Birth-death processes, bushy trees, and a law of weak subsets

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Computability: Fixing Notation

$f : \mathbb{N} \to \mathbb{N}$ or $X \in \{0,1\}^\infty$ is *computable* if there is an algorithm (implemented on a Turing machine) that given n produces $f(n)$ (or $X(n)$).

For $X, Y \in \{0,1\}^\infty$, X is *computable from* Y if there is an algorithm that given n, running in finite (but unlimited) time and space and allowed to now and then query bits among $Y(0), \ldots, Y(k_n)$, produces $X(n)$.

$X \in \{0,1\}^\infty$ is also considered as $X \subseteq \mathbb{N}$.

Example

The complement of X is computable from X.

Example

$0'$, *the halting problem for Turing machines, is not computable.*

Algorithmic randomness

For a finite binary string $\sigma \in \{0,1\}^*$, we let

$$[\sigma] = \{X \in \{0,1\}^\infty : X \text{ starts with } \sigma\}.$$

Fair-coin measure on $\{0,1\}^\infty$ is defined by $\mu([\sigma]) = 2^{-\text{length}(\sigma)}$. Our topology on $\{0,1\}^\infty$ is the product topology obtained from the discrete topology on $\{0,1\}$.

A *Martin-Löf test* is a sequence $\{U_n\}_{n\in\mathbb{N}}$ of open subsets of $\{0,1\}^\infty$ such that $\mu(U_n)\le 2^{-n}$ (equivalently, $\mu(U_n)$ goes to 0 and not "noncomputably slowly") and $\{(\sigma,n):[\sigma]\subseteq U_n\}$ is the range of a computable function from $\mathbb{N}$ to $\{0,1\}^*\times\mathbb{N}$.
The test $\{U_n\}_{n\in\mathbb{N}}$ defines a null set $\cap_n U_n$. X passes the test for randomness $\{U_n\}_{n\in\mathbb{N}}$ if $X\notin\cap_n U_n$.
X is *Martin-Löf random* if it passes all Martin-Löf tests.

A *Martin-Löf test* relative to $0'$ is defined similarly except that we only require that $\{(\sigma, n) : [\sigma] \subseteq U_n\}$ is the range of a function that is computable from $0'$.
X is *Martin-Löf random relative to* $0'$ if it passes all Martin-Löf tests relative to $0'$.

Example

The Strong Law of Large Numbers states that for almost all X according to the measure μ, we have

$$\forall \varepsilon > 0\, \exists N \forall n > N \left| \frac{\text{the \# of 1s up to } n \text{ in } X}{n} - \frac{1}{2} \right| < \varepsilon.$$

Suppose X does not satisfy the SLLN, as witnessed by a number ε_0. Let

$$U_N = \{Z : \exists n > N \left| \frac{\text{the \# of 1s up to } n \text{ in } X}{n} - \frac{1}{2} \right| \geq \varepsilon_0\}.$$

Then U_N is open; $\{(\sigma, n) : [\sigma] \subseteq U_n\}$ is the range of a computable function; and $\mu(U_N)$ goes computably quickly to 0. Thus, X is not Martin-Löf random.

- Almost all X according to μ are Martin-L"of random.
- No computable set X is Martin-Löf random.
- Some Martin-Löf random sets are computable from $0'$.

Theorem 1 (Law of Weak Subsets)

Almost every $X \subseteq \mathbb{N}$, according to μ, has an infinite subset $Y \subseteq X$ such that Y computes no Martin-Löf random set.

(Passing from X to Y we suffer a “loss of randomness beyond algorithmic repair.”)

Equivalently:

For almost all X, the Muchnik degree of

$$\{Y : Y \text{ is infinite and } Y \subseteq X\}$$

is not above $\mathcal{R}_1$, the Muchnik degree of Martin-Löf random sets.

Theorem 2 (The Law of Weak Subsets is Arithmetical)

Every X that is Martin-Löf random relative to $0'$ has an infinite subset $Y \subseteq X$ such that Y computes no Martin-Löf random set.

