Computability, Reverse Mathematics and Combinatorics: Open Problems
Banff International Research Station
Alberta, Canada
Sunday, December 7–Friday, December 12, 2008

Organizers
Peter A. Cholak (University of Notre Dame)
Barbara F. Csima (University of Waterloo)
Steffen Lempp (University of Wisconsin-Madison)
Manuel Lerman (University of Connecticut-Storrs)
Richard A. Shore (Cornell University)
Theodore A. Slaman (University of California at Berkeley)

1 Tim Carlson

Fix a finite set L and an infinite list of variables $v_0, v_1, v_2, \ldots, v_n, \ldots$. For $m, n \leq \omega$, $W(L, m, n)$ is the set of sequences w of elements of $L \cup \{v_i \mid i < m\}$ of length n with the property that v_i occurs in w for each $i < m$ and the first occurrence of v_i is before the first occurrence of v_j whenever $i < j < m$. When $m \in \omega$, $W(L, m)$ is $\bigcup_{n \in \omega} W(L, m, n)$. $W(L)$ is $\bigcup_{m \in \omega} W(L, m)$.

When $w \in W(L, m, n)$ and u is a sequence of length m, let $w(u)$ be the result of simultaneously substituting u_i for v_i in w for all $i < m$. Notice that if $u \in W(k, m)$ then $w(u) \in W(k, n)$.

The following was the **combinatorial core** of the main results of [1].

($\mathbf{CC}_{CS}$) For any coloring of $W(L, 0)$ with finitely many colors there is a $w \in W(L, \omega, \omega)$ such that the collection of *initial parts* of $w(u)$ ($u \in W(L, \omega, \omega)$) is monochromatic.

In the above, the initial part of an element of $W(L, \omega, \omega)$ is the longest initial segment in $W(L, 0)$, i.e., the initial segment before the first occurrence of v_0.

Problem 1.1. Find the proof theoretic strength of $\mathbf{CC}_{CS}$.

$\mathbf{CC}_{CS}$ is established in $\Pi^1_2 - TI_0$ in [3].

The following is the combinatorial core of the simplest of the main results from [2].

$(\mathbf{CC})_{1,1}$ For any coloring of $W(L,1)$ with finitely many colors there is an infinite sequence $\vec{w}$ of elements of $W(L,1)$ such that the collection of elements of $W(L,1)$ of the form

$$\vec{w}_0(a_0) * \vec{w}_1(a_1) * \cdots * \vec{w}_{k-1}(a_{k-1})$$

(where k is the length of $\vec{w}$ and $a_0, \ldots, a_{k-1} \in L \cup \{v_0\}$) is monochromatic.

Problem 1.2. Find the proof theoretic strength of $\mathbf{CC}_{1,1}$.

Since $\mathbf{CC}_{1,1}$ easily implies Hindman's Theorem as long as L is nonempty, $\mathbf{CC}_{1,1}$ implies ACA_0 over RCA_0 by a result in [3]. No upper bounds other than those given by the proof in [2] are known.

A stronger combinatorial result from [2] states that for all $m \in \omega$:

$(\mathbf{CC}_{m,\infty})$ For any coloring of $W(L,m)$ with finitely many colors, there is an infinite sequence $\vec{w}$ with $\vec{w}_k \in W(L, m+k)$ for $k \in \omega$ such that the collection of all elements of $W(L,m)$ of the form $\vec{w}_{i_0}(u_0) * \vec{w}_{i_1}(u_1) * \cdots * \vec{w}_{i_k}(u_k)$ $(i_0 < i_1 < \cdots < i_k; u_0, u_1, \ldots, u_k \in W(L))$ is monochromatic.

Problem 1.3. Find the proof theoretic strength of $\mathbf{CC}_{m,\infty}$.

References

[1] T.J. Carlson and S.G. Simpson, *A dual form of Ramsey's theorem,* Advances in Mathematics 53 (1984), pp. 265-290.

[2] T.J. Carlson, *Some unifying principles in Ramsey theory,* Discrete Mathematics 68 (1988), pp. 117-169.

[3] A.R. Blass, J.L. Hirst and S.G. Simpson, *Logical analysis of some theorems of combinatorics and topological dynamics,* **Logic and Combinatorics (Arcata, CA 1985)**, Contemp. Math., vol. 65, Amer. Math. Soc. Providence, RI, 1987, pp. 125-156.

2 Chi Tat Chong

Problem 2.1. If M is a model of RCA_0+ $\mathrm{B}\Sigma_2$, and A is a Δ_2 subset of M (with parameters from M), is there a solution for A that preserves $\mathrm{B}\Sigma_2$? In other words, is there a G that is either a subset of A or disjoint from A, such that $M[G]$ is a model of $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2$? ($M[G]$ is the model obtained by adjoining G to M and closing under Turing reducibility.)

Problem 2.2. (In reverse recursion theory) Study the proof-theoretic strength of $0'''$-priority theorems.

3 Denis Hirschfeldt

Statements:

Definition 3.1. An *n-coloring* of $[\mathbb{N}]^k$, the unordered k-tuples $(n_1, \ldots, n_k)$ of natural numbers is a map $f : [\mathbb{N}]^k \to n$. A subset H of $\mathbb{N}$ is *homogeneous* for the coloring f if H is infinite and $|f``[H]^k| = 1$.

A coloring f of $[\mathbb{N}]^2$ is *stable* if $(\forall x)(\exists y)(\forall z > y)[f(x, y) = f(x, z)]$.

($\mathbf{RT^2_2}$) Ramsey's Theorem for pairs: Every 2-coloring of $[\mathbb{N}]^2$ has a homogeneous set.

($\mathbf{SRT^2_2}$) Stable Ramsey's Theorem for pairs: Every *stable* coloring of $[\mathbb{N}]^2$ has a homogeneous set.

Definition 3.2. If $\vec{R} = \langle R_i \mid i \in \mathbb{N}\rangle$ is a sequence of sets, an infinite set S is *$\vec{R}$-cohesive* if $(\forall i)(\exists s)[(\forall j > s)(j \in S \to j \in R_i) \vee (\forall j > s)(j \in S \to j \notin R_i)]$.

(COH) Cohesive Principle: For every sequence $\vec{R} = \langle R_i \mid i \in \mathbb{N}\rangle$ there is an $\vec{R}$-cohesive set.

(CAC) Chain-AntiChain: Every infinite partial order $(P, \leq_P)$ has an infinite subset S that is either a *chain*, i.e. $(\forall x, y \in S)(x \leq_P y \vee y \leq_P x)$, or an *antichain*, i.e. $(\forall x, y \in S)(x \neq y \to (x \nleq_P y \wedge y \nleq_P x))$.

(ADS) Ascending or Descending Sequence: Every infinite linear order $(L, \leq_L)$ has an infinite subset S that is either an ascending sequence, i.e. $(\forall s < t)(s, t \in S \to s <_L t)$, and so of order type ω, or a descending sequence, i.e. $(\forall s < t)(s, t \in S \to t <_L s)$, and so of order type ω^*.

Definition 3.3. A partial order $\mathcal{P}$ is *stable* if either

$$((\forall i \in P)(\exists s)[(\forall j > s)(j \in P \to i <_P j) \vee (\forall j > s)(j \in P \to i \mid_P j)]$$

or

$$(\forall i \in P)(\exists s)[(\forall j > s)(j \in P \to i >_P j) \vee (\forall j > s)(j \in P \to i \mid_P j)].$$

(SCAC) Stable CAC: Every infinite stable partial order has an infinite chain or antichain.

Definition 3.4. An infinite linear order in which all nonfirst elements have immediate predecessors and all nonlast ones have immediate successors has type

- ω if every element has finitely many predecessors;.
- ω^* if every element has finitely many successors;
- $\omega + \omega^*$ if it is not of type ω or ω^* and every element has either finitely many predecessors or finitely many successors.

(SADS) Stable ADS: Every linear order of type $\omega + \omega^*$ has a subset of order type ω or ω^*.

(CADS) Cohesive ADS: Every linear order has a subset S of order type ω, ω^*, or $\omega + \omega^*$.

Definition 3.5. An infinite linear order $\mathcal{L}$ with first and last elements (0 and 1, respectively) in which all nonfirst elements have immediate predecessors and all nonlast ones have immediate successors is *strongly of type* $\omega+\omega^*$ if, for every finite ascending sequence $0 = x_0 <_L x_1 <_L \cdots <_L x_n = 1$, there is exactly one infinite subinterval $[x_i, x_{i+1})$, and both $[x_0, x_i]$ and $[x_i, x_n]$ are finite.

(PART) Every linear order of type $\omega + \omega^*$ is strongly of type $\omega + \omega^*$.

Questions:

All implications and nonimplications below are over RCA_0.

ADS and CAC are both natural principles provable in RT^2_2. In the same way that RT^2_2 can be split into SRT^2_2 and COH (i.e., RT^2_2 implies both these principles, and together they imply RT^2_2), ADS can be split into SADS and CADS, and CAC can be split into SCAC and a cohesive version of CAC that is equivalent to COH (see [2]). As noted in [2], CAC implies ADS, SCAC implies SADS, and COH implies CADS.

Problem 3.6. Does ADS imply CAC?

Problem 3.7. Does SADS imply SCAC?

Problem 3.8. Does CADS imply COH?

A positive answer to the second question would imply a positive answer to the first question, since, as shown in [2], ADS implies COH. It was also shown in [2] that CADS and COH are equivalent over $\mathrm{B}\Sigma_2$.

In [2], it is shown that SADS is not Π^1_1-conservative over RCA_0 because it implies PART, which is implied by $\mathrm{B}\Sigma_2$ but not provable in RCA_0. These results raise the following related questions.

Problem 3.9. Does SADS imply $B\Sigma_2$?

Problem 3.10. *(Now solved [[3]])* Does PART imply $B\Sigma_2$ over RCA_0?

Chong, Lempp, and Yang have now shown that, indeed, PART does imply $B\Sigma_2$ over RCA_0.

Finally, the following is an important proof theoretic question left open in [1]. See Section 6 in [2] for a discussion of the potential difficulties in answering it.

Problem 3.11. Is RT^2_2 or SRT^2_2 Π^1_1-conservative over $B\Sigma_2$?

References

[1] Cholak, P. A., Jockusch, Jr., C. G. and Slaman, T. A., *On the strength of Ramsey's Theorem for pairs*, J. Symbolic Logic 2001 **66**, 1–55.

[2] Hirschfeldt, D. R. and Shore, R. A., *Combinatorial principles weaker than Ramsey's Theorem for pairs*J. Symbolic Logic} **72** 2007, 171–206.

[3] Chong, C.T., Lempp, S. and Yang, Y., The collection principle for Σ_2-formulas and the partition principle PART, preprint.

4 Jeff Hirst

We introduce the following shorthand notation for combinatorial principles:

Ramsey's Theorem RT^n_k: If $f : [\mathbb{N}]^n \to k$, then there is an infinite set $X \subset \mathbb{N}$ and a $c < k$ such that for all $\vec{x} \in [X]^n$, $f(\vec{x}) = c$.

Stable Ramsey's Theorem SRT^2_k: Ramsey's theorem for pairs and k colors holds for stable partitions, where $f : [\mathbb{N}]^2 \to k$ is stable if for every m, $\lim_n f(m, n)$ exists.

Ramsey's Theorem on trees TT^n_k: Let $2^{<\mathbb{N}}$ denote the full binary tree and $[2^{<\mathbb{N}}]^n$ denote all n-tuples of comparable nodes in $2^{<\mathbb{N}}$. If $f : [2^{<\mathbb{N}}]^n \to k$, then we can find a $c < k$ and a subtree S such that S is order isomorphic to $2^{<\mathbb{N}}$, and $f(\sigma) = c$ for every n-tuple σ of comparable nodes in S.

Increasing polarized Ramsey's Theorem IPT^n_k: If $f : [\mathbb{N}]^n \to k$, then we can find a $c < k$ and a sequence $H_1, H_2, \dots H_n$ of infinite sets such that $f(\{x_1, x_2, \dots, x_n\}) = c$ for every increasing n-tuple $(x_1, x_2, \dots, x_n) \in H_1 \times \cdots \times H_n$.

