR}_k$. Then

$$\sup(\mathbf{p}, \mathbf{d}^*_2) > \sup(\mathbf{p}, \mathbf{d}^*_{\mathbf{3}}) > \sup(\mathbf{p}, \mathbf{d}^*_{\mathbf{3}}) > \dots$$

In connection with this result, Simpson raised the following question.

**Question.** (Simpson) Is every $\Pi^0_1$ set $P \subseteq 2^\omega$ with $\mu(P) > 0$ strongly reducible to $\mathrm{DNR}_3$?

**Question.** (Joe Miller) Is $\mathcal{R}_1$ (the class of 1-random sets) strongly reducible to $\mathrm{DNR}_3$ ?

Recall that $\mathcal{K}$ is the class of weakly 1-random sets.

**Theorem.** (D-G-J-M) $\mathcal{K}$ is not strongly reducible to $\mathrm{DNR}_3$.

**Corollary.** $\mathcal{R}_1$ is not strongly reducible to $\mathrm{DNR}_3$.

**Proof**. $\mathcal{R}_1 \subseteq \mathcal{K}$.

**Corollary.** There is a $\Pi^0_1$ set $P \subseteq 2^\omega$ with $\mu(P) > 0$ such that $P$ is not strongly reducible to $\mathrm{DNR}_3$.

**Proof.** Since $\mathcal{R}_1$ is a $\Sigma^0_2$ set of positive measure, it has a $\Pi^0_1$ subset $P$ of positive measure. Apply the previous corollary.

## A contrasting result

**Definition.** $\mathcal{H}_1$ is the class of all $A \subseteq \omega$ of effective Hausdorff dimension 1. Thus, $\mathcal{H}_1$ is the family of all $A$ such that

$$\lim_n \frac{K(A \upharpoonright n)}{n} = 1$$

where $K$ is prefix-free Kolmogorov complexity.

**Theorem.** (Greenberg and Miller) For all $k \geq 2$, $\mathcal{H}_1 \leq_s \mathrm{DNR}_k$.

## Outline of proof that $\mathcal{K} \not\leq_s \mathrm{DNR}_3$

Given $e$ such that $\Phi_e^f$ is total for all $f \in \mathrm{DNR}_3$. We must show that there exists $f \in \mathrm{DNR}_3$ such that $\Phi_e^f \notin \mathcal{K}$.

1. We can assume without loss of generality that $\Phi_e^f$ is total and $\{0, 1\}$-valued for all $f \in 3^\omega$ (not just all $f \in \mathrm{DNR}_3$). The reason is that there exists $i$ with the desired properties such that $\Phi_e^f = \Phi_i^f$ for all $f \in \mathrm{DNR}_3$.

2. **Main Step.** Construct a $\Pi^0_1$ class $P \subseteq 3^\omega$ such that $P\cap$ DNR$_3 \neq \emptyset$ and $\Phi_e(P) := \{\Phi_e^f : f \in P\}$ has measure 0.

3. By König's Lemma, if $n \in \omega$ is given and $\sigma \in 3^{<\omega}$ is a sufficiently long finite string, then $\Phi_e^\sigma(n)$ is defined.

4. Using 3, $\Phi_e(P)$ is a $\Pi^0_1$ class. Take $f \in P \cap DNR_3$. Then $\Phi_e^f \in \Phi_e(P)$. $\Phi_e(P)$ is a $\Pi^0_1$ set of measure 0, so $\Phi_e^f \notin \mathcal{K}$.

## Outline of the Main Step

Call a set $S \subseteq 3^{<\omega}$ 2-*bushy* if $S$ is a length-preserving copy of $2^{<\omega}$. Thus, $S$ is closed downwards and every string in $S$ has exactly two **immediate extensions** in $S$.

If $S$ is 2-bushy, then $[S] \cap$ DNR$_3$ is nonempty.

We must construct a computable 2-bushy $S$ with $\Phi_e([S])$ of measure 0. Then $P = [S]$ is the desired $\Pi^0_1$ class with $P \cap$ DNR$_3$ nonempty and $\Phi_e(P)$ of measure 0.

Define $u_n$ recursively. Let $u_n$ be the least number $u > 0$ with $u > u_i$ for all $i < n$ and $\Phi_e^\sigma(i)$ defined for all strings $\sigma \in 3^{<\omega}$ of length $u$ and all $i < n$.

If $\sigma \in 3^{<\omega}$ has length $u_n$, let $t(\sigma)$ be the binary string of length $n$ whose $i$th term is $\Phi_e^\sigma(i)$ for $i < n$.

Given a 2-bushy $S \subseteq 3^{<\omega}$, let

$$c_S(n) = \{t(\sigma) : \sigma \in S \quad \& \quad \sigma \text{ has length } u_n\}$$

To ensure that $\Phi_e([S])$ has measure 0, we require that $\lim_{n\to\infty} |c_S(n)|/2^n = 0$.