Example

Let X be Martin-Löf random and let Y be a "computably chosen" subset of X. Say,

$$Y = \langle X(0), 0, X(2), 0, X(4), 0, \ldots \rangle.$$

Then Y is an infinite subset of X, but Y does compute a Martin-Löf random set, namely

$$Z = \langle X(2), X(4), X(6), X(8), \ldots \rangle.$$

Example

Let X be Martin-Löf random let Y be a "randomly chosen" subset of X. That, is each 1 in X is converted to a 0 with probability $\frac{1}{2}$. Then Y does compute a Martin-Löf random set, as observed by John von Neumann. Namely, let Z be obtained from X by making the following replacements:

$$\langle X(2n), X(2n+1)\rangle \mapsto Z(n)$$

$$\langle 0,0\rangle \mapsto \langle\rangle$$

$$\langle 1,1\rangle \mapsto \langle\rangle$$

$$\langle 0,1\rangle \mapsto \langle 0\rangle$$

$$\langle 1,0\rangle \mapsto \langle 1\rangle$$

Using von Neumann's method of letting Y be a randomly selected subset of X did not work.
Instead, we will let Y be a randomly selected member of (a large subclass of)

$$\{Y : Y \text{ computes no Martin-Löf random set}\},$$

the random choice being carried out by X.

Question 1

Is there a suitable genericity notion such that for each Martin-Löf random set X, if Y is a "generic subset" of X then Y computes no Martin-Löf random set?

An analogue for infinite sets of definitions found in Kumabe/Lewis and Ambos-Spies/Kjos-Hanssen/Lempp/Slaman.

Definition

A subset C of $\mathbb{N}^* = \omega^{<\omega}$ is *n-bushy* if the empty string is in C and every element of C has at least n many immediate extensions in C.

Theorem 3

There is a 3-bushy subset C of $\mathbb{N}^*$ such that (i) for each infinite path Z through C, Z does not compute any Martin-Löf random set; (ii) C is computable from $0'$.

Proof.

A variation of a construction from the paper Ambos-Spies, Kjos-Hanssen, Lempp, Slaman, 2004. Now we ask for sets that are so bushy that there is not just *one* acceptable path through them, but a whole 3-bushy collection of such paths. Then the construction splits up into subconstructions for each of these paths. The construction is still carried out using only the oracle $0'$. □

(By Arslanov's completeness criterion, C cannot be computable.)

Proof of Theorem 2 from Theorem 3

Let X be a subset of $\mathbb{N}^*$ that is Martin-Löf random relative to $0'$. A birth-death process where everyone has 3 children, each with a 50% chance of surviving and themselves having 3 children, gives positive probability to the event of eventual nonextinction of the tribe.
Since C is 3-bushy, almost all X have some finite modification that contains an infinite path through $\mathbb{N}^*$ that is contained in C. Since C is computable from $0'$, the event of extinction is Σ^0_1 relative to $0'$. We apply an effective bijection between $\mathbb{N}^*$ and $\mathbb{N}$.

Question 2

Does every Martin-Löf random set X have an infinite subset Y such that for all Z computable from Y, Z is not Martin-Löf random?

Theorem 2 states that this is true if X is Martin-Löf random relative to $0'$.

If the answer to Question 2 is

no, there is a counterexample X that is computable from $0'$,

and more generally

for each A, there is an X that is ML-random relative to A and computable from A', such that for all infinite subsets Y of X there is a set Z that is ML-random relative to A and computable from the join $Y \oplus A$,

then

Stable Ramsey's Theorem for Pairs implies Weak Weak König's Lemma for ω-models.

Another question about Ramsey theory and WWKL.

Question 3

Does the conjunction of G_δ-Regularity and Weak Weak König's Lemma imply the Rainbow Ramsey Theorem for pairs (over RCA_0)?

This is suggested by recent results of Miller, and would be perhaps the first example in this part of Reverse Mathematics of mathematical theorems A, B, C such that

$$A \not\Rightarrow C$$

$$B \not\Rightarrow C$$

$$A \ \& \ B \Rightarrow C$$