We use RT^n to abbreviate $\forall k \mathsf{RT}^n_k$, and use similar abbreviations for other principles.

Problem 4.1. Do we need Σ^0_2 induction to prove TT^1?

Working in RCA_0, it is easy to show that TT^1 implies RT^1. A proof of TT^1 from Σ^0_2 induction can be found in Chubb, Hirst, and McNicholl [3]. Recently, Corduan, Groszek, and Mileti [5] have shown that RT^1 does not imply TT^1.

Problem 4.2. Does ACA_0 prove Hindman's Theorem?

A proof of Hindman's Theorem in the stronger system ACA_0^+ can be found in Blass, Hirst, and Simpson [1].

Problem 4.3. Can Glazer's proof of Hindman's Theorem be adapted to a countable setting?

Versions of Glazer's proof can be found in Comfort's article [4] and Graham, Rothschild, and Spencer [7]. Glazer's proof uses ultrafilters on the full power set of $\mathbb{N}$, but ultrafilters on countable Boolean algebras suffice for the deduction of Hindman's Theorem, as shown in Hirst [8].

Problem 4.4. Does $\mathsf{RCA}_0 + \mathsf{TT}^2_2 \vdash \mathsf{TT}^2$?

Cholak, Jockusch, and Slaman [2] proved that $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{RT}^2$. It may be possible to provide a negative answer to the problem by emulating their proof.

Problem 4.5. Does $\mathsf{RCA}_0 + \mathsf{TT}^2_2 \vdash \mathsf{RT}^2$?

A positive response to the preceding problem yields this result immediately. This result might be true even if the preceding problem is not. In light of Cholak, Jockusch, and Slaman's work [2], a positive result on this problem would show that TT^2_2 is strictly stronger than RT^2_2.

Problem 4.6. Does SRT^2 imply IPT^2?

Many related problems on the strength of the Polarized Ramsey's Theorem can be found in Dzhafarov and Hirst [6]. A negative response here would show that SRT^2 is strictly weaker than RT^2.

Problem 4.7. Does IPT^2 imply RT^2?

A negative response here would show that SRT^2 is strictly weaker than RT^2.

References

[1] Andreas R. Blass, Jeffry L. Hirst, and Stephen G. Simpson, *Logical analysis of some theorems of combinatorics and topological dynamics*, in: *Logic and combinatorics (Arcata, Calif., 1985)*, volume 65 of *Contemp. Math.*, pages 125–156. Amer. Math. Soc., Providence, RI, 1987.

[2] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman, *On the strength of Ramsey's theorem for pairs*, J. Symbolic Logic **66**(1), 1–55, 2001.

[3] Jennifer Chubb, Jeffry Hirst, and Tim McNicholl, *Reverse mathematics and partitions of trees*, J. Symbolic Logic **74**(1), 201–215, 2009.

[4] W. W. Comfort, *Ultrafilters: some old and some new results*, Bull. Amer. Math. Soc. **83**(4), 417–455, 1977.

[5] Jared Corduan, Marcia Groszek, and Joseph Mileti, *Draft: A note on reverse mathematics and partitions of trees*

[6] Damir Dzhafarov and Jeffry Hirst, *The polarized Ramsey theorem*, Archive for Mathematical Logic **48**(2), 141–157, 2009.

[7] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer, *Ramsey theory*, John Wiley & Sons Inc., New York, 1980, Wiley-Interscience Series in Discrete Mathematics, A Wiley-Interscience Publication.

[8] Jeffry L. Hirst, *Hindman's theorem, ultrafilters, and reverse mathematics*, J. Symbolic Logic **69**(1), 65–72, 2004.

5 Carl G. Jockusch, Jr.

Background. It follows immediately from Ramsey's Theorem for pairs that every infinite partial ordering has an infinite chain or an infinite antichain. These problems concern the complexity of infinite chains and antichains in infinite computable partial orderings. It follows from effective Ramsey theory (see [3]), that every such partial ordering has an infinite Π^0_2 chain or antichain. On the other hand, Herrmann [5] showed that there is an infinite computable partial ordering with no infinite Σ^0_2 chains or antichains. If only chains, or only antichains, are considered, the bounds are much higher. It was shown by Harizanov, Jockusch, and Knight [4] that there is an infinite computable partial ordering which has an infinite chain but none which is Σ^1_1 or Π^1_1, and they also obtained the analogous result for antichains. In the other direction they showed that every infinite computable partial ordering which contains an infinite chain has an infinite chain which is the difference of two Π^1_1 sets. They also showed that every infinite computable partial

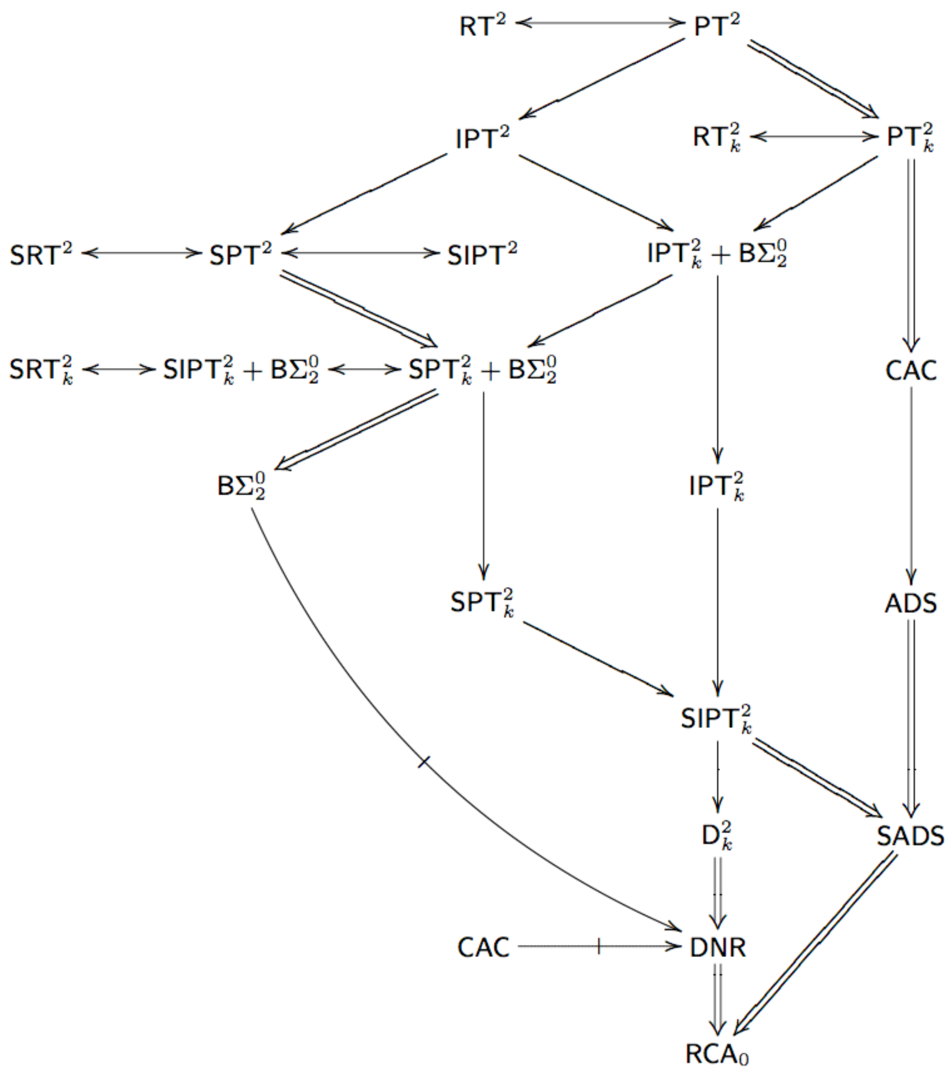


Figure 1: Results contributed by: Cholak, Dzhafarov, Hirschfeldt, Hirst, Jockusch, Kjos-Hanssen, Lempp, Slaman, and Shore The reverse direction of each single headed single arrow corresponds to an open problem.

ordering which contains an infinite antichain has an infinite antichain which is truth-table reducible to Kleene's $\mathcal{O}$.

Problem 5.1. ([4]) Does every computable partial ordering which contains an infinite antichain have an infinite antichain which is the difference of two Π^1_1 sets?

Background. Stable partial orderings were introduced by Hirschfeldt and Shore in [6]. In [7], Jockusch, Kastermans, Lempp, Lerman, and Solomon introduced the possibly more natural notion of a *weakly stable* partial ordering and extended some of the results of Hirschfeldt and Shore. Specifically, they called a partial ordering $(P, \leq)$ *weakly stable* if for all $a \in P$, either $a \leq b$ holds for almost all $b \in P$, or $b \leq a$ holds for almost all $b \in P$, or a is incomparable with almost all $b \in P$ (where "almost all" means for all but finitely many). It was shown in [7] (extending work from [6]) that every infinite computable weakly stable partial ordering has either an infinite low chain or an infinite computable antichain.

Problem 5.2. ([7]) Does every infinite computable weakly stable partial ordering have either an infinite computable chain or an infinite low antichain?

5.1 Open problems from my talk.

Background. Let the combinatorial bounding function $f : \mathbb{N} \to \mathbb{N}$ be defined as follows: $f(n)$ is the least number b such that every complete rooted ternary tree of depth n with every edge labeled 0 or 1 has a complete binary subtree of depth n having at most b path words. (Here a finite rooted tree is "complete" if all of its leaves have the same distance from the root, and this common distance is the depth of the tree. The path words of a rooted edge-labeled tree are those obtained by writing in order the labels on the edges on a path from the root to a leaf.) It is shown in [2] that:

$f(i) = i$ for $1 \leq i \leq 4$

$6 \leq f(5) \leq 8$

$f(n+1) \leq 2f(n)$ for all n

$\lim_n f(n)^{1/n}$ exists, and this limit L satisfies $1.548 \leq 2^{1/log_2 3} \leq L \leq 2$.

Problem 5.3. ([2])

1. What is the value of $f(5)$?
2. Does there exist $n > 1$ with $f(n+1) = 2f(n)$?
3. What is the value of the limit L defined above? Does $L = 2$?

The third problem is by far the most significant of these.

5.2 Well-known open problems about the strength of Ramsey's Theorem for pairs.

Background. Let RT^n_k be Ramsey's Theorem for k-colorings of n-element sets. It was shown by Simpson [9] that RT^n_k follows from ACA_0 for each $n, k \in \omega$, and that it is provable in RCA_0 that RT^n_k implies ACA_0 in RCA_0 when $n \geq 3, k \geq 2$. It was shown by Seetapun [8] that RT^2_2 does not imply ACA_0 in RCA_0. Also, it follows easily from [3] that RT^2_2 is not provable in WKL_0, so RT^2_2 is not equivalent to any of the "big five" systems of reverse mathematics. The following results appear in [1].

1. (J. Hirst) RT^2_2 implies Σ^0_2-Bounding in RCA_0

2. Every Π^1_1 sentence provable from RCA_0 + Σ^0_2-Induction + RT^2_2 is provable from just RCA_0 + Σ^0_2-Induction.

3. RT^2_2 is equivalent over RCA_0 to SRT^2_2 + COH . Here a two-coloring of pairs is called *stable* if for every a, all but finitely many pairs containing a have the same color, and SRT^2_2 is Ramsey's Theorem for pairs restricted to stable colorings. COH is a cohesiveness principle.

Problem 5.4. ([8]) Does RT^2_2 imply WKL_0 in RCA_0 ?

Problem 5.5. ([1]) Does SRT^2_2 imply RT^2_2 in RCA_0 ?

Problem 5.6. ([1]) Does SRT^2_2 imply WKL_0 in RCA_0 ?

Problem 5.7. ([1]) Does RT^2_2 imply Σ^0_2-Induction in RCA_0 ?