## A combinatorial formulation

A *rooted tree* is a connected undirected graph with no cycles having a distinguished vertex called the *root*.

**Definition.** In a rooted tree, the *depth* of a vertex is its distance from the root. A finite rooted tree is *complete* if all of its leaves have the same depth, and this common depth is called the *depth* of the tree. It is *q-ary* if each vertex which is not a leaf has exactly $q$ children. The *depth* of an edge is the depth of its deeper endpoint.

**Monochromatic Subtree Lemma.** (Folklore ?) Suppose that $T$ is a complete rooted ternary tree of depth $n$, and each leaf of $T$ is colored red or blue. Then $T$ has a complete binary subtree $S$ of depth $n$ with all leaves of the same color.

This lemma was used implicitly to show that $\mathrm{DNR}_k$ is not strongly reducible to $\mathrm{DNR}_{k+1}$ for any $k$.

**Definition.** An infinite tree is *complete* if it has no leaves.

**Definition.** Let $T$ be an infinite complete ternary tree, and let $U = \{u_1 < u_2 < \dots\}$ be an infinite set of positive integers. A $U$-*labeling* of $T$ assigns to each edge with depth in $U$ a label which is 0 or 1.

Suppose that $T$ is an infinite complete $U$-labeled ternary tree. We consider infinite paths through $T$, starting at the root. With each such path $p$ we associate the infinite binary word $t(p) = a_1 a_2 \dots$, where $a_i \in \{0, 1\}$ is the label on the unique edge of depth $u_i$ on the path.

Let $T$ be an infinite complete $U$-labeled rooted ternary tree, where $U$ is an infinite set of positive integers.

If $S$ is an infinite complete subtree of $T$, let $c(S) = \{t(p) : p \text{ is a path through } S\}$

Thus, $c(S)$ is the set of infinite binary words associated with paths through $S$.

Our goal is to find an infinite complete binary subtree $S$ of $T$ with $c(S)$ "small". However, in general we cannot make $c(S)$ countable.

**Measure 0 Theorem.** Let $U$ be an infinite set of positive integers, and let $T$ be a $U$-labeled infinite complete ternary tree. Then $T$ has an infinite complete binary subtree $S$ with $c(S)$ of measure 0. Hence, the set of infinite words along the paths through $S$ has measure 0.

If $\sigma$ is a vertex of depth $u_n$ in $T$, let $t(\sigma)$ be the $n$-bit binary word formed by the labels of the edges on the path from the root to $\sigma$. Let

$$c_S(n) = \{t(\sigma) : \\ \sigma \in S \quad \& \quad \sigma \text{ has depth } u_n\}.$$

To ensure that $c(S)$ has measure 0, we arrange that $\lim_n |c_S(n)|/2^n = 0$.

**Definition.**

Let $S \sqsubset T$ mean that $S, T$ are complete finite rooted trees of the same depth, $S$ is a subtree of $T$, $T$ is ternary, and $S$ is binary.

If $S \sqsubset T$, then every leaf of $S$ is a leaf of $T$.

Let $U = \{u_1 < u_2 < \dots\}$ If $S$ is a $U$-labeled tree of depth $u_n$, let

$$c(S) = \{t(\sigma) : \sigma \text{ is a leaf of } S\}$$

Thus, $c(S)$ is the set of binary words occurring along paths from the root of $S$ to leaves of $S$.

**Multiple Tree Lemma.** Let $U = \{u_1 < u_2 < \dots\}$ be an infinite set, and let $T_1, T_2, \dots, T_k$ be complete $U$-labeled ternary trees of depth $u_n$, where $n > 2^k$. Then there exist binary trees $S_1, S_2, \dots, S_k$ with $S_i \sqsubset T_i$ for $1 \leq i \leq k$ such that $| \cup_{i \leq k} c(S_i)| \leq (3/4)2^n$.

## Proof of Multiple Tree Lemma

For $1 \leq i \leq n$ say that a complete tree $T$ of depth $u_n$ is *$i$-good* if there exists $S \sqsubset T$ such that every word in $c(S)$ has a 0 as its $i$-th bit. By the Monochromatic Subtree Lemma, if $T$ is not $i$-good, there exists $S \sqsubset T$ such that every word in $c(S)$ has a 1 as its $i$th bit.

Let $G_i = \{j \leq k : T_j \text{ is } i\text{-good}\}$

By the pigeonhole principle, there exist distinct coordinates $a, b$ between 1 and $n$ with $G_a = G_b$.

Let $a$ and $b$ be distinct coordinates such that $G_a = G_b$.

For $j \in G_a$, choose $S_j \sqsubset T_j$ such that every word in $c(S_j)$ has a 0 in the $a$th coordinate.