Problem 5.8. ([1]) Is RT^2_2 Π^1_1-conservative (as above) over RCA_0 + Σ^0_2-bounding?

References

[1] Peter A. Cholak, Carl G. Jockusch, Jr., and Theodore A. Slaman, *On the strength of Ramsey's theorem for pairs*, J. Symbolic Logic **66** (2001), no. 1, 1–55.

[2] Rod Downey, Noam Greenberg, Carl Jockusch, and Kevin Milans, *Binary subtrees with few labeled paths*, in preparation.

[3] Carl G. Jockusch, Jr., *Ramsey's theorem and recursion theory*, J. Symbolic Logic **37** (1972), no. 2, 268–280.

[4] Valentina S. Harizanov, Carl G. Jockusch, Jr., and Julia F. Knight, *Chains and antichains in partial orderings*, Archive for Mathematical Logic **48** (2009), no. 1, 39–53.

[5] E. Herrmann, *Infinite chains and antichains in computable partial orderings*, J. Symbolic Logic **66** (2001), 923–934.

[6] Denis R. Hirschfeldt and Richard A. Shore, *Combinatorial principles weaker than Ramsey's Theorem for pairs*, J. Symbolic Logic **72** (2007), 171–206.

[7] Carl G. Jockusch, Jr., Bart Kastermans, Steffen Lempp, Manuel Lerman, and Reed Solomon, *Stability and posets*, J. Symbolic Logic **74**(2009), no. 2, 693–711.

[8] David Seetapun and Theodore A. Slaman, *On the strength of Ramsey's theorem*, Notre Dame Journal of Formal Logic, **36**, 570–582.

[9] Stephen G. Simpson, *Subsystems of Second Order Arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.

6 H. Jerome Keisler

The language ${}^*L_1$ of nonstandard arithmetic has sorts N with variables $m, n, \ldots$ and ${}^*N$ with variables $x, y, \ldots$, and the vocabulary $0, 1, +, \dot{-}, \cdot, <$. In ${}^*L_1$, Basic Nonstandard Arithmetic (BNA) has the axioms of $I\Sigma_1$ in sort N, the axioms of linear order in sort ${}^*N$, and the Proper Initial Segment Axioms:

$$\forall n \exists x (x = n), \quad \forall n \forall x [x < n \to \exists m \, x = m], \quad \exists y \forall n [n < y].$$

$\forall$-Transfer is the scheme $\forall \vec{m} \, \varphi(\vec{m}) \leftrightarrow \forall \vec{x} \, \varphi(\vec{x})$ where $\forall \vec{m} \, \varphi(\vec{m})$ is a universal sentence.

L_2 is the language of second order arithmetic, with sorts N and P.

In the combined language $L_2 \cup {}^*L_1$, the Standard Part Principle (STP) is the sentence

$$\forall x \exists X \forall m \, [(p_m | x) \leftrightarrow m \in X] \wedge \forall X \exists x \forall m \, [(p_m | x) \leftrightarrow m \in X].$$

A Δ_0^S formula in ${}^*L_1$ is built from atomic formulas, connectives, and bounded quantifiers of both sorts N and ${}^*N$. Σ_1^S means $\exists m \, \varphi$ where φ is Δ_0^S.

Special Δ_0^S formulas are built from atomic formulas, connectives, divisibility formulas $(n | t(\vec{x}))$, and bounded quantifiers of sort N. Special Σ_1^S means $\exists m \, \varphi$ where φ is special Δ_0^S.

Σ_1^S Induction is the scheme

$$[\psi(0, \vec{u}) \wedge \forall n [\psi(n, \vec{u}) \to \psi(n+1, \vec{u})]] \to \forall n \, \psi(n, \vec{u})$$

where $\psi(n, \vec{u})$ is Σ_1^S. Special Σ_1^S Induction is the same scheme where $\psi(n, \vec{u})$ is special Σ_1^S. Special Σ_1^S Comprehension is the scheme which says that if both $\psi(n, \vec{u})$ and its negation are equivalent to special Σ_1^S formulas, then

$$\exists x \forall n [\psi(n, \vec{u}) \leftrightarrow (p_n | x)]$$

${}^*\mathsf{RCA}_0{}'$ is the theory BNA+Special Σ_1^S Induction + Special Δ_1^S Comprehension +STP. It implies and is conservative over RCA_0. The theory ${}^*\mathsf{RCA}_0{}' + \forall$-Transfer is still conservative over RCA_0. However, many innocent-looking extensions of ${}^*\mathsf{RCA}_0{}'$ imply the weak König lemma, and thus are not conservative over RCA_0.

Problem 6.1. How much can one add to ${}^*\mathsf{RCA_0}' + \forall$-Transfer and still be conservative over $\mathsf{RCA_0}$?

Problem 6.2. Is ${}^*\mathsf{RCA_0}' + \forall$-Transfer$+\Sigma_1^S$ Induction conservative over $\mathsf{RCA_0}$?

Problem 6.3. Is ${}^*\mathsf{RCA_0}'$+Transfer for universal formulas (rather than sentences) conservative over $\mathsf{RCA_0}$?

Note: The above two theories do not imply the Weak König Lemma.

Problem 6.4. Is the analogue of ${}^*\mathsf{RCA_0}'$ with an added symbol for exponentiation conservative over $\mathsf{RCA_0}$?

Problem 6.5. What happens if one uses a different method of coding sets by hyperintegers?

7 Bjørn Kjos-Hanssen

7.1 Computability: Fixing Notation.

$f : \mathbb{N} \to \mathbb{N}$ or $X \in \{0,1\}^\infty$ is *computable* if there is an algorithm (implemented on a Turing machine) that given n produces $f(n)$ (or $X(n)$).

For $X, Y \in \{0,1\}^\infty$, X is *computable from* Y if there is an algorithm that given n, running in finite (but unlimited) time and space and allowed to now and then query bits among $Y(0), \dots, Y(k_n)$, produces $X(n)$.

$X \in \{0,1\}^\infty$ is also considered as $X \subseteq \mathbb{N}$.

Example 7.1. The complement of X is computable from X.

Example 7.2. $0'$, the halting problem for Turing machines, is not computable.

7.2 Algorithmic randomness.

For a finite binary string $\sigma \in \{0,1\}^*$, we let

$$[\sigma] = \{X \in \{0,1\}^\infty : X \text{ starts with } \sigma\}.$$

The fair-coin measure on $\{0,1\}^\infty$ is defined by $\mu([\sigma]) = 2^{-\text{length}(\sigma)}$. Our topology on $\{0,1\}^\infty$ is the product topology obtained from the discrete topology on $\{0,1\}$. A *Martin-Löf test* is a sequence $\{U_n\}_{n\in\mathbb{N}}$ of open subsets of $\{0,1\}^\infty$ such that $\mu(U_n) \le 2^{-n}$ (equivalently, $\mu(U_n)$ goes to 0 and not "noncomputably slowly") and $\{(\sigma, n) : [\sigma] \subseteq U_n\}$ is the range of a computable function from $\mathbb{N}$ to $\{0,1\}^* \times \mathbb{N}$. The test $\{U_n\}_{n\in\mathbb{N}}$ defines a null set $\bigcap_n U_n$. X passes the test for randomness $\{U_n\}_{n\in\mathbb{N}}$ if $X \notin \bigcap_n U_n$. X is *Martin-Löf random* if it passes all Martin-Löf tests.

A *Martin-Löf test relative to* $0'$ is defined similarly except that we only require that $\{(\sigma, n) : [\sigma] \subseteq U_n\}$ is the range of a function that is computable from $0'$. X is *Martin-Löf random relative to* $0'$ if it passes all Martin-Löf tests relative to $0'$.

Example 7.3. The Strong Law of Large Numbers states that for almost all X according to the measure μ, we have

$$\forall \varepsilon > 0 \, \exists N \, \forall n > N \left| \frac{\text{the \# of 1's up to } n \text{ in } X}{n} - \frac{1}{2} \right| < \varepsilon.$$

Suppose X does not satisfy the SLLN, as witnessed by a number ε_0. Let

$$U_N = \{Z : \exists n > N \left| \frac{\text{the \# of 1's up to } n \text{ in } X}{n} - \frac{1}{2} \right| \geq \varepsilon_0\}.$$

Then U_N is open; $\{(\sigma, n) : [\sigma] \subseteq U_n\}$ is the range of a computable function; and $\mu(U_N)$ goes computably quickly to 0. Thus, X is not Martin-Löf random.

Some basic facts include:

- Almost all X according to μ are Martin-Löf random.
- No computable set X is Martin-Löf random.
- Some Martin-Löf random sets are computable from $0'$.

Theorem 7.4 (Law of Weak Subsets [KHb])**.** *Almost every $X \subseteq \mathbb{N}$, according to μ, has an infinite subset $Y \subseteq X$ such that Y computes no Martin-Löf random set.*

(Passing from X to Y we suffer a "loss of randomness beyond algorithmic repair".) Equivalently, for almost all X, the Muchnik degree of

$$\{Y : Y \text{ is infinite and } Y \subseteq X\}$$

is not above $\mathcal{R}_1$, the Muchnik degree of Martin-Löf random sets.

This strengthens the result of Kjos-Hanssen [KHa] that there exists a Martin-Löf random set of integers X and an infinite subset Y such that Y computes no Martin-Löf random set.

Problem 7.5. How "fat" may the subset Y in Theorem 7.4 be? Can we ensure that it has positive relative density within X?

Theorem 7.6 (The Law of Weak Subsets is Arithmetical [KHb])**.** *Every X that is Martin-Löf random relative to $0'$ has an infinite subset $Y \subseteq X$ such that Y computes no Martin-Löf random set.*

The following two examples illustrate how *not* to try to prove Theorem 7.6.

Example 7.7. Let X be Martin-Löf random and let Y be a "computably chosen" subset of X. Say,

$$Y = \langle X(0), 0, X(2), 0, X(4), 0, \dots \rangle.$$

Then Y is an infinite subset of X, but Y *does* compute a Martin-Löf random set, namely,

$$Z = \langle X(2), X(4), X(6), X(8), \dots \rangle.$$

Example 7.8. Let X be Martin-Löf random and let Y be a "randomly chosen" subset of X. That, is each 1 in X is converted to a 0 with probability $\frac{1}{2}$. Then Y *does* compute a Martin-Löf random set, as observed by John von Neumann. Namely, let Z be obtained from X by making the following replacements:

$$\langle X(2n), X(2n+1) \rangle \mapsto Z(n)$$

$$\langle 0, 0 \rangle \mapsto \langle \rangle$$

$$\langle 1, 1 \rangle \mapsto \langle \rangle$$

$$\langle 0, 1 \rangle \mapsto \langle 0 \rangle$$

$$\langle 1, 0 \rangle \mapsto \langle 1 \rangle$$

Problem 7.9. Is there a suitable genericity notion such that for each Martin-Löf random set X, if Y is a "generic subset" of X then Y computes no Martin-Löf random set?

Definition 7.10. A subset C of $\mathbb{N}^* = \omega^{<\omega}$ is *n-bushy* if the empty string is in C and every element of C has at least n many immediate extensions in C.

Theorem 7.11 ([KHb])**.** *There is a 3-bushy subset C of $\mathbb{N}^*$ such that*

1. *for each infinite path Z through C, Z does not compute any Martin-Löf random set; and*

2. *C is computable from $0'$.*

Proof. A variation of a construction of Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman [ASKHLS04]. Now we ask for sets that are so bushy that there is not just *one* acceptable path through them, but a whole 3-bushy collection of such paths. Then the construction splits up into subconstructions for each of these paths. The construction is still carried out using only the oracle $0'$. □

(By Arslanov's completeness criterion, C cannot be computable.)

Proof of Theorem 7.6 from Theorem 7.11. Let X be a subset of $\mathbb{N}^*$ that is Martin-Löf random relative to $0'$. A birth-death process where everyone has 3 children, each with a 50% chance of surviving and themselves having 3 children, gives positive probability to the event of eventual nonextinction of the tribe. Since C is 3-bushy, almost all X have some finite modification that contains an infinite path through $\mathbb{N}^*$ that is contained in C. Since C is computable from $0'$, the event of extinction is Σ^0_1 relative to $0'$. We apply an effective bijection between $\mathbb{N}^*$ and $\mathbb{N}$. □

Problem 7.12. Does every Martin-Löf random set X have an infinite subset Y such that for all Z computable from Y, Z is not Martin-Löf random?

Theorem 7.6 states that this is true if X is Martin-Löf random relative to $0'$. One can also ask Problem 7.12 for Kurtz and Schnorr randomness. If the answer to Problem 7.12 should turn out to be

> no, there is a counterexample X that is computable from $0'$,

and more generally

> for each A, there is an X that is ML-random relative to A and computable from A', such that for all infinite subsets Y of X, there is a set Z that is ML-random relative to A and computable from the join $Y \oplus A$,

then

> *Stable Ramsey's Theorem for Pairs* implies *Weak Weak König's Lemma* for ω-models.

A major problem in Reverse Mathematics:

Problem 7.13. Does Stable Ramsey's Theorem for Pairs (SRT^2_2) imply Weak König's Lemma, or at least Weak Weak König's Lemma?

One can also ask a uniform version of Problem 7.12.

Problem 7.14. Given a Turing reduction Φ, does every Martin-Löf random set X have an infinite subset Y such that Φ^Y is not Martin-Löf random?

Carl Jockusch's talk at the conference, on joint work with Downey, Greenberg, and Milans [DGJM], inspired this problem, which may have an easy "yes" answer whose proof is yet to be found.

Another problem about Ramsey theory and WWKL:

Problem 7.15. Does the conjunction of G_δ-Regularity and Weak Weak König's Lemma imply the Rainbow Ramsey Theorem for pairs (over RCA_0)?

This is suggested by recent results of Miller [Mil], and would be perhaps the first example in this part of Reverse Mathematics of mathematical theorems A, B, C such that

$$A \not\Rightarrow C$$
$$B \not\Rightarrow C$$
$$A \ \& \ B \Rightarrow C$$

For those interested in attempting to answer this problem, we mention that the Rainbow Ramsey Theorem was studied by Csima and Mileti [CM], and relevant results about G_δ-Regularity were given by Kjos-Hanssen, Miller, and Solomon [KHMS].

References

[ASKHLS04] Klaus Ambos-Spies, Bjørn Kjos-Hanssen, Steffen Lempp, and Theodore A. Slaman, *Comparing DNR and WWKL*, J. Symbolic Logic **69**(4), 1089–1104, 2004.

[CM] Barbara F. Csima and Joseph R. Mileti, *The strength of the Rainbow Ramsey Theorem*, to appear.

[DGJM] Rod Downey, Noam Greenberg, Carl G. Jockusch, and Kevin Milans, *Binary subtrees with few path labels*, to appear.

[KHa] Bjørn Kjos-Hanssen, *Infinite subsets of random sets of integers*, Math. Res. Lett., to appear.

[KHb] Bjørn Kjos-Hanssen, *A law of weak subsets*, to appear.

[KHMS] Bjørn Kjos-Hanssen, Joseph S. Miller, and D. Reed Solomon, *Lowness notions, measure, and domination*, to appear.

[Mil] Joseph S. Miller, *What good is always being wrong?*, to appear.

8 Julia Knight

Ketonen and Solovay [1] defined "α-largeness", generalizing the largeness from the Paris-Harrington Theorem [9]. The definition involves fundamental sequences for ordinals.

Definition 8.1 (Fundamental sequences)**.** To each ordinal $0 < \alpha < \epsilon_0$, we assign a *fundamental sequence* $\{\alpha\}(x)$ as follows.

1. For $\alpha = \beta + 1$, $\{\alpha\}(x) = \beta$, for all x.
2. For $\alpha = \omega^{\beta+1}$, $\{\alpha\}(x) = \omega^\beta \cdot x$.
3. For $\alpha = \omega^\beta$, where β is a limit ordinal, $\{\alpha\}(x) = \omega^{\{\beta\}(x)}$.
4. For $\alpha = \omega^\beta \cdot (a+1)$, where $a \neq 1$, $\{\alpha\}(x) = \omega^\beta \cdot a + \{\omega^\beta\}(x)$.
5. For α with Cantor normal form ending in $\omega^\beta \cdot a$, say $\alpha = \gamma + \omega^\beta \cdot a$, $\{\alpha\}(x) = \gamma + \{\omega^\beta \cdot a\}(x)$.

Definition 8.2 (α-largeness)**.** The set X is α-large, for $\alpha < \epsilon_0$, if there is a sequence $C = (\alpha_0, x_0, \alpha_1, x_1, \ldots, \alpha_{r-1}, x_{r-1}, \alpha_r)$ such that

1. $\alpha_0 = \alpha$,
2. $\alpha_r = 0$,
3. x_0 is the first element of X,
4. for $0 < i < r$, x_i is the first element of X that is $> x_{i-1}$, and
5. for $i < r$, $\alpha_{i+1} = \{\alpha_i\}(x_i)$.

Notation: We write $\beta \to (\alpha)^n_r$ if for any β-large set X and any partition of the n-sized subsets of X into r classes, there is an α-large homogeneous set $Y \subseteq X$.

Theorem 8.3 (Ketonen-Solovay)**.** *For each finite n and r and each $\alpha < \epsilon_0$, there exists $\beta < \epsilon_0$ such that $\beta \to (\alpha)^n_r$.*

Ketonen and Solovay connected this version of Ramsey theory with the Wainer functions [10].

Definition 8.4 (Wainer hierarchy)**.** We define the Wainer hierarchy as Ketonen and Solovay did.

For $\alpha < \epsilon_0$, $F_\alpha(x)$ is defined as follows.

1. $F_0(x) = x + 1$,
2. $F_{\alpha+1}(x) = F_\alpha^{(x+1)}(x)$,
3. for a limit ordinal α, $F_\alpha(x) = \max\{F_{\{\alpha\}(j)}(x) : j \leq x\}$.

For a model $\mathcal{A}$ of $I\Delta_0$, $\mathcal{A}$ is a model of PA iff in $\mathcal{A}$, F_α is total for all $\alpha < \epsilon_0$, and $\mathcal{A}$ is a model of $I\Sigma_n$ iff the F_α is total for all $\alpha < \omega_n$, where ω_n is a tower of n ω's.

Problem 8.5. How much Ramsey theory is provable in $I\Sigma_n$?

Sommer [2] developed the theory of ordinals in $I\Delta_0$. Using $I\Delta_0 + \exp$, he developed the connections between α-largeness and the Wainer functions. However, Sommer did not do the Ramsey theory. There is further related work by Kotlarski, Bigorajska, Ratajczyk, Piekart, and Weiermann [3], [4], [5], [6], [7], [8], [11].

References

[1] Ketonen, J. and Solovay, R., *Rapidly growing Ramsey functions*, Annals of Mathematics **113** (1981), 267–314.

[2] Sommer, R., *Transfinite induction within Peano Arithmetic*, Annals of Pure and Appl. Logic **76** (1995), 231–189.

[3] Bigorajska, T., and H. Kotlarski, *A partition theorem for α-large sets*, Fund. Math. **160** (1999), 27–37.

[4] Bigorajska, T., and H. Kotlarski, *Some combinatorics involving ξ-large sets*, Fund. Math. **175** (2002), pp. 119–125.

[5] Bigorajska, T., and H. Kotlarski, *Partitioning α-large sets: some lower bounds*, Trans. Amer. Math. Soc. **358** (2006), 4981–5001.

[6] Kotlarski, H., and Z. Ratajczyk, *Inductive full satisfaction classes*, Annals of Pure and Appl. Logic **47** (1990), 199–233.

[7] Kotlarski, H., and Z. Ratajczyk, *More on induction in the language with a satisfaction class*, Zeitschr. f. math. Logik und Grundlagen d. Math. **36** (1990), 441–454.

[8] Kotlarski, H., B. Piekart, and A. Weiermann, *More on lower bounds for partitioning α-large sets*, preprint.

[9] Paris, J. B., and L. Harrington, *A mathematical incompleteness in Peano arithmetic*, in: *Handbook of Math. Logic*, North-Holland, 1978, pp. 1133–1142.

[10] Wainer, S. S., *A classification of the ordinal recursive functions*, Archiv für mathematische Logik **13** (1970), 136–153.

[11] Weiermann, A., *A classification of rapidly growing Ramsey functions*, Proc. Amer. Math. Soc. **132** (2004), 553–561.

9 Manuel Lerman

A technique frequently used to separate combinatorial principles is to build an ideal of degrees in which one principle always has a solution, but there is an example of the other principle in the ideal with no solution in the ideal. Usually, the ideals constructed are ad hoc. However, there are natural ideals in the arithmetical degrees.

Problem 9.1. Can natural ideals in the arithmetical degrees be used to construct ideals separating combinatorial principles?

One candidate would be the downward closure of the cappable r.e. degrees.

10 Alberto Marcone

10.1 The maximal linear extension theorem

Recall that a poset $(P, \leq_P)$ is a well-partial-order (wpo) if for every $f : \mathbb{N} \to P$ there exists $i < j$ such that $f(i) \leq_P f(j)$.

It is well-known that a poset $(P, \leq_P)$ is a wpo if and only if all its linear extensions are well-orders (see [CMS04] for an analysis of the logical strength of the equivalence between this and other characterizations of wpo). Here a linear extension of $(P, \leq_P)$ is a linear order $\leq_L$ which satisfies that $x \leq_P y$ implies $x \leq_L y$ for all $x, y \in Q$. If the linear extension is a well-order, its order type is the unique ordinal α isomorphic to it.

Definition 10.1. If $(P, \leq_P)$ is a wpo, the *maximal order type* of $(P, \leq_P)$ is

$$o(P, \leq_P) = \sup\{\alpha \mid \alpha \text{ is the order type of a linear extension of } (P, \leq_P)\}.$$

We often write just $o(P)$.

We are interested in the reverse mathematics strength of the following theorem, due to de Jongh and Parikh ([JP77]), which we denote by MLE (for maximal linear extension).

Theorem 10.2. *The supremum in the definition of maximal order type is actually a maximum, i.e., every wpo $(P, \leq_P)$ has a linear extension of order type $o(P)$.*

MLE is provable in ATR_0. To see this, one can either check that de Jongh and Parikh's original proof goes through in ATR_0, or use the tree of bad sequences

$$\mathbf{Bad}(P, \leq_P) = \{s \in P^{<\omega} \mid \forall i < j < |s|\, s(i) \not\leq_P s(j)\}.$$

Problem 10.3. Does MLE imply ATR_0 over RCA_0 or some other weak base theory?

Montalbán [Mon07] studied the existence of maximal linear extensions from the computability-theoretic perspective. He proved that, although every computable wpo has a computable maximal linear extension, there is no hyperarithmetic way of computing the index of this maximal linear extension from the index of the wpo. This suggests a positive answer to Problem 10.3. However, Montalbán's proof exploits the knowledge of the order type of the maximal linear extensions of some wpos and has no straightforward translation in the reverse mathematics setting.

10.2 "3 is a bqo" and its use

We start by giving the definition of a better quasi-order or bqo (for more details see, e.g., [Mar05]).

Definition 10.4. A set $B \subseteq \mathbb{N}^{<\omega}$ is a barrier if

1. $\mathrm{base}(B) = \{n \in \mathbb{N} \mid \exists s \in B\, \exists i < |s|\, s(i) = n\}$ is infinite;

2. $\forall X \in \mathrm{base}(B)^\omega\, \exists s \in B\ (s = X \restriction |s|)$; and

3. $\forall s, t \in B (s \neq t \implies \exists i < |s| \forall j < |t| s(i) \neq t(j))$.

Definition 10.5. Let $s, t \in \mathbb{N}^{<\omega}$; we write $s \triangleleft t$ if there exists $u \in \mathbb{N}^{<\omega}$ such that $s = u \restriction |s|$ and $t = {}_*u \restriction |t|$ where ${}_*u$ is the sequence gotten from u by deleting its first element..