For $j \notin G_a = G_b$ with $j \leq k$ , choose $S_j \sqsubset T_j$ such that every word in $c(S_j)$ has a 1 in the $b$th coordinate.

Then every word in $\cup_{j \leq k} c(S_j)$ has either a 0 in the $a$th coordinate or a 1 in the $b$th coordinate. It follows that $| \cup_{j \leq k} c_{S_j}(n)| \leq (3/4)2^n$.

**Proof of measure 0 path label theorem.**

Let $T$ be an infinite complete $U$-labeled ternary tree. We must construct $S \sqsubset T$ with $\mu(c(S)) = 0$.

*Stage* $s$. Suppose we are given $S_s$, a complete binary subtree of $T$ of depth $u_m$ (say), with $|c(S_s)| \leq (3/4)^s 2^m$. We choose $n$ large and construct $S_{s+1}$ by extending the leaves of $S_s$ to depth $u_{m+n}$ with $|c(S_{s+1})| \leq (3/4)^{s+1} 2^{m+n}$.

The tree $S_s$ has $k$ leaves, where $k = 2^{u_m}$. Let $n = 2^k + 1$. Let $T_1, T_2, \ldots, T_k$ be the subtrees of $T$ above the leaves of $S_s$ to depth $u_{m+n}$, with the induced edge-labeling. By the multiple tree lemma, there exist $L_1 \sqsubset T_1, \ldots, L_k \sqsubset T_k$ with $|\cup\, c(L_i)| \leq (3/4)2^n$. Obtain $S_{s+1}$ by gluing each $L_i$ above the corresponding leaf of $S_s$.

Then $|c(S_{s+1})| \leq |c(S_s)||\cup_{i\leq k} c(L_i)| \leq (3/4)^s 2^m (3/4) 2^n = (3/4)^{s+1} 2^{m+n}$.

This finally completes the proof that $\mathcal{K}$ is not strongly reducible to $\mathrm{DNR}_3$. Identify the vertices of the complete ternary tree with $3^{<\omega}$. Define $U$ from $\Phi_e$ as before, and note that $U$ is computable. Let $U = \{u_1, u_2, \dots\}$, with $u_1 < u_2 < \dots$. If a string $\sigma \in 3^{<\omega}$ has length (depth) $u_n$, label the edge just above it with $\Phi_e^\sigma(n-1)$. The proof of the measure 0 path theorem is effective and so yields a *computable* infinite complete binary subtree $S$ of $T$ with $\mu(c(S)) = 0$. Then $S$ is the desired computable 2-bushy set with $\Phi_e([S])$ of measure 0.

We now consider the case $U = \mathbb{N}$, so *all* edges are colored. We define a combinatorial bounding function $f : \mathbb{N} \to \mathbb{N}$.

First define $f$ on finite complete ternary trees $T$ with all edges labeled 0 or 1:

$f(T) = \min\{|c(S)| : S \sqsubset T\}$

Thus, $f(T)$ is the smallest number of path labels that can be achieved for complete binary subtrees $S$ of the same depth as $T$.

Then we look at the worst case for each depth:

$f(n) = \max\{f(T) :$
$T$ is a complete ternary tree of depth $n\}$

Thus, $f(n)$ is the least number $b$ such that every $\{0, 1\}$ edge-labeled complete ternary tree of depth $n$ has a complete binary subtree $S$ of depth $n$ with at most $b$ path labels.

**Proposition**. Let $m$ and $n$ be positive integers.

(i) $f(m+n) \geq f(m)f(n)$

(ii) $f(n+1) \leq 2f(n)$.

**Proposition.**

(i) $f(i) = i$ for $1 \leq i \leq 4$.

(ii) $6 \leq f(5) \leq 8$.

**Proposition.** $\lim_n (f(n))^{1/n}$ exists and is the supremum of the values of $f(n)^{1/n}$ for $n \in \mathbb{N}$.

**Corollary.** $\lim_n (f(n))^{1/n} \geq \sqrt[3]{3} \geq 1.442$.

**Theorem.** For all $n \in \mathbb{N}$ , $f(n) \geq 2^{\frac{n-2}{\log_2 3}}$.

**Corollary.**
$\lim_n (f(n))^{1/n} \geq 2^{\frac{1}{\log_2 3}} \geq 1.548$

**Theorem.** There are positive constants $\gamma$ and $c$ such that, for all $n \in \mathbb{N}$,

$$f(n) \leq \gamma 2^{n-c\sqrt{n}}$$

**Open Question.** What is $\lim_n (f(n))^{1/n}$? We know that this limit $L$ exists and satisfies

$$1.548 \leq 2^{\frac{1}{\log_2(3)}} \leq L \leq 2$$

**Open Question.** Does there exist $n > 1$ such that $f(n+1) = 2f(n)$?

**Open Question.** What if one considers $p$-ary trees and $q$-ary subtrees in place of ternary trees and binary subtrees?