Definition 10.6. The poset $(P, \leq_P)$ is a bqo if for every barrier B and every map $f : B \to P$, there exist $s, t \in B$ such that $s \triangleleft t$ and $f(s) \leq_P f(t)$.

For $n \in \mathbb{N}$, let $\mathbf{n}$ denote the poset consisting of exactly n pairwise incomparable elements. Using the clopen Ramsey Theorem, it is easy to show that for every fixed n, ATR_0 proves that $\mathbf{n}$ is a bqo.

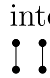
Theorem 10.7 ([Mar05])**.** *1. RCA_0 proves that $\mathbf{2}$ is a bqo; and*

2. for every $n \geq 3$, RCA_0 proves that if $\mathbf{n}$ is bqo then $\mathbf{n+1}$ is a bqo.

Thus the following problem is natural:

Problem 10.8. Does "$\mathbf{3}$ is a bqo" imply ATR_0 over RCA_0 or some other weak base theory?

Problem 10.8 has a graph-theoretic interpretation. We can view $\mathbb{N}^{<\omega}$ as the set of vertices of a graph where s and t are adjacent if $s \triangleleft t$ or $t \triangleleft s$. Then a barrier is identified with the induced subgraph, and the statement "$\mathbf{n}$ is a bqo" amounts to saying that no barrier is n-colorable. Indeed, the RCA_0 proof of "$\mathbf{2}$ is a bqo" consists in showing that every barrier contains a cycle of odd length. The well-known fact that showing that a graph is not 3-colorable is much harder than showing that it is not 2-colorable accounts for the difference between the cases 2 and 3.

A positive answer to Problem 10.8 immediately leads to easy reversals for several theorems in bqo theory. Consider for example the following theorem, which settled a long-standing open problem. (Recall that an interval order is a poset not containing a copy of the partial order with Hasse diagram ; for the reverse mathematics strength of the equivalence between this and other definitions of interval order, see [Mar07]).

Theorem 10.9 ([PS06])**.** *Every interval order which is a wpo is a bqo.*

The proof of Theorem 10.9 can be carried out in ATR_0. Notice that RCA_0 proves that $\mathbf{3}$ is both a wpo and an interval order. Therefore, if Problem 10.8 has a positive answer, the statement of Theorem 10.9 is equivalent to ATR_0.

References

[CMS04] Peter Cholak, Alberto Marcone, and Reed Solomon, *Reverse mathematics and the equivalence of definitions for well and better quasi-orders*, J. Symbolic Logic **69**(3), 683–712, 2004.

[JP77] Dick H. J. de Jongh and Rohit Parikh, *Well-partial orderings and hierarchies*, Nederl. Akad. Wetensch. Proc. Ser. A **80** Indag. Math. 39(3), 195–207, 1977.

[Mar05] Alberto Marcone, *Wqo and bqo theory in subsystems of second order arithmetic*, in: Stephen G. Simpson, editor, *Reverse mathematics 2001*, Lecture Notes in Logic, pp. 303–330, Association for Symbolic Logic, La Jolla, CA, 2005.

[Mar07] Alberto Marcone, *Interval orders and reverse mathematics*, Notre Dame J. Formal Logic **48**(3), 425–448, 2007.

[Mon07] Antonio Montalbán, *Computable linearizations of well-partial-orderings*, Order **24**(1), 39–48, 2007.

[PS06] Maurice Pouzet and Norbert Sauer, *From well-quasi-ordered sets to better-quasi-ordered sets*, Electron. J. Combin. **13**(1), Research Paper 101, 27 pp. (electronic), 2006.

11 Antonio Montalbán

11.1 Fraïssé's conjecture

Let FRA be the statement that the countable linear orderings form a well-quasi-ordering under embeddability. A well-quasi-ordering is a quasi-ordering that has no descending sequences and no infinite antichains. FRA was conjectured by Fraïssé [Fra48] and later proved by Laver [Lav71], using Nash-Williams' notion of a better quasi-ordering [NW68]. The precise reverse mathematics classification of FRA is unknown. Laver's proof of FRA can be carried out in Π^1_2-CA_0, but since FRA is a true Π^1_2 statement, it cannot imply even Π^1_1-CA_0. Shore [Sho93] proved that the restriction of FRA to countable well-orderings is equivalent to ATR_0, and hence FRA implies ATR_0. The following problem remains open.

Problem 11.1. Is FRA equivalent to ATR_0?

Clote conjectured in [Clo90] that the answer is affirmative. Marcone and Montalbán have later done extensive work on this problem [Mar05, Mon07]. They now believe answering the following problem will lead to a solution. Given an ordinal α, let $\mathbb{L}_\alpha$ be the partial ordering obtained by considering the class of linear orderings of Hausdorff rank less than α modulo the relation of equimorphism (bi-embeddability), and ordered by embeddability. It is not hard to show that FRA is equivalent to the statement "for every ordinal α, $\mathbb{L}_\alpha$ is well-founded".

Problem 11.2. Given an ordinal α, what is the well-founded-rank of $\mathbb{L}_\alpha$?

Marcone and Montalbán [MM] have shown that the well-founded rank of $\mathbb{L}_{\omega+1}$ is $\epsilon_{\epsilon_{\epsilon_{\dots}}}$, the first fixed point of the epsilon ordinal function, getting results about the strength of FRA restricted to linear orderings of finite Hausdorff rank. (Marcone and Montalbán actually calculated the length of the well-partial-ordering $\mathbb{L}_\omega$, which can be shown to be the same as the well-founded-rank of $\mathbb{L}_{\omega+1}$.)

11.2 Theories of Hyperarithmetic Analysis

We say that a theory T is a *theory of hyperarithmetic analysis* if all its ω-models are closed under hyperarithmetic reduction and for every $Y \subseteq \omega$, $HYP(Y) \models T$. We say that a statement S is *of hyperarithmetic analysis* if RCA_0+S is a theory of hyperarithmetic analysis. Many theories of hyperarithmetic analysis have been studied. Here are the main examples, from stronger to weaker:

$$\Sigma^1_1\text{-dependent choices} \Rightarrow \Sigma^1_1\text{-choice} \Rightarrow \Pi^1_1\text{-separation} \Rightarrow$$
$$\Delta^1_1\text{-}\mathsf{CA}_0 \Rightarrow \text{Jullien's Indecomposablity Theorem} \Rightarrow$$
$$\text{weak-}\Sigma^1_1\text{-choice} \Rightarrow \text{The jump iteration statement}$$

So far, only one natural mathematical theorem has been found at this level, namely, Jullien's theorem on indecomposable linear orderings [Mon06, Nee]. All the other theories at this level use concepts from logic. A general problem would be the following.

Problem 11.3. Are there other natural theorems of mathematics, outside logic, which are statements of hyperarithmetic analysis?

In [Mon06], Montalbán introduced four statements about finitely terminating games that are also statements of hyperarithmetic analysis. Whether they are natural mathematical theorems is arguable. A finitely terminating game $G(T)$ is given by a well-founded tree $T \subseteq \omega^{<\omega}$, where the players take turns playing natural numbers $x_0, x_1, x_2 ....$, and the first one who leaves the tree, i.e., $(x_0, ..., x_n) \notin T$, loses. Given a countable sequence of trees $\{T_n\}_{n\in\omega}$, we consider the game $\sum_n G(T_n)$ where the first player starts by choosing $n \in \omega$, than then they continue playing $G(T_n)$. We say that a game is *determined* if one of the players has a winning strategy as usual. We say that a game $G(T)$ is *completely determined* if there is a function that for each node $\sigma \in T$ tells us which player has a winning strategy if we started playing from σ. Note that from such a function we could easily build a winning strategy for whichever player has it. Let DG-AC_0 be the statement that says that if $\{G(T_n)\}_{n\in\omega}$ is a sequence of determined finitely terminating games, then so is $\sum G(T_n)$. Let CDG-AC_0 be the statement that if $\{G(T_n)\}_{n\in\omega}$ is a sequence of completely determined finitely terminating games, then so is $\sum G(T_n)$. It is shown in [Mon06] that

$$\begin{array}{rcccl} \Sigma^1_1\text{-}\mathsf{AC}_0 & \rightarrow & DG\text{-}\mathsf{AC}_0 & \rightarrow & \Delta^1_1\text{-}\mathsf{AC}_0 \\ \text{weak-}\Sigma^1_1-\mathsf{AC}_0 & \rightarrow & CDG\text{-}\mathsf{AC}_0 & \Rightarrow & \mathsf{JI}, \end{array}$$

where the arrow $\rightarrow$ means that the reverse is unknown and the arrow $\Rightarrow$ means that the reverse is known to be false.

Problem 11.4. Is DG-AC_0 equivalent to Σ^1_1-AC_0?

Problem 11.5. Is CDG-AC_0 equivalent to weak-Σ^1_1-AC_0?

References

[Clo90] Peter Clote, *The metamathematics of Fraïssé's order type conjecture*, in: *Recursion theory week* (Oberwolfach, 1989), volume 1432 of *Lecture Notes in Math.*, pp. 41–56. Springer, Berlin, 1990.

[Fra48] Roland Fraïssé, *Sur la comparaison des types d'ordres*, Comptes rendus de l'Académie des sciences de Paris **226**, 1330–1331, 1948.

[Lav71] Richard Laver, *On Fraïssé's order type conjecture*, Annals of Mathematics (2) **93**, 89–111, 1971.

[Mar05] Alberto Marcone, *Wqo and bqo theory in subsystems of second order arithmetic*, in: *Reverse mathematics 2001*, volume 21 of *Lect. Notes Log.*, pp. 303–330, Assoc. Symbol. Logic, La Jolla, CA, 2005.

[MM] Alberto Marcone and Antonio Montalbán, *On Fraïssé's conjecture for linear orders of finite Hausdorff rank*, Annals of Pure and Applied Logic, to appear.

[Mon06] Antonio Montalbán, *Indecomposable linear orderings and hyperarithmetic analysis*, Journal of Mathematical Logic **6**(1), 89–120, 2006.

[Mon07] Antonio Montalbán, *On the equimorphism types of linear orderings*, Bulletin of Symbolic Logic **13**(1), 71–99, 2007.

[Nee] Itay Neeman, *The strength of jullien's indecomposability theorem*, submitted for publication.

[NW68] C. St. J. A. Nash-Williams, *On better-quasi-ordering transfinite sequences*, Proc. Cambridge Philos. Soc. **64**, 273–290, 1968.

[Sho93] Richard A. Shore, *On the strength of Fraïssé's conjecture*, in: *Logical methods (Ithaca, NY, 1992)*, volume 12 of *Progr. Comput. Sci. Appl. Logic*, pp. 782–813, Birkhäuser Boston, Boston, MA, 1993.

12 N. W. Sauer

12.1 Age and weak indivisibility

Let $\mathcal{R}$ be a relational structure with base set R. The *skeleton* of $\mathcal{R}$ is the set of finite induced substructures of $\mathcal{R}$. The structure $\mathcal{R}$ is *homogeneous* if every isomorphism between elements of the skeleton of $\mathcal{R}$ has an extension to an automorphism of $\mathcal{R}$. The *age* of $\mathcal{R}$ is the class of finite relational structures isomorphic to an element of the skeleton of $\mathcal{R}$. The subset S of R is *age complete* if the age of the restriction of $\mathcal{R}$ to S is equal to the age of $\mathcal{R}$. For a general introduction to homogeneous structures see [3].

The structure $\mathcal{R}$ is *age indivisible* if for every partition (A, B) of R at least one of A and B is age complete. The structure $\mathcal{R}$ is *weakly indivisible* if for every partition (A, B) of R for which A is not age complete, there is an embedding of $\mathcal{R}$ into B. Clearly, age indivisibility implies weak indivisibility.

Problem 12.1. Let $\mathcal{R}$ be homogeneous and R countable and the relational language of $\mathcal{R}$ be finite. Does then age indivisibility imply weak indivisibility?

For a counterexample in the case of infinitely many binary relations see [5]. For other discussions see [6] and [1].

An *age* is a class of finite relational structures which is updirected, closed under induced substructures and under isomorphic images. The age is countable if the set of isomorphism equivalence classes is countable. An age $\mathfrak{A}$ is *representable* if there is a relational structure $\mathcal{R}$ whose age is $\mathfrak{A}$.

A theorem of Fraïssé, see [2], says that if $\mathfrak{A}$ is a countable age then it is representable. If the age $\mathfrak{A}$ is not countable, then there need not be a relational structure $\mathcal{R}$ whose age is $\mathfrak{A}$. For examples and further discussion see [7].

Problem 12.2. Find a characterization of the uncountable representable ages.

References

[1] N. Sauer, *Age and weak indivisibility*, European Journal of Combinatorics, submitted (link to arxiv article will be established soon).

[2] R. Fraïssé, *Theory of relations*, Studies in Logic and the Foundations of Mathematics **145**(2000), North-Holland Publishing Co., Amsterdam.

[3] N. Sauer, *Appendix in: Theory of Relations, by R. Fraïssé, Revised Edition*, in: *Studies in Logic and the Foundations of Mathematics* **1** 45 (2000), North-Holland Publishing Co., Amsterdam.

[4] C. Delhommé and C. Laflamme and M. Pouzet and Norbert Sauer, *Indivisible Ultrametric Spaces*, Topology and its Applications, special issue, "Workshop on the Urysohn space" May 21-24, 2006, Ben-Gurion University of the Negev, Beer Sheva, Israel.

[5] L. Nguyen Van Thé and N. Sauer, *Some weak indivisibility results in ultrahomogeneous metric spaces*, manuscript submitted (link to arxiv from my web page at the university of Calgary).

[6] M. Pouzet and C. Laflamme and L. Nguyen Van Thé, *Partitions of countable infinite-dimensional vector and affine spaces*, manuscript (link to arxiv article from my web page will be established soon).

[7] C. Delhommé, M. Pouzet, G. Sági, N. Sauer, *Representation of ideals of relational structures*, Discrete Mathematics, in press.

13 Richard Shore

Reverse Mathematics asks how hard is it to prove the theorems of classical (countable) mathematics in terms of what set existence axioms are needed to carry out the proof. The theorems analyzed are typically Π^1_2 assertions, for every structure of some sort there is a function or relation with some desired property, but arbitrary statements can be so analyzed. Quite often analyses proceed by recursion (computability) theoretic methods. In the positive direction, one shows that the desired function or relation is computable or computable from some type of jump operator (Turing, or iterations into the transfinite all the way to the hyperjump). Such proofs generally provide ones in the analogous system of Reverse Mathematics (at times with more induction needed than the usual minimum of Σ^0_1). In the other direction, one often demonstrates that one principle or mathematical assertion Φ does not follow from another Ψ by providing an ideal in the Turing degrees (i.e., a collection of sets or degrees closed under Turing reducibility and join and perhaps the jump operator relevant to the discussion) such that Ψ holds in (the model of second order arithmetic consisting of the sets in) the ideal but Φ does not. This, of course, proves that Ψ does not imply Φ over RCA_0 (or over the system (ACA_0, ATR_0 or Π^1_1-CA_0) corresponding to the jump closure condition). It actually provides a stronger independence result that, for example, applies to the base systems with full induction and more.

We propose a direct formulation of this computability theoretic measure based on the difficulty of computing the desired output (function) from the input (as in the typical case of Π^1_2 theorems). Making this view explicit formalizes the intuition of "being harder to prove" meaning that it is harder to compute the sets that the theorem asserts exist. It also provides a different expository route into the subject suitable for a mathematical or computer science audience that intuitively understands computability but may find formal proof systems foreign or less appealing. More interestingly, for the practitioners already familiar with this approach, it provides an opportunity to deal with uncountable structures and higher order statements that are out of reach of the standard proof theoretic methods. The route here is to use one (or more) of the studied definitions of computability on uncountable structures.

Definition 13.1. If $\mathcal{C}$ is a *closed class* of sets, i.e., closed under Turing reducibility and join, we say that $\mathcal{C}$ *computably satisfies* Ψ (a sentence of second order arithmetic) if Ψ is true in the standard model of arithmetic whose second order part consists of the sets with degrees in $\mathcal{C}$. We say that Ψ *computably entails* Φ, $\Psi \vDash_c \Phi$, if every closed $\mathcal{C}$ satisfying Ψ also satisfies Φ. We say that Ψ and Φ are *computably equivalent* if each computably entails the other.

One can now express the equivalence of some Ψ with, e.g., ACA_0, ATR_0 or Π^1_1-CA in this way. One can also describe entailment or equivalence over one of these systems by either adding them on to the sentences Ψ and Φ or by requiring that the classes $\mathcal{C}$ be closed under the appropriate operators and reductions (Turing jump, hyperarithmetic in and hyperjump, respectively). More interestingly, one can directly express the relationships between two mathematical statements without going through any formal proof systems.

Turning now to uncountable structures, one can simply interpret computability as some version of generalized computability and then immediately have notions appropriate to uncountable settings. For example, if one is interested in algebraic or combinatorial structures where the usual mathematical setting assumes that an uncountable structure is given with its cardinality, i.e., the underlying set for the structure (vector space, field, graph, etc.) may as well be taken to be a cardinal κ, then a plausible notion of computation is given by α-recursion theory. In this setting, one carries out basic computations (including an infinitary sup operation) for α (or, in our situation, κ) many steps, assuming some closure properties such as admissibility on α. (Note that every infinite cardinal is admissible.)

For settings such as analysis, where the basic underlying set is the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$, it seems less natural to assume one has a well-ordering of the structure and one wants a different model of computation. Natural possibilities include Kleene recursion in higher types, E-recursion (of Normann and Moschovakis) and Blum-Shub-Smale computability. (See for example [8] or [3] for α-recursion theory; [8], [7] or [4] for the various versions of recursion in higher types or E-recursion; and [2] for the Blum-Shub-Smale model.)

Problem 13.2. Develop a computability theoretic type of reverse mathematical analysis of mathematical theorems on uncountable structures using whichever generalized notion of computability seems appropriate to the subject being analyzed.

Note that the formulation of the basic yardsticks for this analysis will not, in general be the same as for the countable case. An obvious example is Weak König's Lemma. For uncountable cardinals κ, the assertion that every binary tree of size κ (or even just quite simple ones) has a size κ branch is equivalent to κ being weakly compact. Thus such a principle is not even a candidate yardstick for most cardinals. On the other hand, there are natural candidates for analogs of ACA_0 once one has the right notion for the jump operator or enough closure to make sense of closing under first order definability (as over some L_κ). We have worked out [9] a few standard examples in the setting of α-recursion theory for arbitrary cardinals κ inside L.

Theorem 13.3. *The existence of bases for vector spaces of size κ over fields of size κ is computably equivalent (in the sense of α-recursion theory) to closure under κ-jump (suitably defined) or under first order comprehension (over L_κ).*

For WKL_0, the appropriate tree formulation seems to be the following:

Definition 13.4. A binary tree T on a cardinal κ (i.e., a subset of $2^{<\kappa}$ closed downward under initial segments) is *of finite character* if T is continuous at limit levels, i.e., for any $\gamma \in 2^{<\kappa}$ of length a limit ordinal λ, if $\gamma \upharpoonright \delta \in T$ for every $\delta < \lambda$ then $\gamma \in T$ and for every $\sigma \in T$, if there is a $\gamma > |\sigma|$ such that σ has no successors on T at level γ then there is a $\tau \subseteq \sigma$ of length a successor ordinal such that τ has no successors on T of length γ.

Definition 13.5. The *finite character tree property for a cardinal κ, $FCTP_\kappa$*, says that every binary tree T on κ of finite character has a path of length κ.

Theorem 13.6. *The following are computably equivalent in the sense of α-recursion theory for each cardinal κ:*

1. *$FCTP_\kappa$*
2. *Compactness for first order logic for languages and theories of size κ.*
3. *Every commutative ring of size κ has a prime ideal.*

A natural candidate for the analog of Π^1_1-CA_0 is, of course, closure under definability by formulas with a single second order quantifier. We have not yet looked for any equivalences at this level.

Problem 13.7. What mathematical theorems are computably equivalent (in the sense of α-recursion theory) to closure under definability over L_κ by formulas with a single second order quantifier.

It is not clear what the appropriate basic yardsticks corresponding to ATR_0 should be. A candidate for analysis here is König's Duality Theorem (KDT), which is equivalent to ATR_0 [1],[10]. The arguments of [1] show that it is strictly stronger than closure under first order definability for every κ.

Problem 13.8. What is the right standard in α-recursion theory that corresponds to ATR_0, and is it equivalent to KDT for every κ? What about comparability of well-orderings (of subsets of κ)? (Note that if $\mathrm{cf}(\kappa) > \omega$ then being well-founded is a co-κ-r.e. relation, so the situation for well-orderings is quite different than in the countable case.)

Turning to analysis and related subjects about $\mathbb{R}$, we just note that an old result of Grilliot [5] can be seen from our point of view as showing that the existence of a noncontinuous functional is computably equivalent to the existence of ${}^2E$. In this setting, there are also proof theoretic approaches that correspond to Kleene recursion in higher types as the classical proof theoretic systems do to Turing computability (see [6]).

Problem 13.9. Analyze the classical theorems of analysis in terms of computable entailment and equivalence with computation taken to be Kleene Recursion in higher types or Blum-Shub-Smale computation.

References

[1] Aharoni, R., Magidor, M., and Shore, R. A., *On the strength of König's duality theorem for infinite bipartite graphs*, J. Combin. Theory *Ser. B* **54** (1992) 257–290.

[2] Blum, Lenore, Cucker, Felipe, Shub, Michael and Smale, Steve, *Complexity and Real Computation*, Springer-Verlag, New York, 1998.

[3] Chong, C. T., *Techniques of Admissible Recursion Theory*, Lecture Notes in Mathematics **1106**, Springer-Verlag, Berlin, 1984.

[4] Fenstad, Jens Erik, *General Recursion Theory: An Axiomatic Approach*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin-New York, 1980.

[5] Grilliot, T. J., *On effectively discontinuous type-*2 *objects*, J. Symbolic Logic **36** (1971) 245–248.

[6] Kohlenbach, U., *Higher order reverse mathematics*, in: *Reverse mathematics 2001*, S. Simpson ed., 281–295, Lect. Notes in Logic **21**, Assoc. Symbol. Logic and A.K. Peters, Wellesley, MA, 2005.

[7] Moldestad, J., *Computations in Higher Types*, Lecture Note in Mathematics **574**, Springer-Verlag, Berlin, 1977.

[8] Sacks, Gerald E., *Higher recursion theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1990.

[9] Shore, R. A., *Reverse mathematics of the uncountable: a computational approach*, in preparation.

[10] Simpson, S., *On the strength of König's duality theorem for countable bipartite graphs*, J. Symbolic Logic **59** (1994) 113–123.

14 Stephen G. Simpson

14.1 The Lebesgue Differentiation Theorem

The problem is to determine the computable analysis and/or reverse mathematics status of the Lebesgue Differentiation Theorem, LDT.

The textbook statement of LDT reads as follows.

> Let f be a Lebesgue measurable real-valued function on the d-dimensional unit cube, $[0,1]^d$. Then for almost all $x \in [0,1]^d$ we have
>
> $$f(x) = \lim_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)}$$
>
> where the limit is taken over all d-dimensional cubes $Q \ni x$ as the diameter tends to 0. Here μ is Lebesgue measure on $[0,1]^d$.

We say that f is *L_1-computable* if there exists a computable sequence f_n, $n = 0, 1, 2, \ldots$, of polynomials with rational coefficients such that $\|f - f_n\|_1 \leq 1/2^n$ for all n. Here $\|f\|_1 = \int_{[0,1]^d} |f| d\mu$. My student Noopur Pathak [3] has proved that if x is Martin-Löf random then LDT holds at x for all L_1-computable f. It is open whether the converse holds. Instead of Martin-Löf randomness, I am now thinking that weaker notions of randomness may be relevant here, e.g., Schnorr randomness.

Problem 14.1. Let x be a point in $[0,1]^d$. If LDT holds at x for all L_1-computable f, does x have to be random in the sense of Martin-Löf?

In reverse mathematics terms, WWKL_0 proves LDT, and it is an open question whether the reversal holds. Here WWKL_0 is Weak Weak König's Lemma, a subsystem of Z_2 which is strictly between RCA_0 and WKL_0 and which has been very useful in the reverse mathematics of measure theory. See, for instance, [4, 5, §X.1].

Problem 14.2. Is LDT equivalent to WWKL_0 over RCA_0?

14.2 The Density Theorem for Muchnik Degrees

The problem is to prove or disprove the Muchnik degree analog of the Sacks Density Theorem.

Background: If P and Q are sets of reals, P is said to be *Muchnik reducible* to Q if for all $y \in Q$ there exists $x \in P$ such that $x \leq_T y$. The *Muchnik degrees* are the equivalence classes of sets of reals under mutual Muchnik reducibility, partially ordered by Muchnik reducibility. The Muchnik degrees form a complete distributive lattice, denoted $\mathcal{D}_w$. The study of $\mathcal{D}_w$ was originally motivated by intuitionistic considerations going back to Kolmogorov. We define $\mathcal{P}_w$ to be the sublattice of $\mathcal{D}_w$ consisting of the Muchnik degrees of nonempty Π^0_1 sets of reals.

Note that $\mathcal{P}_w$ is the Muchnik degree analog of the recursively enumerable Turing degrees. See also [7], where it is shown that the recursively enumerable Turing degrees are naturally embedded in $\mathcal{P}_w$. However, $\mathcal{P}_w$ is much better than the recursively enumerable Turing degrees, because $\mathcal{P}_w$ contains many specific, natural degrees which are closely related to foundationally interesting topics. Among these topics are: algorithmic randomness, diagonal nonrecursiveness, almost everywhere domination, Kolmogorov complexity, effective Hausdorff dimension, and hyperarithmeticity. See the references below.

The Sacks Density Theorem says that given a pair of recursively enumerable Turing degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} < \mathbf{b}$, we can find a recursively enumerable Turing degree $\mathbf{c}$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$. The problem that we are posing is to prove or disprove the same statement with the recursively enumerable Turing degrees replaced by $\mathcal{P}_w$.

Problem 14.3. Prove or disprove the following conjecture. Given $\mathbf{a}, \mathbf{b} \in \mathcal{P}_w$ such that $\mathbf{a} < \mathbf{b}$, we can find $\mathbf{c} \in \mathcal{P}_w$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.

References

[1] Stephen Binns and Stephen G. Simpson, *Embeddings into the Medvedev and Muchnik lattices of* Π^0_1 *classes*, Archive for Mathematical Logic **43** 2004, 399–414.

[2] Joshua A. Cole and Stephen G. Simpson, *Mass problems and hyperarithmeticity*, Journal of Mathematical Logic **7** 2008, 125–143.

[3] Noopur Pathak, *A computational aspect of the Lebesgue Differentiation Theorem*, 12 pages, 2008, accepted for publication in Journal of Logic and Analysis.

[4] Stephen G. Simpson, *Subsystems of Second Order Arithmetic*, Springer-Verlag, 1999, xiv + 445 pages.

[5] Stephen G. Simpson, *Subsystems of Second Order Arithmetic*, 2nd edition, Association for Symbolic Logic, Cambridge University Press, 2009, to appear.

[6] Stephen G. Simpson, *Mass problems and randomness*, Bulletin of Symbolic Logic **11** 2005, 1–27.

[7] Stephen G. Simpson, *An extension of the recursively enumerable Turing degrees*, Journal of the London Mathematical Society **75** 2007, 287–297.

[8] Stephen G. Simpson, *Some fundamental issues concerning degrees of unsolvability*, in: *Computational Prospects of Infinity, Part II: Presented Talks*, edited by C.-T. Chong, Q. Feng, T. Slaman, H. Woodin, and Y. Yang, Lecture Notes Series, Number 15, Institute for Mathematical Sciences, National University of Singapore, World Scientific, 2008, 313–332.

[9] Stephen G. Simpson, *Mass problems and almost everywhere domination*, Mathematical Logic Quarterly **53** 2007, 483–492.

[10] Stephen G. Simpson, *Mass problems and intuitionism*, Notre Dame Journal of Formal Logic **49** 2008, 127–136.

15 Henry Towsner

Let c be a finite coloring of the integers. Take F to be the collection of finite sets of integers all of whose finite sums are monochromatic under c, and order F by reverse inclusion. (That is, $s \prec t$ iff $t \subseteq s$.) Hindman's Theorem is essentially the statement that F is not well-ordered by $\prec$. We define the *length* of c, $o(c)$, to be the order type of the well-founded part of $\prec$. This is precisely the order type of the tree of finite partial witnesses to Hindman's Theorem which cannot be extended to full witnesses.

Recent results suggest that the order type of c is related to the difficulty of proving that Hindman's Theorem holds for c, and in particular that a c which has no arithmetic solution to Hindman's Theorem is likely to require at least length ω^2. By contrast, the canonical example that Hindman's Theorem implies ACA_0 over RCA_0 has order type at most $\omega + \omega$. We can ask:

Problem 15.1. What is an example of a coloring c with large length?

Problem 15.2. What is the supremum of the order type for recursive/arithmetic/arbitrary colorings of the integers?

The most interesting examples would have the additional property of being homogeneously long, in the sense that the restriction of the coloring to any infinite set is either monochromatic except on finitely many elements, or is also long.

16 Andreas Weiermann

16.1 Maximal order types for well partial orders

According to de Jongh and Parikh [3] there exists for any well partial order a linear (and automatically well-ordered) extension on the same set having a maximal possible order type. A very general problem is to establish a sort of rule of thumb formula for calculating the maximal order type of a concretely given well partial order. An even more general problem is to explain the connection between concrete well orders and concrete well partial orders in general. To this end we offer several informal principles and it would be an interesting problem to see how far they would lead. Our first suggestion concerns a formula (which has been developed in joint research with M. Rathjen) for computing maximal order types of tree embeddability relations and the long term hope is to give an analysis of Friedman's extended Kruskal theorem [11] (of even Kríž's theorem [7]) by it.

To explain this formula informally let us consider a given explicit operator W which maps a (countable) wpo X to a (countable) wpo $W(X)$ so that the elements of $W(X)$ can be described as generalized terms in which the variables are replaced by constants for the elements of X. We assume that the ordering between elements of $W(X)$ is induced effectively by the ordering from X. (This resembles Feferman's notion of effective relative categoricity.) In concrete situations W may for example stand for an iterated application of basic constructions like disjoint union and Cartesian product, the set of finite sequences construction, the multiset construction, or a tree constructor and the like. We assume that for W we have an explicit knowledge of $o(W(X))$ such that $o(W(X)) = o(W(o(X)))$ and such that this equality can be proved using an effective reification as in [8].

Using W we then build the set of W-constructor trees $T(W(Rec))$ (similarly as in [5]) as follows:

1. $\cdot \in T(W(Rec))$.
2. If (s_i) is a sequence of elements in $T(W(Rec))$ and $w((x_i))$ is a term from $W(X)$ then $\cdot(w((s_i))) \in T(W(Rec))$.

The embeddability relation $\trianglelefteq$ on $T(W(Rec))$ is defined recursively as follows:

1. $\cdot \trianglelefteq t$.
2. If $s \trianglelefteq t_i$ then $s \trianglelefteq \cdot(w((t_i)))$.
3. If $w((s_i)) \leq w'((t_j))$ mod $W(T(W(Rec)))$ is induced recursively by $\trianglelefteq$ then $\cdot(w((s_i))) \trianglelefteq \cdot(w'((t_j)))$.

The general principle now is that

$$T(W(Rec)) \text{ is a wpo}$$

(cf. [5]) and

$$o(\langle T(W(Rec)), \trianglelefteq \rangle) \leq \vartheta o(W(\Omega)) \tag{1}$$

for $o(W(\Omega)) \in dom(\vartheta)$ with $o(W(\Omega)) \geq \Omega^3$. [Moreover the reverse inequality should also hold in most cases.]

To understand this formula some additional information on ordinals and in particular about a so called *collapsing function* is required (cf., e.g., [13]). Let Ω denote the first uncountable ordinal and $\varepsilon_{\Omega+1}$ the first epsilon number above Ω. Then any ordinal $\alpha < \varepsilon_{\Omega+1}$ can be described uniquely in terms of its Cantor normal form:

$$\alpha = \Omega^{\alpha_1} \cdot \beta + \cdots + \Omega^{\alpha_n} \cdot \beta_n$$

where $\alpha_1 > \ldots > \alpha_n$ and $0 < \beta_1, \ldots, \beta_n < \Omega$. In this situation we define the countable subterms $K\alpha$ of α recursively via

$$K\alpha := K\alpha_1 \cup \ldots \cup K\alpha_n \cup \{\beta_1, \ldots, \beta_n\}$$

where $K0 := 0$. Let $AP = \{\omega^\delta : \delta \in ON\}$ be the class of additive principal numbers. With $K\alpha < \beta$ we abbreviate $\forall \xi \in K\alpha \xi < \beta$. We can then put

$$\vartheta\alpha := \min\{\beta \in AP : \beta \geq \max K\alpha \wedge \forall \gamma < \alpha (K\gamma < \beta \rightarrow \vartheta\gamma < \beta\}. \quad (2)$$

One easily verifies $\vartheta\alpha < \Omega$ by induction on α using a cardinality argument. It is easy to verify that then $\varepsilon_0 = \vartheta\Omega$ and $\Gamma_0 = \vartheta\Omega^2$.

To investigate the maximal order types of Friedman style Kruskal theorems (which rely on a so called gap condition) one has to extend the domain of ϑ to intrinsically larger domains but this is rather easy (cf., e.g., [13]). In a first step one defines a function $\vartheta_1 : \varepsilon_{\Omega_2+1} \rightarrow [\Omega, \Omega_2[$ in the same way as $\vartheta : \varepsilon_{\Omega+1} \rightarrow \Omega$ was defined previously. Here Ω_2 denotes the second uncountable cardinal and ε_{Ω_2+1} the next epsilon number above Ω_2. On the segment determined by $\vartheta_1\varepsilon_{\Omega_2+1}$ on can define the countable coefficient sets $K\alpha$ similarly as before using $K\vartheta_1\alpha = K\alpha$. Using this one can then define $\vartheta : \vartheta_1\varepsilon_{\Omega_2+1} \rightarrow \Omega$ by (2). This process can be iterated through the first ω number classes to provide a function $\vartheta : \vartheta_1 \ldots \vartheta_n\varepsilon_{\Omega_{n+1}} \rightarrow \Omega$ giving an end-extension of the previously defined versions of ϑ [13]. The limiting value of $\vartheta\vartheta_1 \ldots \vartheta_n\varepsilon_{\Omega_{n+1}}$ as $n \rightarrow \infty$ is known to be the ordinal related to the union of Friedman's assertions FKT_n which rely on an embeddability relation satisfying a gap condition.

The idea is now to approximate Friedman's FKT_n via the well partial orders

$$\underbrace{T(\ldots(T}_{n\text{-times}}(Rec^*))\ldots) \quad (3)$$

for which our formula would predict a maximal order type

$$\vartheta(\vartheta_1(\ldots\vartheta_{n-1}(\Omega_n^\omega)\ldots)). \quad (4)$$

Here we work with an operator W which is defined by iteration. Let X^* be the Higman ordering on the set of finite sequences of elements from a poset X. Then the partial order in (3) is $T(W(Rec))$ for the operator $W(X) := \underbrace{T(\ldots(T}_{n-1\text{-times}}(X^*))$. (We intend to verify prediction (4) in a joint project with M. Rathjen.)

Problem 16.1. How far does the general formula (1) lead? Are there natural situations in which it fails?

Problem 16.2. Assume that W as before is a natural operator mapping (countable) wpo's to (countable) wpo's and assume that $o(W(\Omega) \geq \Omega^3$. In what generality does the following principle hold? Principle: The proof-theoretic ordinal of $\mathsf{RCA}_0 + \forall X(\mathsf{WPO}(X) \rightarrow \mathsf{WPO}(T(W(X)))$ is equal to $\vartheta o(W(\Omega))$?

Problem 16.3. Assume that W as before is a natural operator mapping (countable) wpo's to (countable) wpo's and assume that $o(W(\Omega) \geq \Omega^3$. In what generality does the following principle hold? Principle: $\mathsf{RCA}_0 \vdash \mathsf{WPO}(T(W(Rec))) \leftrightarrow \mathsf{WO}(\vartheta(o(W(\Omega))))$?

From a more general perspective we believe that a general connection between well orders and well partial orders might be as follows. Assume that $(X, \leq)$ is a recursive well order represented by a proof-theoretic ordinal notation system. Assume that $\trianglelefteq$ is a restriction of $\leq$ on X such that $x \trianglelefteq x'$ holds if $x \leq x'$ holds (hereditarily) due to a graph-theoretic reason, like a subterm or monotonicity property. Then the rule of thumb principle would suggest that $(X, \trianglelefteq)$ is a wpo having maximal order type given by the order type of $\leq$.

A very interesting test case is provided by Kríž's theorem.

I. Kríž proved in [7] the following wqo/wpo result generalizing a result by H. Friedman [11]: Let E_α be the class of all rooted trees whose edges are labelled by ordinals below a given ordinal α. Given T in E_α and vertices x and y of T, agree that $x \leq y$ if and only if x is on the path between y and the root. Given trees T and S in E_α, agree that $S \leq T$ if and only if there is an order- and inf-preserving embedding φ of the vertices of S into the vertices of T satisfying the following "gap-condition": If x, y is an edge of S and α is the ordinal labelling x, y, then the labels of all edges on the path in T from $\varphi(x)$ to $\varphi(y)$ are at least as large as α .

Theorem 16.4 (Kríž). *If $\langle T_n \rangle_{n=1}^{\infty}$ is a sequence in E_α then for some $n < m$ one has $T_n \leq T_m$ (that is, E_α is well-quasiordered).*

Gordeev proved 1993 [4] that the assertion

$$\forall X(\mathsf{WO}(X) \rightarrow \mathsf{WQO}(E_X)) \tag{5}$$

is provable in $\Pi^1_1 - \mathsf{TR}_0$.

Problem 16.5. How strong is $\mathsf{WQO}(E_\alpha)$ for a given fixed (infinite) proof-theoretic ordinal α? Is the maximal order type of E_α roughly equal to $\vartheta \Omega_\alpha^\omega$ when the domain of ϑ is suitably extended a la [10]?

Problem 16.6. Do there exist natural wpo assertions which are proof-theoretically stronger than (5), i.e. Kríž's theorem?

To obtain $\mathsf{WQO}/\mathsf{WPO}$-statements of larger proof-theoretic complexity a natural idea coming from strong ordinal notation systems would be to replace the ordinal labels by (previously generated) trees. To make this idea work one first would need a natural version of Kríž's result where the labels are stemming from a wqo/wpo.

Problem 16.7. Does such a version of (5) exist which in a later generalization would allow for an insertion of trees for the labels and which still would preserve wqo/wpo'ness.

We expect that this problem is rather difficult. A positive answer will presumably shed light on the notoriously difficult natural well-ordering problem of Feferman.

16.2 Parameter free comprehension in a weak environment

Let $(\Pi^1_1 - \mathsf{CA})$ denote lightface Π^1_1-comprehension. This principle is useful to carry out minimal bad sequence arguments. In a joint investigation with M. Rathjen we investigate $(\Pi^1_1 - \mathsf{CA})$ in the context of not too strong theories from reverse mathematics. A particularly interesting problem concerns Simpson's theory $\mathsf{RCA}_0^\star$ [11] which is RCA_0 where Σ^0_1 induction is replaced by Σ^0_0 induction plus additional axioms expressing the totality of the exponential function. Moreover let φ denote the binary Veblen function [10].

Problem 16.8. Is it true that $(\Pi^1_1 - \mathsf{CA})^- + \mathsf{RCA}_0^* \nvdash \mathsf{WO}(\varphi\omega 0)$? (We know that $(\Pi^1_1 - \mathsf{CA})^- + \mathsf{RCA}_0^* \vdash \mathsf{WO}(\alpha)$ for all $\alpha < \varphi\omega 0$.) If the answer to this question is no: Is it true that $(\Pi^1_1 - \mathsf{CA})^- + (\Delta^0_0 - \mathsf{CA}) \nvdash \mathsf{WO}(\varphi\omega 0)$? This problem is interesting since, if the answer would be yes, then $(\Pi^1_1 - \mathsf{CA})^-$, which is commonly considered to be the prototype of an impredicative comprehension, will have a predicative interpretation in a weak context.

16.3 How strong is Voronin universality in reverse mathematics?

Problem 16.9. How strong is Voronin's universality theorem about the Riemann ζ-function with respect to reverse mathematics? How strong are the universality theorems proved later for variants of the Riemann zeta function (confer, e.g., the corresponding results proved in [6]).

Voronin's theorem [12, 6] is as follows: Let $0 < r < \frac{1}{4}$; let $f(s)$ be a function that is analytic inside the disc $|s| \leq r$ and continuous up to the boundary of the disc; if $f(s)$ has no zeros inside the disc $|s| \leq r$, then for every $\varepsilon > 0$ there exists a real number $T = T(\varepsilon)$ such that $max_{|s| \leq r} |f(s) - \zeta(s + (\frac{3}{4} + iT))| < \varepsilon$.

This theorem has proven useful in joint work with A. Bovykin on independence results for arithmetic [2] and the intuition is that Voronin's result carries reasonable strength.

16.4 How strong is the Erdös Moser principle in reverse mathematics?

A tournament is a complete directed simple graph. Let CAC be the statement that every infinite partial order has an infinite subchain and let EM be the statement that every infinite tournament contains an infinite transitive subtournament.(A finitary analogue of EM has been investigated by Erdös and Moser.) Then RT^2_2 proves CAC and EM. But we also have a reversal: $\mathsf{RCA}_0 + \mathsf{EM} + \mathsf{CAC} \vdash \mathsf{RT}^2_2$. The problem (which emerges from joint work with A. Bovykin [1]) is:

Problem 16.10. How strong is EM in the context of reverse mathematics? Is $\mathsf{RCA}_0 + \mathsf{EM}$ strictly weaker than $\mathsf{RCA}_0 + \mathsf{RT}^2_2$?

(There exists a recursive tournament without an infinite recursive transitive subtournament and so EM is not provable in RCA_0 alone.)

References

[1] A. Bovykin and A. Weiermann, *The strength of infinitary Ramseyan principles can be accessed by their densities*, preprint 2005.

[2] A. Bovykin and A. Weiermann, *Unprovable statements based on diophantine approximation and distribution of values of zeta-functions*, preprint 2007.

[3] D.J.H. De Jongh and R. Parikh, *Well-partial orderings and hierarchies*, Nederl. Akad. Wetensch. Proc. Ser. A 80 Indag. Math. **39** 1977, 195–207.

[4] L. Gordeev, *Strong well-quasi-ordering tree theorem*, preprint (1993), talk at Fourteenth British Combinatorial Conference (Keele, 1993) `http://www-ls.informatik.uni-tuebingen.de/gordeew/publikationen/QuasiOrdering.pdf`

[5] R. Hasegawa, *Two applications of analytic functors. Theories of types and proofs* (Tokyo, 1997). Theoret. Comput. Sci. **272**, 2002 113–175.

[6] A. A. Karatsuba and S. M. Voronin, *The Riemann zeta-function*, translated from the Russian by Neal Koblitz. de Gruyter *Expositions in Mathematics* **5**. Walter de Gruyter and Co., Berlin, 1992. xii+396 pp.

[7] I. Kríž, *Well-quasiordering finite trees with gap-condition. Proof of Harvey Friedman's conjecture*, Ann. of Math. (2) **130** 1989, 215–226.

[8] M. Rathjen and A. Weiermann, *Proof-theoretic investigations on Kruskal's theorem*, Annals of Pure and Applied Logic **60** 1993, 49–88.

[9] D. Schmidt, *Well-Partial Orderings and Their Maximal Order Types*, Habilitationsschrift, Heidelberg, 1979.

[10] K. Schütte, *Proof Theory,*. Translated from the revised German edition by J. N. Crossley, *Grundlehren der Mathematischen Wissenschaften* **225**. Springer-Verlag, Berlin-New York, 1977. xii+299.

[11] S. G. Simpson, *Nonprovability of certain combinatorial properties of finite trees*, in *Harvey Friedman's research on the foundations of mathematics*, L. A. Harrington, M. D. Morley, A. Ščedrov and S. G. Simpson eds., ,*Studies in Logic and the Foundations of Mathematics* **117**, Amsterdam, North-Holland, 1985, 87-117.

[12] S. M. Voronin, *A theorem on the "universality" of the Riemann zeta-function*, Izv. Akad. Nauk SSSR Ser. Mat. **39** 1975, 475–486, 703.

[13] G. Wilken, *Ordinal arithmetic based on Skolem hulling*, Annals of Pure and Applied Logic **145** 2007, 130-161

17 Keita Yokoyama

ns-WKL_0 is a system of non-standard second order arithmetic which consists of the following axioms:

1. The standard universe $V = (M, S)$ satisfies WKL_0.
2. There exist a non-standard universe $V^* = (M^*, S^*)$ and an embedding $\pi : V \to V^*$. Moreover, π is elementary with respect to Σ^0_0 formulas and $\pi(M)$ is an initial segment of M^*.
3. V and V^* are elementary equivalent with respect to Σ^1_2 sentences.
4. For any $X \in S^*$, $X \cap M$ is an element of S, i.e., the standard part of X exists.

Note that ns-WKL_0 is a conservative extension of WKL_0. (For systems of non-standard second order arithmetic, see [1,2].)

Within ns-WKL_0, we can develop a part of non-standard analysis (see [2]). In particular, we can define the standard part map $\mathrm{st} : \mathbb{R}^* \to \mathbb{R} \cup \{\pm\infty\}$. We next consider an axiom for the transfer principle.

- (Σ^0_1-TP) π is elementary with respect to Σ^0_1 formulas.

ns-WKL_0 + (Σ^0_1-TP) proves ACA_0, and it is a conservative extension of ACA_0.

Within ns-WKL_0 + (Σ^0_1-TP), we can prove the following non-standard version of the Bolzano/Weierstrass theorem:

Theorem 17.1. *(nsBWT) : Let $f : \mathbb{N} \to \mathbb{R}$ be a bounded real sequence. Then, for any $w \in \mathbb{N}^* \setminus \mathbb{N}$, $st(\pi(f)(w))$ is an accumulation value of f.*

What about the reversal of this theorem?

Problem 17.2. Is (nsBWT) equivalent to (Σ^0_1-TP) over ns-WKL_0?

Another problem was raised at the meeting:

Problem 17.3. *(Now solved)* Let (M, S) be a countable model of WWKL_0. Can we then find $\bar{S} \supseteq S$ such that $(M, \bar{S})$ is a countable model of WKL_0 and every closed set of positive measure which is coded in $\bar{S}$ contains points in S?

This result was needed to prove that the formal system for nonstandard analysis with Loeb measures, ns-BASIC + LMP, is conservative over WWKL_0. The problem was solved (positively) at the meeting by Stephen Simpson. Yokoyama and Simpson are writing a joint paper which will include this result.

References

[1] Keita Yokoyama, *Formalizing non-standard arguments in second order arithmetic*, preprint.

[2] Keita Yokoyama, *Reverse Mathematics for non-standard analysis*, in preparation.